# LECTURE 6: THE LEVI-CIVITA CONNECTION

## 1. Induced Linear connections on tensors

**¶ Linear connections on the trivial line bundle.**

Let $M$ be a smooth manifold, and $E$ a vector bundle over $M$. As we have seen, a linear connection on $E$ is a bilinear map

$$\nabla : \Gamma^\infty(TM) \times \Gamma^\infty(E) \to \Gamma^\infty(E), \quad (X, s) \mapsto \nabla_X s$$

such that for any $f \in C^\infty(M)$,

$$\nabla_{fX} s = f \nabla_X s$$

and

$$\nabla_X(fs) = f\nabla_X s + (Xf)s.$$

Again there are numerous choices of linear connections on any vector bundle.

Consider the simplest vector bundle, the trivial line bundle $M \times \mathbb{C}$, which will be regarded as $\otimes^{0,0}TM$ below. Since $\Gamma^\infty(\otimes^{0,0}TM) = C^\infty(M)$, by definition a linear connection on this bundle is a bilinear map

$$\nabla : \Gamma^\infty(TM) \times C^\infty(M) \to C^\infty(M)$$

that satisfies the two conditions above. Since we are only considering "directional derivative of smooth functions", we have an obvious and perfect candidate, namely

$$\nabla : \Gamma^\infty(TM) \times C^\infty(M) \to C^\infty(M), \quad (X, f) \mapsto \nabla_X f := Xf, \tag{1}$$

which obviously satisfies the two conditions, and is canonical in the sense that it depends only on the smooth structure of $M$.

Although we will use the canonical linear connection $\nabla$ defined by (1) for smooth functions, we should point out that there exist many other interesting linear connections. In fact, for any smooth 1-form $\omega \in \Omega^1(M)$, we have

$$\nabla_X(gf) = X(gf) + gf\omega(X) = (Xg)f + g(Xf + f\omega(X)) = (Xg)f + g\nabla_X f,$$

which implies

**Lemma 1.1.** *For any 1-form $\omega$ on $M$,*

$$\nabla^\omega_X f := Xf + f\omega(X)$$

*is a linear connection on $\otimes^{0,0}TM$.*

Equivalently, we can write this connection as $\nabla = d + \omega$.

**¶ The induced linear connection on cotangent bundle.**

Suppose we are given a linear connection $\nabla$ on $\otimes^{1,0}TM = TM$. Together with the canonical linear connection $\nabla$ on $\otimes^{0,0}TM = M \times \mathbb{R}$, next let's try to find a reasonable linear connection on the cotangent bundle $T^*M$. By definition the covariant derivative we want to construct is a bilinear map

$$\nabla : \Gamma^\infty(TM) \times \Gamma^\infty(T^*M) \to \Gamma^\infty(T^*M), \quad (X, \omega) \mapsto \nabla_X \omega$$

with given properties. The idea is simple and natural: we need to apply the pairing between $T^*M$ and $TM$. Note that the linear connection $\nabla$ on $TM$ gives rise to a parallel transport map $P^\gamma_{0,t} : T_{\gamma(0)}M \to T_{\gamma(t)}M$, and by taking dual one gets a linear isomorphism

$$(P^\gamma_{0,t})^* : T^*_{\gamma(t)}M \to T^*_{\gamma(0)}M.$$

With this map at hand, it is thus natural to define the covariant derivative to be

$$\nabla_X \omega(p) := \lim_{t \to 0} \frac{(P^\gamma_{0,t})^* \omega_{\gamma(t)} - \omega_p}{t}, \tag{2}$$

where $\gamma$ is any curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. To get a clear sense of this formula using $\nabla$ on $TM$ instead of using $P^\gamma$, let's pair the 1-form $\nabla_X\omega$ with any vector field $Y$, to get

$$(\nabla_X \omega)(Y) = \lim_{t \to 0} \frac{(P^\gamma_{0,t})^* \omega_{\gamma(t)}(Y_p) - \omega_p(Y_p)}{t} = \lim_{t \to 0} \frac{\omega_{\gamma(t)}(P^\gamma_{0,t}(Y_p)) - \omega_p(Y_p)}{t}$$

We have

$$\begin{aligned}\omega_{\gamma(t)}(P^\gamma_{0,t}(Y_p)) - \omega_p(Y_p) &= \omega_{\gamma(t)}(P^\gamma_{0,t}(Y_p)) - \omega_{\gamma(t)}(Y_{\gamma(t)}) + \omega_{\gamma(t)}(Y_{\gamma(t)}) - \omega_p(Y_p) \\ &= -\omega_{\gamma(t)}\Big(P^\gamma_{0,t}\big((P^\gamma_{0,t})^{-1}(Y_{\gamma(t)}) - Y_p\big)\Big) + \omega_{\gamma(t)}(Y_{\gamma(t)}) - \omega_p(Y_p).\end{aligned}$$

So in view of the facts

$$\lim_{t \to 0} \frac{(P^\gamma_{0,t})^{-1}(Y_{\gamma(t)}) - Y_p}{t} = \nabla_X Y$$

and

$$\lim_{t \to 0} \frac{\omega_{\gamma(t)}(Y_{\gamma(t)}) - \omega_p(Y_p)}{t} = \frac{d}{dt}|_{t=0} \omega(Y)(\gamma(t)) = \dot\gamma(0)(\omega(Y)) = X(\omega(Y)) = \nabla_X(\omega(Y))$$

we get the desired formula

$$(\nabla_X \omega)(Y) = \nabla_X(\omega(Y)) - \omega(\nabla_X Y). \tag{3}$$

Note that although it looks like our "definition formula" (2) may depend on the curve $\gamma$, the formula (3) shows that it is independent of the choice of $\gamma$.

**¶ Induced linear connection for tensors.**

One can continue this process. Let

$$(P_{0,t}^{\gamma})^{(r,s)}: \otimes^{r,s}T_{\gamma(0)}M \to \otimes^{r,s}T_{\gamma(t)}M$$

be the naturally induced linear isomorphism (which equals $P_{0,t}^{\gamma}$ on tangent components, and equals $((P_{0,t}^{\gamma})^*)^{-1}$ on cotangent components). Then for any tensor field $T \in \Gamma^\infty(\otimes^{r,s}TM)$, one may naturally define

$$\nabla_X T(p) := \lim_{t\to 0} \frac{\left((P_{0,t}^{\gamma})^{(r,s)}\right)^{-1} T_{\gamma(t)} - T_p}{t}, \tag{4}$$

where $\gamma$ is any curve with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$.

After some standard but messy computations as above, one can convert the conceptional definition above to a "computable" formula

$$\begin{aligned}(\nabla_X T)(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s) =& \nabla_X(T(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s)) \\ &- \sum_i T(\omega_1, \cdots, \nabla_X\omega_i, \cdots, \omega_r, Y_1, \cdots, Y_s) \\ &- \sum_j T(\omega_1, \cdots, \omega_r, Y_1, \cdots, \nabla_X Y_j, \cdots, Y_s).\end{aligned} \tag{5}$$

*Example.* Let $\nabla$ be a linear connection on $M$, and $g$ be a Riemannian metric which is a $(0,2)$-tensor field on $M$. Applying the induced linear connection to $g$ we get

$$(\nabla_X g)(Y, Z) = X\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle.$$

**¶ Parallel tensors.**

As in the case of vector fields, the linear connection $\nabla$ on $\otimes^{r,s}TM$ satisfies the three locality properties. We may also talk about parallel tensors:

**Definition 1.2.** A tensor field $T$ is called *parallel along* $\gamma$ if $\nabla_{\dot\gamma}T = 0$, and is called *parallel* (in all directions) if $\nabla_X T = 0$ for all $X \in \Gamma^\infty(TM)$.

*Example.* Under the natural pairing between $T_p^*M$ with $T_pM$, we may view the identity map $\mathrm{Id}: \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ as a $(1,1)$-tensor via

$$I(\omega, Y) = \omega(Y).$$

It is not surprising that $I$ (which comes from the identity map) is parallel:

$$(\nabla_X I)(\omega, Y) = X(\omega(Y)) - (\nabla_X\omega)(Y) - \omega(\nabla_X Y) = 0,$$

which gives a second explanation of (3).

**¶ Compatibility of the induced linear connection.**

Now let $M$ be a smooth manifold, and $\nabla$ a linear connection on (the tangent bundle of) $M$. As we have seen, $\nabla$ induces linear connections on all tensor bundles $\otimes^{r,s}TM$ over $M$. It turns out that the induced connections are consistent in the sense that they are compatible with the two natural operations on tensors: the tensor product and the contraction.

To see this, let's consider two examples:

*Example.* For any $Y \in \Gamma^\infty(TM)$ and $\omega \in \Omega^1(M) = \Gamma^\infty(T^*M)$, applying $\nabla$ to the $(1,1)$-tensor field $Y \otimes \omega$ we get

$$\begin{aligned}\nabla_X(Y\otimes\omega)(\eta, Z) &= X(\eta(Y)\omega(Z)) - (\nabla_X\eta)(Y)\omega(Z) - \eta(Y)\omega(\nabla_X Z)\\ &= X(\eta(Y))\omega(Z) - (\nabla_X\eta)(Y)\omega(Z) + \eta(Y)X(\omega(Z)) - \eta(Y)\omega(\nabla_X Z)\\ &= ((\nabla_X Y)\otimes\omega)(\eta, Z) + (Y\otimes(\nabla_X\omega))(\eta, Z).\end{aligned}$$

In other words,

$$\nabla_X(Y\otimes\omega) = (\nabla_X Y)\otimes\omega + Y\otimes(\nabla_X\omega).$$

*Example.* Here is another way to understand the fact $\nabla I = 0$: Let $C_1^1$ be the contraction map that pairs the first tangent component to the first cotangent component, then

$$X(\omega(Y)) = \nabla_X(C_1^1(Y\otimes\omega)),$$

and by the previous example,

$$C_1^1(\nabla_X(Y\otimes\omega)) = C_1^1((\nabla_X Y)\otimes\omega + Y\otimes\nabla_X\omega) = \omega(\nabla_X Y) + (\nabla_X\omega)(Y).$$

In other words, the fact "the identity map $I$ being parallel" implies the fact "$\nabla$ commutes with $C_1^1$" for $(1,1)$-tensor. Similarly one can show that for an $(r,s)$-tensor, $\nabla$ commutes with all contraction $C_j^i$'s.

Now we can state the compatibility of $\nabla$ with the two tensor operations:

**Theorem 1.3.** *Given a linear connection $\nabla$ on $TM$, the induced linear connection*

$$\nabla : \Gamma^\infty(TM)\times\Gamma^\infty(\otimes^{r,s}TM) \to \Gamma^\infty(\otimes^{r,s}TM), \quad (X,T)\mapsto \nabla_X T,$$

*on tensor bundles $\otimes^{r,s}TM$ above is compatible with the tensor product operation* [1]

$$\nabla_X(T_1\otimes T_2) = (\nabla_X T_1)\otimes T_2 + T_1\otimes(\nabla_X T_2) \tag{6}$$

*and commutes with the contractions*

$$C_j^i(\nabla_X T) = \nabla_X C_j^i(T), \tag{7}$$

*where $1\le i\le r, 1\le j\le s$, and*

$$C_j^i : \Gamma^\infty(\otimes^{r,s}TM)\to\Gamma^\infty(\otimes^{r-1,s-1}TM)$$

*is the contraction map that pairs the $i$-th vector with the $j$-th covector.*

[1]This fact has another beautiful explanation: For any $X$, the covariant derivative operator $\nabla_X$ is a derivation on the (graded tensor) algebra of all tensor fields on $M$!

The proof is merely a simple but messy computation which we will omit. Instead, we will show how do we recover (3) using compatibility conditions (6) and (7):

$$\begin{aligned}\nabla_X(\omega(Y)) &= \nabla_X(C_1^1(Y\otimes\omega)) = C_1^1(\nabla_X(Y\otimes\omega))\\ &= C_1^1(Y\otimes\nabla_X\omega + \nabla_X Y\otimes\omega) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y)\end{aligned}$$

which is another way to write (3).

Moreover, by a tedious messy induction argument in the same philosophy, one can even recover (5) by using (6) and (7). In other words,one has

**Theorem 1.4.** *Given any linear connection $\nabla$ on the tangent bundle $TM$, there is a unique linear connection on all tensor fields that coincides with $\nabla$ on $TM$, coincides with* (1) *on functions, and satisfies compatibility conditions* (6) *and* (7) *above.*

## ¶ The Hessian of a function.

Let $M$ be a smooth manifold and $\nabla$ a linear connection on $M$. One may equivalently write the induced linear connections on tensor bundles as maps

$$\nabla : \Gamma^\infty(\otimes^{r,s}TM) \to \Gamma^\infty\big(T^*M\otimes(\otimes^{r,s}TM)\big) = \Gamma^\infty(\otimes^{r,s+1}TM)$$

with the understanding that

$$\nabla T(\cdots, X) = (\nabla_X T)(\cdots).$$

Then one may iterate $\nabla$ to get

$$\nabla^2 : \Gamma^\infty(\otimes^{r,s}TM) \to \Gamma^\infty(\otimes^{r,s+2}TM)$$

(or even higher order powers) in the understanding that

$$\nabla^2 T(\cdots, X, Y) = (\nabla_Y\nabla T)(\cdots, X) = (\nabla_Y\nabla_X T)(\cdots) - (\nabla_{\nabla_Y X}T)(\cdots). \tag{8}$$

[Note that $\nabla^2 T(\cdots, X, Y) \neq (\nabla_Y\nabla_X T)(\cdots)$ in general.]

In particular, if we take $r = s = 0$, i.e. consider functions $f \in C^\infty(M)$, we get

$$\nabla^2 f(X, Y) = (\nabla_Y df)(X) = YXf - (\nabla_Y X)f.$$

The bilinear form $\nabla^2 f$ is known as the *Hessian* of $f$ with respect to $\nabla$.

## ¶ Torsion tensors of a linear connection.

For a general linear connection $\nabla$, the Hessian is not interesting, since it might be non-symmetric. A natural question is: when will $\nabla^2 f$ symmetric? We calculate:

$$\begin{aligned}\nabla^2 f(X,Y) - \nabla^2 f(Y,X) &= (\nabla_X Y)f - (\nabla_Y X)f - XYf + YXf\\ &= (\nabla_X Y - \nabla_Y X - [X,Y])f\end{aligned}$$

It follows that the vector field

$$\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{9}$$

measures how far $\nabla^2 f$ from being symmetric. A direct computation shows

$$\mathcal{T}(fX, Y) = \mathcal{T}(X, fY) = f\mathcal{T}(X, Y).$$

In other words, $\mathcal{T}$ is really a $(1,2)$-tensor (where we identify $\mathcal{T}$ with the (1,2)-tensor $\widetilde{\mathcal{T}}(\omega, X, Y) := \omega(\mathcal{T}(X,Y))$).

**Definition 1.5.** For any linear connection $\nabla$ on $TM$, the map

$$\mathcal{T} : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$$

defined by (9) is called the *torsion tensor* of $\nabla$.

*Example.* Consider the connection $\nabla$ defined on $\mathbb{R}^3$ so that with respect to the standard frame $e_1, e_2, e_3$,

$$\nabla_{e_i} e_j = e_i \times e_j,$$

where $\times$ is the cross product. Then

$$\mathcal{T}(e_i, e_j) = e_i \times e_j - e_j \times e_i = 2 e_i \times e_j.$$

To understand the effect of the torsion, let's parallel transport the vector $e_2$ along the $e_1$-axis starting at the origin. Let $X = a(x)e_1 + b(x)e_2 + c(x)e_3$ be the parallel transport of $e_2$ along the $e_1$-axis. Then we have

$$0 = \nabla_{e_1} X = a'(x)e_1 + (b'(x) - c(x))e_2 + (c'(x) + b(x))e_3,$$

i.e.

$$a'(x) = 0, \quad b'(x) = c(x), \quad c'(x) = -b(x).$$

Together with the initial condition $a(0) = c(0) = 0, b(0) = 1$, we will get

$$X(x) = (\cos x)e_2 - (\sin x)e_3.$$

From this formula one can see that in the presence of a torsion, how the vector $e_2$ "twist" when we parallel transport it.

Back to the Hessian $\nabla^2 f$. We have seen that for $\nabla^2 f$ to be symmetric for all $f$, one need the linear connection to have vanishing torsion tensor.

**Definition 1.6.** If $\mathcal{T} = 0$, we call $\nabla$ a *torsion free* (or *symmetric*) connection.

The name "symmetric connection" comes from local computation: if we write

$$\widetilde{\mathcal{T}} = T^k{}_{ij} \partial_k \otimes dx^i \otimes dx^j,$$

i.e. we let $T^k{}_{ij}$ to be the functions such that

$$\mathcal{T}(\partial_i, \partial_j) = T^k{}_{ij} \partial_k,$$

then from

$$\mathcal{T}(\partial_i, \partial_j) = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i - [\partial_i, \partial_j] = \Gamma^k{}_{ij} \partial_k - \Gamma^k{}_{ji} \partial_k$$

one gets

$$T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}.$$

As a consequence,

**Corollary 1.7.** *$\nabla$ is torsion free if and only if $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$ for all $i, j$.*

*Remark.* In particular, we see that the symmetry-condition "$\Gamma^k_{ij} = \Gamma^k_{ji}$ for all $i, j$" is independent of the choice of local coordinates.

## 2. The Levi-Civita connection

We know that for any smooth manifolds, there are numerous choices of metric structures and measure structures. But with a Riemannian metric structure $g$ at hand, one can define a unique canonical metric structure and measure structure associated to $g$. The same phenomena happens for linear connections: Given a Riemannian metric $g$, there a unique canonical linear connection associated to $g$, known as the Levi-Civita connection, which has many nice properties.

**¶ Metric compatible linear connection.**

Let $(M, g)$ be a Riemannian manifold. Before we write down the definition of the Levi-Civita connection, we may ask ourselves a question: what kind of nice properties do we want?

First, we may want the linear connection to be torsion free, since under this condition, the Hessian is symmetric (and in fact as we will see later, there will be many other nice symmetry properties under the torsion free condition). Second, we may want the linear connection to be "compatible with the Riemannian metric $g$".

Now a natural question is:

> Question: when we say a linear connection is "compatible with the Riemannian metric $g$", what do we really mean?

Let's explore this question from the geometric point of view. With a Riemannian metric $g$ at hand, we get an inner product on each $T_pM$. So a natural requirement would be

> Answer: we may require that each parallel transport
> $$P_{0,t}^{\gamma} : (T_{\gamma(0)}, g_{\gamma(0)}) \to (T_{\gamma(t)}, g_{\gamma(t)})$$
> preserves the given inner product structure (i.e. is an isometry between the two inner product spaces).

It turns out that the geometric requirement above is equivalent to an algebraic equation on $\nabla_X$ (which is easy to use) and also to an analytic equation on $g$:

**Proposition 2.1.** *Let $\nabla$ be a linear connection on a Riemannian manifold $(M, g)$. Then the following statements are equivalent:*

1. *All the parallel transports $P_{0,t}^{\gamma} : (T_{\gamma(0)}, g_{\gamma(0)}) \to (T_{\gamma(t)}, g_{\gamma(t)})$ are isometries.*
2. *For any smooth vector fields $X, Y, Z \in \Gamma^\infty(TM)$, one has*
$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$
3. *$g$ is parallel, i.e. $\nabla g = 0$.*

*Proof.* $\boxed{(1) \Longrightarrow (2)}$ Let $\nabla$ be a linear connection such that $P_{0,t}^{\gamma}$ are isometries. For any vector fields $X, Y, Z \in \Gamma^\infty(TM)$, and any $p \in M$, take a curve $\gamma$ such that $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Take an orthonormal basis $\{e_i\}$ of $(T_pM, g_p)$, and let $e_i(t)$

be the parallel transport of $e_i$ along $\gamma$. By assumption, $\{e_i(t)\}$ is an orthonormal basis at $\gamma(t)$. If we denote $Y = Y^i(t)e_i(t)$ and $Z = Z^i(t)e_i(t)$, then

$$\langle Y, Z\rangle = \sum Y^i(t)Z^i(t)$$

along $\gamma$. So

$$\nabla_{X_p}\langle Y, Z\rangle = \sum X_p(Y^i(t))Z^i(0) + Y^i(0)X_p(Z^i(t)) = \langle \nabla_{X_p}Y, Z_p\rangle + \langle Y_p, \nabla_{X_p}Z\rangle,$$

i.e. $\nabla$ satisfies the desired equation.

$\boxed{(2) \Longrightarrow (1)}$ Conversely, suppose $\nabla$ be a linear connection on $M$ such that $X(\langle Y, Z\rangle)$ equals $\langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ for all $X, Y, Z$. Fix any curve $\gamma$, let $\{e_i\}$ be an orthonormal basis at $p = \gamma(0)$, and let $e_i(t)$ be the parallel transport of $e_i$ along $\gamma$, then

$$\frac{d}{dt}\langle e_i(t), e_j(t)\rangle = \dot\gamma(t)(\langle e_i(t), e_j(t)\rangle) = \langle \nabla_{\dot\gamma(t)}e_i(t), e_j(t)\rangle + \langle e_i(t), \nabla_{\dot\gamma(t)}e_j(t)\rangle = 0.$$

It follows that $\{e_i(t)\}$ remains to be an orthonormal basis for $(T_{\gamma(t)}, g_{\gamma(t)})$. So the linear map $P^\gamma_{0,t}$ is an isometry.

$\boxed{(2) \Longleftrightarrow (3)}$ Recall that the Riemannian metric $g$ is a $(0, 2)$-tensor, and thus one can take its covariant derivative $\nabla g$, which by definition is given by

$$(\nabla_X g)(Y, Z) = X\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle.$$

So the conclusion follows. □

**Definition 2.2.** We say a linear connection $\nabla$ on a Riemannian manifold $(M, g)$ is *compatible* with $g$ if one (and thus all) of the three equivalent conditions in Proposition 2.1 hold.

*Remark.* Note that by the geometric condition, if $\nabla$ is a metric-compatible linear connection on $(M, g)$, and if $X, Y$ are vector fields parallel along a curve $\gamma$, then $\langle X, Y\rangle$ is a constant on $\gamma$.

**¶ The Levi-Civita connection.**

Finally we define

**Definition 2.3.** A connection $\nabla$ is on $(M, g)$ is called a *Levi-Civita connection* (also called a *Riemannian connection*) if it is torsion-free and is compatible with $g$.

We example two simple examples:

*Example.* Let $M = \mathbb{R}^m$, equipped with the canonical Riemannian metric $g_0$, then the canonical linear connection (i.e. the one with all Christoffel symbols ${\Gamma^l}_{ij} = 0$ under the canonical basis) is a Levi-Civita connection.

*Example.* Equip $M = S^m$ with the round metric $g = g_{round}$, i.e. the induced metric from the canonical metric in $\mathbb{R}^{m+1}$. We denote by $\overline{\nabla}$ the canonical (Levi-Civita) connection in $\mathbb{R}^{m+1}$. For any $X, Y \in \Gamma^\infty(TS^m)$, one can extend $X, Y$ to smooth vector fields $\bar{X}$ and $\bar{Y}$ on $\mathbb{R}^{m+1}$. By localities we proved last time, the vector

$$\overline{\nabla}_{\bar{X}}\bar{Y}$$

at any point $p \in S^m$ depends only on the vector $\bar{X}(p) = X(p)$ and the vectors $\bar{X}(q) = X(q)$ for $q \in S^m$ near $p$. In other words, it is independent of the choice of the extension. So for simplicity we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\bar{X}}\bar{Y}$ for points on $S^m$. It is a vector that is not necessary tangent to $S^m$. We define $\nabla_X Y$ be the "orthogonal projection" of $\overline{\nabla}_X Y$ onto the tangent space of $S^m$, i.e.

$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \vec{n}\rangle \vec{n},$$

where $\vec{n} = (x^1, x^2, \cdots, x^{m+1})$ is the unit out normal vector on $S^m$. Observe that

$$\overline{\nabla}_X \vec{n} = X^i \partial_i(x^j)\partial_j = X.$$

It follows $\langle \overline{\nabla}_X Y, \vec{n}\rangle\vec{n} = -\langle Y, \overline{\nabla}_X \vec{n}\rangle \vec{n} = -\langle X, Y\rangle \vec{n}$ and thus

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle \vec{n}.$$

We claim that $\nabla$ is a Levi-Civita connection of $(S^m, g_{round})$. To prove this, first notice that $\nabla$ is bilinear, and $\nabla_{fX} Y = f\nabla_X Y$. Also

$$\nabla_X(fY) = \overline{\nabla}_X(fY) + \langle X, fY\rangle\vec{n} = (Xf)Y + f\overline{\nabla}_X Y + f\langle X, Y\rangle \vec{n} = (Xf)Y + f\nabla_X Y.$$

This connection is torsion free because

$$\nabla_X Y - \nabla_Y X = \overline{\nabla}_X Y + \langle X, Y\rangle\vec{n} - \overline{\nabla}_Y X - \langle Y, X\rangle \vec{n} = \overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

Finally this connection is compatible with the metric $g$, since

$$X\langle Y, Z\rangle = \langle \overline{\nabla}_X Y, Z\rangle + \langle Y, \overline{\nabla}_X Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

where we used the fact that $Z$ is perpendicular to $\vec{n}$.

*Remark.* If $(M, g)$ is a Riemannian manifold, with a Levi-Civita connection $\nabla^M$, and if $(N, \iota^* g)$ is a Riemannian submanifold of $(M, g)$, then we can define a connection on $N$ by the same trick, namely orthogonally project $\nabla^M$ onto $TN$,

$$\nabla^N_X Y := (\nabla^M_{\bar{X}} \bar{Y})^T,$$

One can prove that it is the Levi-Civita connection on $(N, \iota^* g)$.

## ¶ The fundamental theorem of Riemannian geometry.

Since any Riemannian manifold can be embedded to the standard Euclidian space isometrically, the arguments in the previous remark implies that on any Riemannian manifold, there exists a Levi-Civita connection! In what follows we will give two direct elementary proofs of this fact, and also prove the uniqueness:

**Theorem 2.4** (The fundamental theorem of Riemannian geometry)**.** *On any Riemannian manifold $(M, g)$, there is a unique Levi-Civita connection.*

*First proof (local coordinate).* We first prove uniqueness. Let $\nabla$ be a Levi-Civita connection. Pick a coordinate chart and let $\Gamma^k{}_{ij}$ be the Christoffel symbols. It is enough to prove that the $\Gamma^k{}_{ij}$'s are determined by $g_{ij}$'s. The trick already appeared in Lecture 1. First we note that by torsion free property, $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$. Second we calculate

$$\begin{aligned}\partial_i g_{jk} = \partial_i(g(\partial_j, \partial_k)) &= g(\nabla_{\partial_i}\partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i}\partial_k) \\ &= g(\Gamma^l{}_{ij}\partial_l, \partial_k) + g(\partial_j, \Gamma^l{}_{ik}\partial_l) = \Gamma^l{}_{ij}g_{lk} + \Gamma^l{}_{ik}g_{jl}.\end{aligned}$$

Similarly one can prove

$$\partial_j g_{ki} = \Gamma^l{}_{jk}g_{li} + \Gamma^l{}_{ji}g_{kl} \quad \text{and} \quad \partial_k g_{ij} = \Gamma^l{}_{ki}g_{lj} + \Gamma^l{}_{kj}g_{il}.$$

So we get

$$\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij} = 2g_{lk}\Gamma^l{}_{ij}.$$

It follows

$$2\Gamma^l{}_{ij} = g^{lk}(\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij}). \tag{10}$$

This proves the uniqueness. [This is essentially the same as we did in Lecture 1.]

For the existence, we can define locally (for $X = X^i\partial_i$ and $Y = Y^j\partial_j$)

$$\nabla_X Y = X^i(\partial_i Y^j)\partial_j + X^i Y^j \Gamma^l{}_{ij}\partial_l,$$

where $\Gamma^l{}_{ij}$ is the function given by (10). By tedious computations one can check that this give a Levi-Civita connection whose Christoffel symbols are the $\Gamma^l{}_{ij}$'s. □

*Second proof (coordinate free).* Again we first prove the uniqueness. Assume the Levi-Civita connection exists. Then (use torsion-free and metric-compatibility in turns)

$$\begin{aligned}\langle\nabla_X Y, Z\rangle =& X(\langle Y, Z\rangle) - \langle Y, \nabla_X Z\rangle \\ =& X(\langle Y, Z\rangle) - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle \\ =& X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle\nabla_Z Y, X\rangle - \langle Y, [X, Z]\rangle \\ =& X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + \langle\nabla_Y Z, X\rangle + \langle[Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ =& X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \langle Z, \nabla_Y X\rangle \\ &+ \langle[Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ =& X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) - \langle Z, \nabla_X Y\rangle - \langle Z, [Y, X]\rangle \\ &+ \langle[Z, Y], X\rangle - \langle Y, [X, Z]\rangle.\end{aligned}$$

It follows that $\nabla_X Y$ must be the vector satisfying

$$\begin{aligned}2\langle\nabla_X Y, Z\rangle =& X(\langle Y, Z\rangle) - Z(\langle Y, X\rangle) + Y(\langle Z, X\rangle) \\ &- \langle Z, [Y, X]\rangle + \langle[Z, Y], X\rangle - \langle Y, [X, Z]\rangle.\end{aligned} \tag{11}$$

The right hand side is determined by the metric. So the uniqueness is proved. [(11) is called the *Koszul* formula, which reduce to (10) if we take $X, Y, Z$ to be $\partial_i, \partial_j$ and $\partial_l$.]

To prove the existence, one "only need" to check that the $\nabla_X Y$ defined by the above formula satisfies all conditions of Levi-Civita connections. □