LECTURE 4: THE RIEMANNIAN MEASURE

1. The Riemannian measure

¶ The Riemannian volume in tangent space.

Not only a Riemannian metric g (as an infinitesimal distance, i.e. a distance defined in each tangent space) on M gives rise to a canonical metric structure on M, but also it defines a canonical measure structure (or to be more precise, a volume density) on M through an "infinitesimal volume" (i.e. volume defined in each tangent space). The idea is standard: as in multi-variable calculus, to define the volume or integrate a function over M, one simply start with a coordinate chart, using which one can divide M into small coordinate pieces, and then approximate each small piece $\{(x^1, \cdots, x^m) \mid a^i \le x^i \le a^i + h^i\}$ by the parallelepiped in T_pM (where $p = (x^1, \cdots, x^m)$) generated by $(h^1\partial_1, \cdots, h^m\partial_m)$.

Now the problem is reduces to: how do we define a volume of a parallelepiped in a finite dimensional inner product space? Well, one can always define the volume of a unit cube to be 1 (here we use not only the lengths of vectors, but also the angles between vectors), and then use multi-linearity to extend the definition to more general parallelepipeds. So to compute the volume of the parallelepiped generated by $\partial_1, \partial_2, \cdots, \partial_m$, we start with any an orthonormal basis $e_1, \cdots, e_m$ of (T_pM, g_p), and define the volume of the parallelotope generated by $e_1, \cdots, e_m$ to be

$$V_p(e_1, e_2, \cdots, e_m) = 1.$$

Then we write $\partial_i = a_i^j e_j$, which implies

$$V_p(\partial_1, \partial_2, \cdots, \partial_m) = |\det(a_i^j)|.$$

For simplicity we denote $A = (a_i^j)$. From the observation

$$g_{ij} = g(\partial_i, \partial_j) = g(a_i^k e_k, a_j^l e_l) = \sum_k a_i^k a_j^k = (AA^T)_{ij},$$

we conclude $(g_{ij}) = AA^T$, and thus the "infinitesimal volume" we are calculating is

$$V_p(\partial_1, \partial_2, \cdots, \partial_m) = |\det(a_i^j)| = \sqrt{G},$$

where $G = \det(g_{ij})$.

Remark. Alternatively, one can define $V_p(\partial_1, \partial_2, \cdots, \partial_m)$ as "the length of the vector $\partial_1 \wedge \partial_2 \wedge \cdots \wedge \partial_m$ in the space $\otimes^m T_pM$" (with respect to the induced metric on tensors that we introduced in Lecture 2), and similar computation yields the same result.

¶ Integrals of compactly supported continuous functions.

Now let (M, g) be a Riemannian manifold. We start with a continuous function f with compact support, so that $\mathrm{supp}(f)$ is contained in one chart (φ, U, V). As motivated by the previous computation, we may define

$$\int_M f dV_g := \int_V (f\sqrt{G}) \circ \varphi^{-1} \; dx^1 \cdots dx^m,$$

where $dx^1 \cdots dx^m$ the Lebesgue measure on $\mathbb{R}^m$.

Lemma 1.1. *The definition above is independent of the choices of coordinate charts containing* $\mathrm{supp}(f)$*.*

Proof. Let $(\tilde{\varphi}, \widetilde{U}, \widetilde{V})$ be another coordinate chart containing $\mathrm{supp}(f)$, on which the coordinates are denoted by $y^1, \cdots, y^m$. Then as we have seen in Lecture 2,

$$(g_{ij}) = J^T(\tilde{g}_{kl})J,$$

where $J = (\frac{\partial y^i}{\partial x^j})$ is the Jacobian of the map $\widetilde{\varphi} \circ \varphi^{-1}$. As a consequence, we get

$$\sqrt{G(p)} = \sqrt{\widetilde{G}(p)} \; |\det(J(\varphi(p)))|$$

for $p = \varphi^{-1}(x) = \tilde{\varphi}^{-1}(y)$, and thus by change of variables in $\mathbb{R}^m$,

$$\sqrt{\widetilde{G} \circ \tilde{\varphi}^{-1}} dy^1 \cdots dy^m = \sqrt{\widetilde{G} \circ \tilde{\varphi}^{-1}(\tilde{\varphi} \circ \varphi^{-1})} \; |\det(J)| dx^1 \cdots dx^m = \sqrt{G \circ \varphi^{-1}} dx^1 \cdots dx^m.$$

The conclusion follows. □

Of course in general, even if f is compactly supported, one cannot assume that $\mathrm{supp}(f)$ is contained in one single chart. However, one can extend the above definition to general $f \in C_c(M)$ easily by using partition of unity: Let $\{(\varphi_\alpha, U_\alpha, V_\alpha)\}$ be a system of locally finite coordinate charts that cover M, with local coordinates $\{x_\alpha^1, \cdots, x_\alpha^m\}$ on each U_α, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$. Then we define

$$\int_M f dV_g := \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (f \rho_\alpha \sqrt{G^\alpha}) \circ (\varphi_\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m,$$

Note that by locally finiteness of U_α and compactness of $\mathrm{supp}(f)$, the sum is in fact a finite sum. Moreover, if $\{(\tilde{\varphi}_\beta, \widetilde{U}_\beta, \widetilde{V}_\beta)\}$ is another atlas, the by Lemma 1.1,

$$\int_{\varphi_\alpha(U_\alpha \cap \widetilde{U}_\beta)} (f \rho_\alpha \tilde{\rho}_\beta \sqrt{G^\alpha}) \circ (\varphi_\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m = \int_{\tilde{\varphi}_\beta(U_\alpha \cap \widetilde{U}_\beta)} (f \rho_\alpha \tilde{\rho}_\beta \sqrt{G^\beta}) \circ (\varphi_\beta)^{-1} dx_\beta^1 \cdots dx_\beta^m$$

since both sides equal to $\int_M \rho_\alpha \tilde{\rho}_\beta f dV_g$, which implies

$$\sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (f \rho_\alpha \sqrt{G^\alpha}) \circ (\varphi_\alpha)^{-1} dx_\alpha^1 \cdots dx_\alpha^m = \sum_\beta \int_{\varphi_\beta(U_\beta)} (f \rho_\beta \sqrt{G^\beta}) \circ (\varphi_\beta)^{-1} dx_\beta^1 \cdots dx_\beta^m.$$

In other words, $\int_M f dV_g$ is well-defined for any $f \in C_c(M)$.

¶ The Riemannian measure.

Since manifolds are always locally compact and Hausdorff, and since the linear functional

$$\mu : C_c(M) \to \mathbb{R}, \quad f \mapsto \mu(f) = \int_M f dV_g$$

is positive (i.e. $f \geq 0$ implies $\mu(f) \geq 0$), by Riesz representation theorem, μ gives rise to a unique Radon measure on M. Now one can further extend the integral to more general functions using the standard machinery developed in real analysis:

- first define the (upper) integral of a lower semi-continuous positive function f to be the supremum of integrals of compactly-supported functions that are no more than f,
- then define the (upper) integral of a positive function f as the infimum of the (upper) integral of all lower semi-continuous positive function f that are greater than f,
- a function f is said to be *integrable* if there exists a sequence g_n in $C_c(M)$ so that the (upper) integrals of the sequence $|g_n - f|$ converge to 0.

As usual we denote the space of integrable functions as $L^1(M, g)$, which by definition is the completion of $C_c(M)$ with respect to suitable norm.

As usual, for any $1 \leq p < \infty$ one can define the L^p norm on C_c^∞ via

$$\|f\|_{L^p} := \left(\int_M |f|^p dV_g \right)^{1/p},$$

and define $L^p(M, g)$ to be the completion of C_c^∞ under the L^p norm. Similarly one can define $L^\infty(M, g)$. It is not hard to extend the theory to complex-valued functions. In the special case $p = 2$, one can define an inner product structure on $L^2(M, g)$ by

$$\langle f_1, f_2 \rangle_{L^2} := \int_M f_1 \bar{f}_2 dV_g$$

which make $L^2(M, g)$ into a Hilbert space.

One can also talk about the volume of any Borel set (or more generally, measurable subsets) A in M, which is defined to be

$$\mathrm{Vol}(A) = \int_M \chi_A dV_g$$

Remark. In the above definition, we don't assume M to be oriented or compact. What we really get is a volume density, which, on a local chart, can be written as

$$dV_g = \sqrt{G} \circ \varphi^{-1} dx^1 \cdots dx^m.$$

We will call $d\mathrm{Vol}$ the *Riemannian volume element* (or *volume density*) on (M, g).

Remark. In the special case where M is oriented, then we may choose an orientation-compatible coordinate patch near each point, and define (locally on each chart)

$$\omega_g = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m.$$

One can check that ω_g is a well-defined global volume form on M, which is called the *Riemannian volume form* for the oriented Riemannian manifold (M, g).

Remark. Suppose (M, g) is an m-dimensional Riemannian manifold, and S an r-dimensional submanifold of M, where $r < m$. Then the Riemannian submanifold metric $g_S := \iota^* g$ on M gives a natural measure (an r-dimensional volume density) on S. Here are two special cases:

- If $\gamma : I \to M$ is a simple smooth curve, then with respect to the coordinates t (from the parametrization), we have $g_\gamma = g(\partial_t, \partial_t)dt \otimes dt = |\dot{\gamma}(t)|^2 dt \otimes dt$, and thus the induced 1-dimensional volume density (i.e. *length density*) on γ is simply $|\dot{\gamma}|dt$, which is exactly what we used to calculate the length of γ.
- If M is a smooth manifold with boundary, in which case the boundary ∂M is a smooth submanifold of dimension $m-1$, then one gets a natural Riemannian submanifold metric and thus a volume density on ∂M. In this case the volume density on ∂M is usually called a *surface density* (or *hypersurface density*) and will be denoted by dS_g.

¶ The change of variable formula.

By using the standard change of variable formula for the Lebesgue measure in $\mathbb{R}^m$, together with a partition of unity argument, one can easily prove the following

Proposition 1.2 (Change of variables in Riemannian setting)**.** *Let $\varphi : M \to N$ be a diffeomorphism, and h a Riemannian metric on N. Then*

$$\int_M f \circ \varphi \quad dV_{\varphi^* h} = \int_N f \ dV_h, \quad \forall f \in L^1(N, h).$$

In particular, we see isometries preserve the Riemannian volume densities.

As another consequence, suppose $\dim M \le \dim N$, $\varphi : M \to N$ is an embedding, and $\iota : \varphi(M) \to N$ is the inclusion map. Let g be a Riemannian metric on M and h be a Riemannian metric on N, then [1]

$$\int_M f \circ \varphi \ \frac{dV_{\varphi^* h}}{dV_g} dV_g = \int_{\varphi(M)} f \ dV_{\iota^* h}, \quad \forall f \in L^1(N, h),$$

where $\frac{dV_{\varphi^* h}}{dV_g}$ is the Radon-Nikodyn derivative of the two corresponding Riemannian measures on M (which are by definition σ-finite measures). In particular, one may find the area of $\varphi(M)$ (or integrals over $\varphi(M)$) in the target space by doing computations in the source space M. It is a very special case of the so-called area formula in

[1]Note that it makes no sense to write an expression like $\int_M f \circ \varphi \ |\det d\varphi| \ dV_g$ even if φ is a diffeomorphism, since $d\varphi$ is a linear map between different vector spaces.

geometric measure theory where φ is only supposed to be Lipschitz and need not be injective, and the measures encountered are replaced by the Hausdorff measure.

There is a "dual" version of the area formula above, known as the *co-area formula*, in which, with the help of a map $\varphi : M \to N$ with $\dim M \geq \dim N$, one could use integrals over level sets $\varphi^{-1}(q)$ in target space to compute integrals over the source space M. In the very general version of co-area formula in geometric measure theory, φ is only supposed to be a Lipschitz map, and people use the Hausdorff measures. In what follows we will prove a simplest version of co-area formula (for $N = \mathbb{R}$) that is already very useful in Riemannian geometry. To state the theorem, we need the concept of gradient vector fields associated to a function.

¶ The gradient.

Let (M, g) be a Riemannian manifold. For any smooth function f on M, the differential df is a smooth 1-form on M. By using the musical isomorphism $\sharp : T^*M \to TM$, we will get a smooth vector field on M:

Definition 1.3. The *gradient vector field* of f is $\nabla f = \sharp(df)$.

It is not hard to find out ∇f in local charts: By definition, ∇f is the vector field so that for any vector field $X = X^i \partial_i$,

$$g(\nabla f, X) = df(X) = Xf = X^i \partial_i f.$$

It follows that locally

$$\nabla f = g^{ij} \partial_i f \partial_j.$$

In particular, for $g = g_0$ in $\mathbb{R}^m$, we get the ordinary gradient of f.

As in multivariable calculus, the gradient vector field of a function is always perpendicular to its regular level sets:

Lemma 1.4. *Suppose f is a smooth function on M and c is a regular value of f. Then the gradient vector field ∇f is perpendicular to the level set $f^{-1}(t)$.*

Proof. Since c is a regular value, by the regular level set theorem, $f^{-1}(c)$ is a smooth submanifold of M. Let X be a vector field tangent to $f^{-1}(c)$. Then we learned from manifold theory that $Xf = 0$ on $f^{-1}(c)$. It follows

$$g(\nabla f, X) = Xf = 0$$

on $f^{-1}(c)$. So ∇f is perpendicular to $f^{-1}(c)$. □

¶ The Coarea formula: a simple version.

Fix a smooth function $u \in C^\infty(M)$ and let

$$\Omega_t := u^{-1}((-\infty, t)), \quad \Gamma_t := u^{-1}(t).$$

For any regular value t of u, Γ_t is a smooth submanifold of dimension $m - 1$ in M. By Sard's theorem, critical values of u form a measure zero set in $\mathbb{R}$ (and thus can be ignored in the integration $\int_{\mathbb{R}}$ below). Now we can prove

Theorem 1.5 (The co-area formula, a simple version)**.** *Let (M, g) be a Riemannian manifold. For any regular value t of u, let g_t be the induced Riemannian metric on Γ_t. and denote the corresponding Riemannian volume density on Γ_t by dS_t. Then for any integrable function f on M, one has*

$$\int_M f|\nabla u| dV_g = \int_{\mathbb{R}} \left(\int_{\Gamma_t} f dS_t \right) dt.$$

Proof. First note that if we let C be the set of critical points of u, then C is closed. It follows that $M \setminus C$ is an open submanifold in M, and obviously

$$\int_M f|\nabla u| dV_g = \int_{M\setminus C} f|\nabla u| dV_g.$$

So we may replace M by $M \setminus C$ without changing both sides. In other words, we may assume u admits no critical point on M.

Now consider the vector field

$$X = \frac{\nabla u}{|\nabla u|^2}$$

on M. By Lemma 1.4, X is perpendicular to $T_q\Gamma_c$ at any $q \in \Gamma_c$ for any c. Let φ_t be the (local) flow generated by X. Then by definition,

$$\frac{d}{dt} u(\varphi_t(q)) = du(X(\varphi_t(q))) = \langle \nabla u, X \rangle_{\varphi_t(q)} = 1.$$

It follows that if $q \in \Gamma_c$, then $\varphi_t(q) \in \Gamma_{c+t}$ for t small enough. Now we choose a neighborhood A of q in Γ_c so that

$$\psi : (-\varepsilon, \varepsilon) \times A \to M, \quad (y, t) \mapsto \varphi_t(y)$$

is a diffeomorphism onto an open subset $U = \psi((-\varepsilon, \varepsilon) \times A)$ in M. By shrinking A if necessary, we may suppose A is a coordinate patch on Γ_c and let $y^1, \cdots, y^{m-1}$ be corresponding coordinate functions. Then $\{t, y^1, \cdots, y^{m-1}\}$ form a set of coordinate functions on U. With respect to these coordinates, and in view of the facts $\partial_t = X$ and $X \perp \partial_{y^i}$ for all i, the Riemannian metric g has the form

$$g = \langle X, X \rangle dt \otimes dt + h_{ij} dy^i \otimes dy^j,$$

where $h_{ij} = g(\partial_{y^i}, \partial_{y^j})$. Since $\langle X, X \rangle = \frac{1}{|\nabla u|^2}$, the volume density

$$dV_g = \frac{1}{|\nabla u|} \sqrt{\det (h_{ij})} dt dy^1 \cdots dy^{m-1} = \frac{1}{|\nabla u|} dt dS_t.$$

So we conclude that for any $\rho \in C_c(U)$,

$$\int_M \rho f |\nabla u| dV_g = \int_U \rho f \sqrt{\det G_t} dt dy^1 \cdots dy^{m-1} = \int_{c-\varepsilon}^{c+\varepsilon} \left(\int_{\Gamma_t \cap U} \rho f dS_t \right) dt.$$

Now the conclusion follows from a standard partition of unity argument. □

As a corollary, we get

Corollary 1.6. *Suppose the critical values of u form a closed subset*[2] *in $\mathbb{R}$, and $\mathrm{Vol}(\Omega_t) < \infty$, then the function $t \mapsto \mathrm{Vol}(\Omega_t)$ is smooth at regular value t, and*

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = \int_{\Gamma_t} \frac{1}{|\nabla u|} dS_t.$$

Proof. For any regular t, take $\varepsilon > 0$ so that $(t, t+\varepsilon)$ is free of critical values. By taking $f = \frac{1}{|\nabla u|}$ we get, for $h \in (0, \varepsilon)$,

$$\mathrm{Vol}(\Omega_{t+h}) - \mathrm{Vol}(\Omega_t) = \int_t^{t+h} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} dS_t \right) dt.$$

It follows

$$\frac{d}{dt}\mathrm{Vol}(\Omega_t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \left(\int_{\Gamma_t} \frac{1}{|\nabla u|} dS_t \right) dt = \int_{\Gamma_t} \frac{1}{|\nabla u|} dS_t.$$

□

2. The Laplace-Beltrami operator

¶ The divergence of a vector field.

Let X be a smooth vector field on M. Take a coordinate patch $(U, x^1, \cdots, x^m)$ (which is of course orientable) on M, then the volume element

$$\omega_g = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m$$

is locally an n-form on U. Of course one may choose other coordinates on U, then the corresponding volume forms are either the same, or differ by a negative sign. As a result, the following definition is independent of the choice of coordinate charts:

Definition 2.1. The *divergence* of X is the function $\mathrm{div}(X)$ on M such that

$$(\mathrm{div} X)\omega_g = d\big(\iota(X)\omega_g\big).$$

Remark. According to Cartan's magic formula, the definition above is equivalent to

$$\mathcal{L}_X(\omega_g) = \mathrm{div}(X)\omega_g,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. This coincides with the geometric definition of divergence in the case of $\mathbb{R}^m$: the divergence of a vector field is the infinitesimal rate of change of the volume element along the vector field.

Let's calculate $\mathrm{div}(X)$ locally. Let $X = X^i \partial_i$, then

$$\begin{aligned}(\mathrm{div} X)\sqrt{G} dx^1 \wedge \cdots \wedge dx^m &= d\left(\iota(X^i\partial_i)\sqrt{G} dx^1 \wedge \cdots \wedge dx^m\right) \\ &= d\Big(\sum_i X^i \sqrt{G} (-1)^{i-1} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m\Big) \\ &= \partial_i(X^i\sqrt{G}) dx^1 \wedge \cdots \wedge dx^m,\end{aligned}$$

[2] This condition holds if u is a proper function.

so we conclude

$$\mathrm{div}(X^i\partial_i) = \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G}).$$

We may replace X by fX to get

$$\mathrm{div}(fX) = f\mathrm{div}X + (\partial_i f)X^i = f\mathrm{div}X + g(\nabla f, X).$$

In other words,

Corollary 2.2. *For any smooth vector field $X \in \Gamma^\infty(TM)$ and any smooth function $f \in C^\infty(M)$, one has*

$$\mathrm{div}(fX) = f\mathrm{div}X + g(\nabla f, X).$$

As an application, we prove

Theorem 2.3 (The Divergence theorem I)**.** *Let X be a smooth vector field with compact support on a Riemannian manifold (M, g), then*

$$\int_M \mathrm{div}(X)dV_g = 0.$$

Proof. First we assume that X is supported in a local chart (φ, U, V) and $X = X^i\partial_i$ with $X^i \in C_c^\infty(U)$. Then

$$\begin{aligned}\int_M \mathrm{div}(X)dV_g &= \int_U \frac{1}{\sqrt{G}}\partial_i(X^i\sqrt{G})dV_g \\ &= \int_{\varphi(U)} \partial_i(X^i\sqrt{G}\circ\varphi^{-1})dx^1\cdots dx^m = 0.\end{aligned}$$

The general case follows from partition of unity and Corollary 2.2:

$$\sum_\alpha \rho_\alpha \mathrm{div}(X) = \sum_\alpha \mathrm{div}(\rho_\alpha X) - g(\nabla(\sum_\alpha \rho_\alpha), X) = \sum_\alpha \mathrm{div}(\rho_\alpha X)$$

and thus

$$\int_M \mathrm{div}(X)dV_g = \int_M \sum \rho_\alpha \mathrm{div}(X)dV_g = \int_M \sum_\alpha \mathrm{div}(\rho_\alpha X)dV_g = 0.$$

□

¶ The Laplace-Beltrami operator.

Let (M, g) be a Riemannian manifold.

Definition 2.4. For any smooth function f, we define the *Laplacian* of f to be

$$\Delta f = -\mathrm{div}(\nabla f).$$

Locally, Δf is given by

$$\Delta f = -\text{div}(g^{ij}\partial_i f \partial_j) = -\frac{1}{\sqrt{G}}\partial_i(\sqrt{G} g^{ij}\partial_j f),$$

i.e.

$$\Delta = -\frac{1}{\sqrt{G}}\partial_i(\sqrt{G} g^{ij}\partial_j).$$

We shall call Δ the *Laplace-Beltrami* operator. It is a second order differential operator on M, and is the most important differential operator on Riemannian manifolds. It plays an essential role on the analysis of Riemannian manifolds.

Theorem 2.5 (Green's formula I)**.** *Suppose f and h are smooth function on M and either f or h is compactly supported. Then*

$$\int_M f\Delta h dV_g = \int_M g(\nabla f, \nabla h) dV_g = \int_M h\Delta f \ dV_g.$$

Proof. We have seen

$$\text{div}(fX) = f\text{div}X + g(\nabla f, X).$$

It follows

$$\text{div}(f\nabla h) = -f\Delta h + g(\nabla f, \nabla h).$$

Now the theorem follows from the fact that $f\nabla h$ is compactly supported. □

In particular if M is compact (without boundary), then any smooth function is compactly supported. Replacing h by $\overline{h}$ if they are complex-valued, we can rewrite the above formula as

$$\langle f, \Delta h\rangle_{L^2} = \langle \Delta f, h\rangle_{L^2}.$$

In other words, we get

Corollary 2.6. *If M is compact, then Δ is densely defined symmetric operator on $L^2(M, g)$.*

As another immediate consequence, we see that Δ is a positive operator:

Corollary 2.7. *If M is compact, then $\langle \Delta f, f\rangle_{L^2} \geq 0$.*

Remark. Both the divergence theorem and the Green's formula can be generalized to the case where M is a *compact Riemannian manifold with boundary*, i.e. M is

- an m dimensional smooth manifold with boundary
- M is also a compact subset of an m dimensional Riemannian manifold N
- The Riemannian structure on M coincide with that of N

So ∂M carries

(1) an outward normal vector field ν
(2) an induced Riemannian metric from g_N, and thus a volume density dA.

Then for any smooth vector field X on M and any smooth functions f, h on M,

- (Divergence Theorem II) $\displaystyle\int_M \mathrm{div}(X) dV_g = \int_{\partial M} g(X, \nu) dA,$
- (Green's formula II) $\displaystyle\int_M f\Delta h\ dV_g = \int_M g(\nabla f, \nabla h)\ dV_g - \int_{\partial M} g(\nu, \nabla h) f\ dA.$

Details will be left as an exercise.

¶ Laplacian v.s. isometry.

Why the operator Δ is so important in Riemannian geometry? Since differential operators are local, it is quite obvious that if $\varphi : (M, g_M) \to (N, g_N)$ is a local isometry, then $\psi^*(\Delta_N f) = \Delta_M(\psi^* f)$. Conversely,

Proposition 2.8. *A diffeomorphism $\psi : M \to N$ is an isometry between (M, g_M) and (N, g_N) if and only if it commutes with the Beltrami-Laplace operators, i.e.*

$$\psi^*(\Delta_N f) = \Delta_M(\psi^* f), \quad \forall f \in C^\infty(N).$$

Proof. Obviously if φ is an isometry, then it commutes with the Beltrami-Laplace operators. Conversely, suppose the diffeomorphism ψ commutes with the Beltrami-Laplace operators. Take a chart (φ, U, V) on M so that $(\varphi \circ \psi^{-1}, \psi(U), V)$ is a chart on N. Denote the coordinates by $x^1, \cdots, x^m$ and $y^1, \cdots, y^m$ respectively. Then under these coordinates, for $y = \psi(x)$ we have

$$\partial_i^M (f \circ \psi)(x) = \frac{\partial((f \circ \psi) \circ \varphi^{-1})}{\partial x^i}(\varphi(x)) = \frac{\partial(f \circ (\varphi \circ \psi^{-1})^{-1})}{\partial x^i}(\varphi \circ \psi^{-1}(y)) = \partial_i^N f(y)$$

and thus

$$(\partial_i^N \partial_j^N f) \circ \psi = \partial_i^M \partial_j^M (f \circ \psi).$$

On the other hand, we have

$$\begin{aligned}(\psi^* \Delta_N f)(x) = (\Delta_N f)(\psi(x)) &= -\frac{1}{\sqrt{G_N}} \partial_i^N (\sqrt{G_N} g_N^{ij} \partial_j^N f)(\psi(x)) \\ &= -(g_N^{ij} \partial_i^N \partial_j^N f)(\psi(x)) + \cdots .\end{aligned}$$

and

$$\Delta_M(\psi^* f)(x) = -\frac{1}{\sqrt{G}} \partial_i^M (\sqrt{G} g_M^{ij} \partial_j^M (f \circ \psi))(x) = -(g_M^{ij} \partial_i^M \partial_j^M (f \circ \psi))(x) + \cdots .$$

where $\cdots$ represents terms that involve only first order derivatives of f. So by comparing the coefficients of second order terms, we get $g_M^{ij}(x) = g_N^{ij}(\psi(x))$, as desired. □