LECTURE 1: INTRODUCTION

1. Riemann's inaugural lecture

On June 10, 1854, B. Riemann gave one of the most famous job talk in the history of mathematics, with title "**On the hypothesis which lie at the foundation of geometry**". This talk not only gained a job for him (as a privatdocent at Göttingen University), but also offered jobs for many of us including me: two of our courses, *Manifolds* and *Riemannian geometry*, born in this probationary inaugural lecture.

What Riemann did in this talk was trying to develop a higher dimensional intrinsic geometry. It is a very broad and abstract generalization of the intrinsic differential geometry of surfaces in $\mathbb{R}^3$ developed by Gauss[1].

At the beginning of Riemann's talk was a brief "plan of investigation", in which he started with the sentence "geometry presuppose the concept of space". To clear the confusion over non-Euclidean geometry at that time, he proposed to distinguish metric properties from the topological properties of the Space. The major part of the talk was divided into three parts. In part one Riemann introduced the conception of manifolds, characterized as locally looks like n-dimensional Euclidean space[2]. Part two is the major part of the talk, in which Riemann developed the desired intrinsic geometry, started by introducing a positive definite quadratic form (the Riemannian metric)[3] at each point. The crucial question Riemann asked himself in this part was: when does two Riemannian metrics locally isometric? By a dimension counting argument, Riemann argues that there should be a set of $\frac{n(n-1)}{2}$ functions which will determine the metric completely. They are nothing else but sectional curvatures (as a generalization of Gauss curvature for surfaces in $\mathbb{R}^3$) associated to

[1]In 1827, Gauss published a famous paper "**General investigation of curved surfaces**", in which he proved his Theorema Egregium ("remarkable theorem" in Latin): the Gauss curvature of a surface can be determined entirely by measuring distances along paths on the surface (intrinsic), and does not depend on how the surface might be embedded in 3-dimensional space (extrinsic).

[2]Riemann's definition of manifold is a very primitive form. Since most of his audience were non-mathematicians (faculty of Göttingen University), Riemann tried his best to make his lecture intelligible to general audience. The modern abstract definition of manifolds as "topological spaces that are Hausdorff, second countable and locally Euclidean" was introduced by H. Weyl in 1912.

[3]In fact Riemann was also aware of the existence of more general "metrics" that could be used to measure the length of tangent vectors, including the so-called Finsler metric that was developed by Finsler in 1918.

2-dimensional vector subspaces of the tangent space! Finally in part three, Riemann dealt with possible applications, especially to questions in physics. [4]

2. Riemannian geometry for Euclidean submanifolds: a quick survey on undergraduate differential geometry

Before we introduce the abstract conception of Riemannian metric on a smooth manifold next time, let's start with some basic geometry that we learned in undergraduate differential geometry course (in a higher dimensional fashion). As one can imagine, differential geometry starts by taking derivative. It turns out that all those important geometric quantities appears by this way.

¶ Curves in $\mathbb{R}^N$.

Let $\gamma : I \to \mathbb{R}^N$ be a smooth curve defined on a finite interval $I = [0, T]$. By definition the arc length $s = s(t)$ is given by

$$s(t) = \int_0^t \|\gamma'(\tau)\| d\tau.$$

Since s is strictly increasing, we may change variable and write γ as

$$\gamma = \gamma(s), s \in [0, l],$$

where l is the length of γ.

We start with the unit tangent vector $\gamma'(s)$: since $\|\gamma'(s)\| = 1$, i.e.

$$\langle \gamma'(s), \gamma'(s) \rangle = 1,$$

taking derivative one gets

$$\langle \gamma''(s), \gamma'(s) \rangle = 0,$$

i.e. $\gamma''(s) \perp \gamma'(s)$. In other words, $\gamma''(s)$ is a normal vector.

By definition,

$$\kappa(s) := \|\gamma''(s)\|$$

is called the *curvature* of γ at $\gamma(s)$, and the vector

$$n(s) := \frac{\gamma''(s)}{\|\gamma''(s)\|}$$

is called the *principal normal* of γ at $\gamma(s)$.

Remark. Note that $n(s)$ is again a unit vector. So we may repeat this process. What we will get is the *torsion* and the *binormal.* If we continue this process for the binormal, we will get Frenet formula.

[4]About 60 years later, Einstein used the theory of pseudo-Riemannian manifolds (a generalization of Riemannian manifolds) to develop his general theory of relativity. In particular, his equations for gravitation are constraints on the curvature of spacetime.

¶ The first fundamental form.

Now let M be an n-dimensional manifold embedded into $\mathbb{R}^N$. For simplicity we suppose $U \subset \mathbb{R}^n$ is an open set, and suppose

$$\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^N$$

is an injective immersion such that $\varphi(U) = M$ (or a portion of M). In what follows we denote

$$\varphi_j = \frac{\partial \varphi}{\partial x^j}, \quad 1 \le j \le n.$$

Then $T_{\varphi(x)}M = \mathrm{span}(\varphi_1, \cdots, \varphi_n)$. Now let $\mu = (\mu^1, \cdots, \mu^n) : I \to U$ be a curve in U, so that $\gamma = \varphi \circ \mu : I \to M$ be a curve in $\mathbb{R}^N$ that sits in M. Then

$$\gamma'(t) = \frac{d(\varphi \circ \mu)}{dt} = \sum_{j=1}^n \frac{d\mu^j}{dt} \varphi_j(\mu(t))$$

The arc length of γ is again given by

$$s(t) = \int_0^t \|\gamma'(\tau)\| d\tau.$$

If we denote $v^j := \frac{d\mu^j}{dt}$, $x = \mu(t)$ and

$$g_{jk}(x) := \langle \varphi_j(x), \varphi_k(x) \rangle,$$

then we get

$$\Big(\frac{ds}{dt}\Big)^2 = \|\gamma'(t)\|^2 = \sum_{j,k=1}^n g_{jk}(\mu(t)) v^j v^k.$$

After polarizing, we get a quadratic form

$$\mathrm{I}\Big(\sum_j v^j \varphi_j, \sum_k w^k \varphi_k\Big) := \sum_{j,k=1}^n g_{jk}(x) v^j w^k$$

defined on $T_{\varphi(x)}M$, which is known as the *first fundamental form* of M.

¶ The Second fundamental form.

We may continue to calculate the second derivative to get

$$\gamma''(t) = \frac{d^2(\varphi \circ \mu)}{dt^2} = \sum_{j=1}^n \frac{d^2\mu^j}{dt^2} \varphi_j(\mu(t)) + \sum_{j,k=1}^n \frac{d\mu^j}{dt}\frac{d\mu^k}{dt} \varphi_{jk}(\mu(t)),$$

where $\varphi_{jk}(x) = \frac{\partial^2 \varphi}{\partial x^j \partial x^k}(x)$. Note that the first term lies in $T_{\gamma(t)}M$. So when projecting to the *normal plane* $N_{\gamma(t)}M = (T_{\gamma(t)}M)^\perp$, and denoting

$$h_{jk} = \mathrm{Proj}_{N_{\gamma(t)}M}(\varphi_{jk}),$$

one gets

$$\sum_{j,k=1}^{n} h_{jk} v^j v^k = \mathrm{Proj}_{N_{\gamma(t)}M}(\gamma''(t)).$$

For simplicity let's take arc length parametrization, so that $\sum v^j \varphi_j$ is a unit vector (i.e. $\sum g_{jk} \frac{d\mu^j}{ds} \frac{d\mu^k}{ds} = 1$). Then we get

$$\sum_{j,k=1}^{n} h_{jk} v^j v^k = \kappa(s) \mathrm{Proj}_{N_{\gamma(s)}M}(n(s)).$$

After polarizing, the resulting quadratic form

$$\mathrm{I\!I}(\sum v^j \varphi_j, \sum w^k \varphi_k) := \sum h_{jk} v^j v^k$$

(defined on $T_x M$ with value in $N_x M$) is known as the *second fundamental form* of M. In the case M is a hypersurface (i.e. $n = N - 1$), by fixing an orientation on M one may identify $N_{\gamma(t)}M$ with $\mathbb{R}$, and thus $\mathrm{I\!I}$ can be viewed as a real-valued quadratic form.

¶ The Christoffel symbols.

Interesting quantities also appears when we study the tangent component of $\gamma''(t)$. Since $\varphi_{jk}(x) - h_{jk}(x) \in T_{\varphi(x)}M = \mathrm{span}(\varphi_1, \cdots, \varphi_n)$, one may write

$$\varphi_{jk}(x) = \sum_{l=1}^{n} \Gamma^l{}_{jk} \varphi_l(x) + h_{jk}(x).$$

Paring with the vector φ_i, one gets

$$\langle \varphi_{jk}, \varphi_i \rangle = \sum_{l=1}^{n} \Gamma^l{}_{jk} g_{li}.$$

A miracle is that the mysterious coefficients $\Gamma^l{}_{jk}$ can be calculated via g_{jk}'s: From

$$\partial_k g_{ij} = \langle \varphi_{ik}, \varphi_j \rangle + \langle \varphi_i, \varphi_{jk} \rangle$$

one gets

$$\langle \varphi_{jk}, \varphi_i \rangle = \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{kj}).$$

So if we denote $(g^{ij}) = (g_{ij})^{-1}$, then

$$\Gamma^l{}_{jk} = \sum_i g^{il} \langle \varphi_{jk}, \varphi_i \rangle = \frac{1}{2} \sum_i g^{il} (\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{kj}).$$

The functions $\Gamma^l{}_{jk}$ are known as *Christoffel symbols*. Note that they are determined by the first fundamental form. In summary, we see that

$$\gamma''(t) = \sum_{j=1}^{n} \left(\frac{d^2 \mu^j}{dt^2} + \sum_{i,k=1}^{n} \Gamma^j{}_{ik} \frac{d\mu^i}{dt} \frac{d\mu^k}{dt} \right) \varphi_j(\mu(t)) \bmod N_{\gamma(t)}M.$$

¶ The covariant derivative and geodesics.

In particular, if γ is parametrized by arc length s (i.e. $\sum g_{jk}\frac{d\mu^j}{ds}\frac{d\mu^k}{ds}=1$), then

$$\sum_{j=1}^{n}\left(\frac{d^2\mu^j}{ds^2}+\sum_{i,k=1}^{n}\Gamma^j{}_{ik}\frac{d\mu^i}{ds}\frac{d\mu^k}{ds}\right)\varphi_j(\mu(s))=\kappa(s)\mathrm{Proj}_{T_{\gamma(t)}M}(n(s)).$$

The length

$$\kappa_g(s):=\left\|\sum_{j=1}^{n}\left(\frac{d^2\mu^j}{ds^2}+\sum_{i,k=1}^{n}\Gamma^j{}_{ik}\frac{d\mu^i}{ds}\frac{d\mu^k}{ds}\right)\varphi_j(\mu(s))\right\|$$

is known as the *geodesic curvature* of γ. If $\kappa_g(s)\equiv 0$, then γ is called a *geodesic*. They are locally shortest paths (generalizations of straight lines in Euclidean space and great circles in sphere) in M.

More generally, given any vector field $X=\sum X^j(x)\varphi_j(x)$ along γ (which, by definition, is tangent to M everywhere), the same computation yields

$$\frac{dX}{dt}=\sum_{j,k=1}^{n}\left(\partial_k X^j+\Gamma^j{}_{ik}X^i\right)\frac{d\mu_k}{dt}\varphi_j(\mu(t)) \bmod N_{\gamma(t)}M,$$

which is known as the *covariant derivative* of the vector field X along γ.

¶ The Riemann curvature.

What about vector fields N that are normal to M? We may calculate the tangential component of the derivative of $N(x)$ in a similar way. For this purpose we write

$$\partial_i N(x)=\sum_{k=1}^{n}N_i^k(x)\varphi_k(x) \mod N_{\varphi(x)}M.$$

Start with the equation $\langle N(x),\varphi_j(x)\rangle=0$. By taking derivative we get

$$\langle\partial_i N(x),\varphi_j(x)\rangle+\langle N,\varphi_{ij}\rangle=0,$$

i.e.

$$\sum_k N_i^k g_{kj}=-\langle h_{ij},n\rangle.$$

It follows

$$N_i^k=-\sum_j\langle h_{ij},N\rangle g^{kj}$$

and thus we get, for any normal vector field N on M,

$$\partial_i N(x)=-\sum_{j,k}\langle h_{ij},N\rangle g^{kj}\varphi_k(x) \mod N_{\varphi(x)}M.$$

Applying this formula to the normal vector fields h_{ij}, we may calculate the tangential component of $\varphi_{ijk} = \partial_i \partial_j \partial_k \varphi$. Since $\varphi_{ij} = \sum \Gamma^l{}_{ij} \varphi_l + h_{ij}$, we get

$$\varphi_{kij} = \partial_k \varphi_{ij} = \sum_{m=1}^{n} \big(\partial_k \Gamma^m{}_{ij} + \sum_l \Gamma^l{}_{ij} \Gamma^m{}_{lk} - \sum_l \langle h_{ij}, h_{kl} \rangle g^{lm} \big) \varphi_m \mod N_{\varphi(x)} M.$$

Since $\varphi_{kij} = \varphi_{jik}$, we get

$$\partial_j \Gamma^m{}_{ik} - \partial_k \Gamma^m{}_{ij} + \sum_{l=1}^{n} \big(\Gamma^l{}_{ik} \Gamma^m{}_{lj} - \Gamma^l{}_{ij} \Gamma^m{}_{lk} \big) = \sum_{l=1}^{n} \big(\langle h_{ik}, h_{jl} \rangle - \langle h_{ij}, h_{kl} \rangle \big) g^{lm}.$$

We define

$$R_{ijk}{}^m := \partial_j \Gamma^m{}_{ik} - \partial_k \Gamma^m{}_{ij} + \sum_{l=1}^{n} \big(\Gamma^l{}_{ik} \Gamma^m{}_{lj} - \Gamma^l{}_{ij} \Gamma^m{}_{lk} \big)$$

and let

$$R_{ijkl} := \sum_m g_{lm} R_{ijk}{}^m,$$

then we get $R_{lijk} = \langle h_{ik}, h_{jl} \rangle - \langle h_{ij}, h_{kl} \rangle$. The $(0,4)$-tensor

$$R(\sum X^l \varphi_l, \sum Y^i \varphi_i, \sum Z^j \varphi_j, \sum W^k \varphi_k) := \sum R_{lijk} X^l Y^i Z^j W^k$$

on $T_x M$ is called the *Riemann curvature tensor*. It admits many nice symmetry properties from which one can show that the quantity

$$\frac{R(X,Y,X,Y)}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2}$$

depends only on the two dimensional plane $\mathrm{span}(X,Y)$. It is known as the *sectional curvature* of M with respect to the plane. By taking a basis there are $\frac{n(n-1)}{2}$ such functions, and they are the $\frac{n(n-1)}{2}$ functions first studied by Riemann!