LECTURE 2: THE RIEMANNIAN METRIC

As we have seen last time, in Riemannian geometry there will be lots of summations for quantities with many indices. To simplify notions/computations, from now on we will follow the

Einstein Summation Convention: If an expression is a product of several terms with indices, and if an index variable appears twice in this expression, once as an upper index in one term and once as a lower index in another term[a], then (unless otherwise stated) the expression is understood to be a summation over all possible values of that index (usually from 1 to the space dimension). For example,

$$a_i b^i := \sum_i a_i b^i, \quad a^{ijkl} b_{il}^m c_j := \sum_{i,j,l} a^{ijkl} b_{il}^m c_j.$$

[a]Note: an upper index in the denominator will be regarded as a lower index, and vice versa.

One should also be aware of how we choose upper and lower indices in this course (trying to meet Einstein summation convention). For example, vector fields are always indexed by lower indices (like $X_1, X_2, \cdots$) while the coefficients of vector fields will be indexed by upper indices (e.g. $a^1 X_1 + a^2 X_2$). Similarly a collection of 1-forms will be indexed by upper indices while the coefficients of their linear combinations will be indexed by lower indices.

1. The Riemannian metric

¶ Definition of Riemannian metric.

Let M be a smooth manifold of dimension m, in other words, M is a second countable Hausdorff topological space such that near every point $p \in M$, there is a neighborhood U of p which is diffeomorphic to a domain in $\mathbb{R}^m$. Moreover, if we denote by $\{x^1, \cdots, x^m\}$ the coordinate functions on U, then the tangent space T_pM is spanned by the vectors $\{\partial_1, \cdots, \partial_m\}$, and its dual T_p^*M (the cotangent space) is spanned by $\{dx^1, \cdots, dx^m\}$.

Definition 1.1. A *Riemannian metric* g on M is an assignment of an inner product

$$g_p(\cdot, \cdot) = \langle \cdot, \cdot \rangle_p$$

on T_pM for each $p \in M$, such that the assignment depends smoothly on p.

Remarks. (1) As we have seen last time, the Riemannian metric g is motivated by the first fundamental form of a surface in space. They will be used to measure the length of curves in M.

(2) "Smooth dependence" $\iff$ if X, Y are two smooth vector fields on an open subset $U \subset M$, then $f(p) = \langle X_p, Y_p \rangle_p$ is a smooth function on U.

(3) The Riemannian metric g itself is NOT a *metric* (aka a *distance function*) on M. Recall that a distance function on M is a continuous function $d : M \times M \to \mathbb{R}$ so that for all $p, q, r \in M$,

- $d(p,q) \geq 0$, and $d(p,q) = 0$ if and only if $p = q$;
- $d(p,q) = d(q,p)$;
- $d(p,r) \leq d(p,q) + d(q,r)$.

However, we will see soon that g induces a natural distance function d on M, and the topology generated by d on M coincides with its original manifold topology.

¶ Riemannian metric as a tensor field.

We may also use the language of tensors. By definition

$$g : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to C^\infty(M)$$

defined in the obvious way is $C^\infty(M)$-bilinear, and thus can be viewed as a $(0,2)$-tensor on M. The remaining conditions of being an inner product (i.e. symmetric and positive definite) at each point now becomes, in the language of tensors, that the $(0,2)$-tensor g is symmetric and positive definite. So we get another description of a Riemannian metric g:

A Riemannian metric g is a smooth symmetric $(0,2)$-tensor field that is positive definite.

We remark that many geometric structures on smooth manifold M are defined as a special tensor field. For example, an almost complex structure on M is a special $(1,1)$-tensor field, a symplectic structure on M is a special $(0,2)$ tensor field, while a Poisson structure on M is a special $(2,0)$ tensor field.

¶ Riemannian metric via local coordinates.

One can represent the Riemannian metric g using local coordinates as follows. Let $\{U, x^1, \cdots, x^m\}$ be a coordinate patch. We denote

$$g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p.$$

It is easy to see that the functions g_{ij} have the following properties:

- For all i, j, $g_{ij}(p)$ is smooth in p.
- $g_{ij} = g_{ji}$, so the matrix $(g_{ij}(p))$ is symmetric at any p.
- The matrix $(g_{ij}(p))$ is also positive definite for any p.

Note that although g is intrinsically defined, the functions g_{ij} depend on the choice of coordinate system. If $\{\tilde{x}^1, \cdots, \tilde{x}^n\}$ is another coordinated system on U, then

$$\tilde{\partial}_i = \frac{\partial x^k}{\partial \tilde{x}^i} \partial_k.$$

It follows that

$$\tilde{g}_{ij} := \langle \tilde{\partial}^i, \tilde{\partial}^j \rangle = \frac{\partial x^k}{\partial \tilde{x}^i} g_{kl} \frac{\partial x^l}{\partial \tilde{x}^j}.$$

In other words, the matrices $(\tilde{g}_{ij})$ and (g_{ij}) are related by the matrix equation

$$(\tilde{g}_{ij}) = J^T (g_{ij}) J$$

where J is the Jacobian matrix whose (i,j)-element is $(\frac{\partial x^i}{\partial \tilde{x}^j})$.

Since for any smooth vector fields $X = X^i \partial_i$ and $Y = Y^j \partial_j$ in U,

$$\langle X_p, Y_p \rangle_p = X^i(p) Y^j(p) \langle \partial_i, \partial_j \rangle_p = g_{ij}(p) X^i(p) Y^j(p),$$

so locally we can write the 2-tensor g as

$$g = g_{ij} dx^i \otimes dx^j.$$

¶ The dual Riemannian metric on the cotangent space.

Since each matrix (g_{ij}) is positive definite, it is invertible. We will denote by (g^{ij}) the inverse matrix of (g_{ij}), i.e. they satisfy

$$g_{ij} g^{jk} = \delta_i^k.$$

Then the matrix (g^{ij}) is again positive definite, and we can use it to define a *dual* inner product structure g^* on T_p^*M for each p. More explicitly, for any 1-forms

$$\omega = \omega_i dx^i \quad \text{and} \quad \eta = \eta_i dx^i$$

on U, we define

$$g^*(\omega, \eta) = \langle \omega, \eta \rangle_p^* := g^{ij}(p) \omega_i(p) \eta_j(p).$$

We will leave as a simple exercise for the reader to check that this definition is independent of the choices of coordinates.

¶ The musical isomorphisms.

Since g is non-degenerate and bilinear on T_pM, it gives us an isomorphism between T_pM and T_p^*M via[1]

$$\flat : T_pM \to T_p^*M, \qquad \flat(X_p)(Y_p) := g_p(X_p, Y_p).$$

(Pronunciation of $\flat$: flat)

It is not hard to see that $\flat$ maps smooth vector fields to smooth 1-forms, and gives rise to a vector bundle isomorphism between TM and T^*M.

In local coordinates, if we denote $X = X^i \partial_i$ and take $Y = \partial_j$ for each j, then

$$\flat(X)(\partial_j) = g(X, \partial_j) = g_{ij} X^i,$$

so we conclude

$$\flat(X^i \partial_i) = g_{ij} X^i dx^j.$$

[1] Although $\dim T_pM = \dim T_p^*M$, without using a Riemannian metric or some other extra structure, we don't have a *natural* isomorphism between T_pM and T_p^*M.

In other words, $\flat$ "lowers the indices" via g_{ij}, i.e. changes the coefficients from X^i to $X_i := g_{ij}X^i$.

We will denote the inverse map of $\flat$ by

$$\sharp : T_p^*M \to T_pM.$$

(Pronunciation of $\sharp$: sharp)

Then in local coordinates,

$$\sharp(w_i dx^i) = g^{ij} w_i \partial_j.$$

So $\sharp$ "raises the indices" via g^{ij}. We will call $\flat$ and $\sharp$ the *musical isomorphisms*[2].

Note that for any 1-form ω and η,

$$g_p(\sharp\omega, \sharp\eta) = g_{ij} g^{ki} \omega_k g^{lj} \eta_l = \delta_j^k \omega_k \eta_l g^{lj} = g^{kl} \omega_k \eta_l = \langle \omega, \eta \rangle_p^*.$$

In other words, the dual inner product $g_p^*(\omega, \eta)$ on T_p^*M we mentioned above can be defined as $g_p(\sharp\omega, \sharp\eta)$, which is a coordinate-free definition of g^*.

¶ Riemannian metric for tensors.

Given the Riemannian inner product g on T_pM and the induced inner product g^* on T^*M, one may further define a natural inner product $T_l^k(g)$, also denoted by g if there is no ambiguity, on the tensor product space $(T_pM)^{\otimes k} \otimes (T_p^*M)^{\otimes l}$ as follows:

> Let $W = (T_pM)^k \times (T_p^*M)^l$ (the Cartesian product). Consider the map $W \times W \to \mathbb{R}$ given by
>
> $$((X_1,\cdots,X_k,\omega_1,\cdots,\omega_l),(Y_1,\cdots,Y_k,\eta_1,\cdots,\eta_l)) \mapsto g(X_1,Y_1)\cdots g(X_k,Y_k) g^*(\omega_1,\eta_1)\cdots g^*(\omega_l,\eta_l).$$
>
> It is a multi-linear map which is linear in each entry. By universality of tensor product, it gives rise to a unique bilinear map
>
> $$(T_pM)^{\otimes k} \otimes (T_p^*M)^{\otimes l} \times (T_pM)^{\otimes k} \otimes (T_p^*M)^{\otimes l} \to \mathbb{R}$$
>
> which can be proven to be an inner product.

This inner product can be characterized by the following property: Suppose $e_1, \cdots, e_m$ is an orthonormal basis of (T_pM, g_p), and $e^1, \cdots, e^m$ its dual basis of (T_p^*M, g_p^*). Then the induced inner product on $(T_pM)^{\otimes k} \otimes (T_p^*M)^{\otimes l}$ is defined so that

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes e^{j_1} \otimes \cdots \otimes e^{j_l}\}$$

form an orthonormal basis.

In local coordinates, if $T = T^{i_1\cdots i_k}_{j_1\cdots j_l} \partial_{i_1} \otimes \cdots \otimes \partial_{i_k} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_k}$ and likewise for a (k,l)-tensor S, then

$$\langle T, S \rangle = g^{j_1 b_1} \cdots g^{j_l b_l} g_{i_1 a_1} \cdots g_{i_k a_k} T^{i_1\cdots i_k}_{j_1\cdots j_l} S^{a_1\cdots a_k}_{b_1\cdots b_l}.$$

As an example, we see that the length square of the metric tensor g itself is

$$|g|^2 = \langle g, g \rangle = g^{ik} g^{jl} g_{ij} g_{kl} = \delta_j^k \delta_k^j = m.$$

[2]In music, the symbol $\flat$ means lower in pitch while the symbol $\sharp$ means higher in pitch.

2. Riemannian manifolds

¶ Riemannian manifolds: Definition and simplest example.

Let M be a smooth manifold.

Definition 2.1. Let g be Riemannian metric on M. Then we call the pair (M, g) a *Riemannian manifold.* (Sometimes we omit g and say M is a Riemannian manifold.)

Example. The simplest manifold of dimension m is $\mathbb{R}^m$, on which we can endow many Riemannian metrics:

(1) The standard inner product on $\mathbb{R}^m$ defines a *canonical Riemannian metric* g_0 on $\mathbb{R}^m$ via
$$g_0(X, Y) = \sum_i X^i Y^i.$$
Alternatively, this means the matrix (g_{ij}) is the identity matrix:
$$(g_0)_{ij} = \delta_{ij}.$$
In the notion of tensors, we can write
$$g_0 = dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m.$$

(2) More generally, for any positive definite $m \times m$ matrix $A = (a_{ij})$, the formula
$$g_p^A(X_p, Y_p) := X_p^T A Y_p$$
defines a Riemannian metric on $\mathbb{R}^m$ in which case $g_{ij}^A = a_{ij}$. Equivalently,
$$g^A = \sum_{i,j} a_{ij} dx^i \otimes dx^j.$$

(3) Since $\mathbb{R}^m$ admits a global coordinate system, one may even describe all possible Riemannian metrics on $\mathbb{R}^m$: Endow the space $\mathrm{Sym}(m)$ of all $m \times m$ symmetric matrices (which is linearly isomorphic to $\mathbb{R}^{m(m+1)/2}$) the standard smooth structure, then the subset $\mathrm{PosSym}(m)$ of all positive definite $m \times m$ matrices is open and thus again a smooth manifold. By definition, any smooth map
$$g : \mathbb{R}^m \to \mathrm{PosSym}(m) \subset \mathrm{Sym}(m)$$
defines a Riemannian metric on $\mathbb{R}^m$, and vice versa.

Example. On the torus $\mathbb{T}^m = (S^1)^m$, one has the following flat Riemannian metric
$$g_0 = d\theta^1 \otimes d\theta^1 + \cdots + d\theta^m \otimes d\theta^m.$$

Example. Consider the upper half plane $\mathbb{H}^2 = \{(x, y) \mid y > 0\}$. On $\mathbb{H}^2$ the Riemannian metric
$$g_{(x,y)} = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$
is known as the *Hyperbolic metric*, and $(\mathbb{H}^2, g)$ is known as the *hyperbolic plane.*

¶ Constructing new Riemannian manifolds.

There are many ways to construct new Riemannian manifolds from old ones, for example,

(1) Let (M, g_M) and (N, g_N) be two Riemannian manifolds, then $g_M \oplus g_N$ defined by
$$(g_M \oplus g_N)_{(p,q)}((X_p, Y_q), (X'_p, Y'_q)) = (g_M)_p(X_p, X'_p) + (g_N)_q(Y_q, Y'_q)$$
is a Riemannian metric on $M \times N$, whose matrix is simply
$$\begin{pmatrix} (g_1)_{m_1\times m_1} & 0 \\ 0 & (g_N)_{m_2\times m_2} \end{pmatrix},$$
where m_1, m_2 are the dimensions of M and N respectively.

Definition 2.2. We will call $(M \times N, g_M \oplus g_N)$ the *product Riemannian manifold* of (M, g_M) and (N, g_N).

For example,
- The Euclidean space $(\mathbb{R}^m, g_0)$ is the Riemannian product of m copies of $(\mathbb{R}, g_0)$.
- The torus $(\mathbb{T}^m, g_0)$ is the Riemannian product of m copies of of the standard circle $(S^1, d\theta \otimes d\theta)$.
- The hyperbolic plane $(\mathbb{H}, g)$ is NOT a Riemannian products of two 1-dimensional manifolds.

(2) Let (N, g_N) be a Riemannian manifold, and $f : M \to N$ a smooth *immersion*, i.e. $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. Then the "pull-back metric" f^*g_N on M defined by
$$(f^*g_N)_p(X_p, Y_p) = (g_N)_{f(p)}(df_p(X_p), df_p(Y_p))$$
is a Riemannian metric on M.

Definition 2.3. We call $g_M := f^*g_N$ the *induced metric* or the *pulled-back metric* on M with respect to f, and call $f : (M, g_M) \to (N, g_N)$ an *isometric immersion*. If f is an embedding, then f is called an *isometric embedding*.

(3) Let (N, g_N) be a Riemannian manifold, and $M \subset N$ be an immersed/embedded submanifold. Then the inclusion map $\iota : M \to N$ is an immersion, which defines an induced Riemannian metric on M.

Definition 2.4. We call (M, ι^*g_N) an immersed/embedded *Riemannian submanifold* of (N, g_N). (Usually "Riemannian submanifold" refers to "embedded Riemannian submanifold").

Note that under the identification of T_pM with $d\iota_p(T_pM) \subset T_pN$, the induced metric $(\iota^*g_N)_p$, viewed as an inner product or a tensor field, is just the restriction of g_N onto the subspace $T_pM \subset T_pN$.

(4) Let (M, g) be any Riemannian manifold, and $u : M \to \mathbb{R}$ an arbitrary smooth function on M. Then $e^{2u}g$ defined by

$$(e^{2u}g)_p(X_p, Y_p) = e^{2u(p)}g_p(X_p, Y_p)$$

is a Riemannian metric on M.

Definition 2.5. We say a Riemannian metric g' on M is *conformal* to g if

$$g' = e^{2u}g$$

for some $u \in C^\infty(M)$.

By definition, if two Riemannian metrics g' and g are conformal, then when we replace g by g', for each p, all vectors in T_pM are stretched in length by the same constant $e^{u(p)}$, while the angle between any pair of vectors in T_pM keeps the same.

¶ S^2 as a Riemannian submanifold of $\mathbb{R}^3$.

Example. Let $M = S^2$ be the unit 2-sphere in $\mathbb{R}^3$. The induced Riemannian metric g (from the canonical Riemannian metric g_0 on $\mathbb{R}^3$) is known as the *round metric*. To calculate g locally, we need to choose a coordinate patch.

For example, we can use cylindrical coordinates θ and z to parametrize S^2,

$$x = \sqrt{1-z^2}\cos\theta, \quad y = \sqrt{1-z^2}\sin\theta, \quad z = z,$$

with $0 < \theta < 2\pi, -1 < z < 1$. Then

$$dx = \frac{-z}{\sqrt{1-z^2}}\cos\theta dz - \sqrt{1-z^2}\sin\theta d\theta$$

and

$$dy = \frac{-z}{\sqrt{1-z^2}}\sin\theta dz + \sqrt{1-z^2}\cos\theta d\theta.$$

It follows

$$\begin{aligned} g_{S^2} &= [dx \otimes dx + dy \otimes dy + dz \otimes dz]|_{S^2} \\ &= \frac{z^2}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta + dz \otimes dz \\ &= \frac{1}{1-z^2} dz \otimes dz + (1-z^2) d\theta \otimes d\theta. \end{aligned}$$

Alternatively, one may use the colatitude $\theta \in (0, \pi)$ and the longitude $\varphi \in (0, 2\pi)$ to parametrize S^2 as

$$x = \sin\theta\cos\varphi, \quad y = \sin\theta\sin\varphi, \quad z = \cos\theta.$$

A similar computation as above will give us

$$g_{S^2} = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi.$$

¶ Isometries and local isometries.

Next let's define the notion of "equivalence" in the Riemannian world.

Definition 2.6. Let (M, g_M) and (N, g_N) be two Riemannian manifolds.

(1) If $\varphi : M \to N$ is a local diffeomorphism such that $g_M = \varphi^* g_N$, then we call φ a *local isometry*.
(2) If a local isometry $\varphi : (M, g_M) \to (N, g_N)$ is invertible, then we call φ an *isometry*, in which case we say (M, g_M) and (N, g_N) are *isometric* Riemannian manifolds.

Isometries are crucial in Riemannian geometry since isometric Riemannian manifolds will be viewed as the same. Local isometries are also important in studying local invariants like curvatures.

Remark. A map $\varphi : (M, g_M) \to (N, g_N)$ is a local isometry if and only if for any $p \in M$, there exists a neighborhood U of p in M so that $\varphi : U \to \varphi(U)$ is an isometry.

Example. For any $m \times m$ positive definite matrix A, $(\mathbb{R}^m, g^A)$ is isometric to $(\mathbb{R}^m, g_0)$. [Can you write down the isometry?]

Example. On the set $M = \mathbb{R}_{>0} \times (0, 2\pi)$, consider the Riemannian metric

$$g = dr \otimes dr + r^2 d\theta \otimes d\theta.$$

Then the map

$$\varphi : M \to \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}, \quad (r, \theta) \mapsto (r \cos\theta, r \sin\theta)$$

(where the latter is endowed with the standard Euclidean metric) is an isometry. [Obviously (M, g) is really the polar coordinate system for $\mathbb{R}^2$.]

Example. For the standard metrics on $\mathbb{R}^m$ and T^m: if we regard $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, then the projection $\pi : (\mathbb{R}^m, g_0) \to (\mathbb{T}^m, g_0)$ is a local isometry but not a global isometry.

¶ The isometry group.

Obviously isometries satisfies the following functorality:

- If $\varphi : (M, g_M) \to (N, g_N)$ is an isometry, the φ^{-1} is an isometry.
- If $\varphi : (M, g_M) \to (N, g_N)$ and $\psi : (N, g_N) \to (P, g_P)$ are two isometries, then the composition $\psi \circ \varphi : (M, g_M) \to (P, g_P)$ is again an isometry.

In particular, if we let

$$\mathrm{Isom}(M, g) = \{\varphi : (M, g) \to (M, g) \mid \varphi \text{ is an isometry}\}.$$

Then $\mathrm{Isom}(M, g)$ is a group. It is a subgroup of the diffeomorphism group

$$\mathrm{Diff}(M) = \{\varphi : M \to M \mid \varphi \text{ is a diffeomorphism}\}.$$

Definition 2.7. We call $\mathrm{Isom}(M, g)$ the *isometry group* of (M, g).

For example,

- The isometry group of $(\mathbb{R}^m, g_0)$ is the Euclidean group $E(m) = O(m) \ltimes \mathbb{R}^m$.
- The isometry group of (S^2, g_{round}) is the orthogonal group $O(3)$.

Remark. A remarkable theorem proved by Myers and Steenrod in 1939 claims

> **Theorem 2.8** (Myers-Steenrod). *Let (M, g) be any Riemannian manifold. Then with respect to the compact open topology, there is a smooth structure on* $\mathrm{Isom}(M, g)$ *so that* $\mathrm{Isom}(M, g)$ *is a Lie group, which is compact if M is compact. Moreover, the obvious action of* $\mathrm{Isom}(M, g)$ *on M is smooth.*

On the other hand, as we have learned in the course of smooth manifold, the diffeomorphism group $\mathrm{Diff}(M)$ can be regarded as an "infinite dimensional Lie group" whose Lie algebra is the Lie algebra of all smooth vector fields on M (for simplicity we may assume M is compact). So the isometry group $\mathrm{Isom}(M, g)$ of a Riemannian manifold (M, g), as a Lie group which is finite dimensional and carries a smooth structure, is much nicer than the diffeomorphism group $\mathrm{Diff}(M)$ of the underlying manifold M. What is the Lie algebra of $\mathrm{Isom}(M, g)$? They are nothing else but those vector fields whose flow are isometries, known as *Killing vector fields*.

By the remark above we see that the Riemannian structure is much more *rigid* than the smooth structure. As we know, locally manifolds of the same dimension are always the same. However, this is not the case for Riemannian manifolds: there are rich local geometry in the Riemannian world.

¶ The existence of Riemannian metric.

The first remarkable theorem in this course is

Theorem 2.9. *On any smooth manifold M, there exist (<u>many</u>) Riemannian metrics on any smooth manifold M.*

We shall give two proofs of this theorem.

The first proof. We first take a locally finite covering of M by coordinate patches $\{U_\alpha, x_\alpha^1, \cdots, x_\alpha^m\}$. It is clear that one can choose a Riemannian metric g_α on each U_α, e.g. one may take

$$g_\alpha = \sum_i dx_\alpha^i \otimes dx_\alpha^i.$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinate the chosen covering $\{U_\alpha\}$. We define

$$g = \sum_\alpha \rho_\alpha g_\alpha.$$

Note that this is in fact a finite sum in the neighborhood of each point. It is positive definite since for any $p \in M$, there always exist some α such that $\rho_\alpha(p) > 0$. So it is a Riemannian metric on M. $\square$

Second proof. According to the famous Whitney embedding theorem, any smooth manifold M of dimension m can be embedded into $\mathbb{R}^{2m+1}$ as a smooth submanifold, and thus each Riemannian metric on $\mathbb{R}^{2m+1}$ will induce a Riemannian submanifold metric on M. □

Remark. One may ask: How large is the space of all Riemannian metrics on a given smooth manifold? Let

$$\mathrm{Riem}(M) = \{g \mid g \text{ is a Riemannian metric on } M\}$$

be the set of all Riemannian metrics on M. Motivated by the first proof, it is easy to see that if g_1, g_2 are two Riemannian metrics on M, so is $ag_1 + bg_2$ for $a, b > 0$. As a consequence, $\mathrm{Riem}(M)$ (as a subset in the infinite dimensional vector space of all symmetric $(0,2)$-tensor fields on M) is a positive convex cone.

Of course a natural question is: Given a manifold, can one find a Riemannian metric that is "best" in some sense? This is one of the main targets in Riemannian geometry. In this course we shall define various kind of invariants (curvatures) of Riemannian metrics, and we shall study the relations between these invariants and the topology of the underlying manifold.

Remark. In the second proof we used the Whitney embedding theorem. For Riemannian manifolds (M, g), there is a much stronger embedding theorem proved by the famous Nobel prize (in Economics) winner John Nash in 1956,

Theorem 2.10 (Nash embedding theorem)**.** *Any m-dimensional Riemannian manifold (M, g) can be isometrically embedded into the standard $(\mathbb{R}^N, g_0)$ as a Riemannian submanifold, where*

$$N = \begin{cases} \frac{m(3m+11)}{2}, & \text{if } M \text{ is compact,} \\ \frac{m(m+1)(3m+11)}{2}, & \text{if } M \text{ is noncompact.} \end{cases}$$

For compact manifolds, the dimension N was lowered by Gromov in 1986 to

$$N = \frac{m^2 + 5m + 6}{2},$$

and then was further lowered by Günther in 1989 to

$$N = \max\{\frac{m^2 + 5m}{2}, \frac{m^2 + 3m + 10}{2}\}.$$

In particular, any 2-dimensional smooth Riemannian manifold can be isometrically embedded into $\mathbb{R}^{10}$ (instead of $\mathbb{R}^{17}$ by Nash). It is still not known whether this dimension can be further lowered. [3]

[3] However, if instead of C^∞ maps (or C^r-maps for $r \geq 3$), one only require a "C^1-isometries", i.e. C^1-diffeomorphism $\varphi : M \to N$ with $\varphi^* g_N = g_M$, then Nash showed in 1955 that any (M^m, g) can be C^1-isometrically embedded into $\mathbb{R}^{2m+1}$. Of course "C^1-isometries" are not natural objects in Riemannian geometry, because as we have seen last time, basic Riemannian geometric quantities like curvatures need third order derivatives of the embedding (second order derivatives of the Riemannian metric).