LECTURE 7: THE CURVATURE TENSOR

1. The curvature tensor of a linear connection

¶ Derivations on the graded tensor algebra.

Let M be a smooth manifold endowed with a linear connection ∇. As we have seen last time, ∇ induces a linear connection

$$\nabla : \Gamma^\infty(TM) \times \Gamma^\infty(\otimes^{k,l}TM) \to \Gamma^\infty(\otimes^{k,l}TM)$$

on each tensor bundle $\otimes^{k,l}TM$. Moreover, all these linear connections are compatible as a whole set of connections in the sense that they are compatible with the tensor product operation and the contraction operation for tensors.

Let's take a closer look of the "tensor product compatibility". Denote by $\Gamma^\infty(\otimes^{*,*}TM)$ the graded tensor algebra of all smooth tensor fields on M. Then the tensor product compatibility means that for any smooth vector field $X \in \Gamma^\infty(TM)$, the map

$$\nabla_X : \Gamma^\infty(\otimes^{*,*}TM) \to \Gamma^\infty(\otimes^{*,*}TM)$$

satisfies $\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$. In other words, ∇_X is a derivation on $\Gamma^\infty(\otimes^{*,*}TM)$.

Now let $\mathcal{D}$ be the set of all derivations on the tensor algebra $\Gamma^\infty(\otimes^{*,*}TM)$, which are by definition linear maps D such that $D(S\otimes T) = DS\otimes T + S\otimes DT$. A standard fact (which is easy to verify via definition) is that $\mathcal{D}$ is a Lie algebra (with respect to commutator), namely if D_1, D_2 are two derivations, so is their commutator

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

Example. For any smooth vector field X, the Lie derivative $\mathcal{L}_X$ is a derivation on $\Gamma^\infty(\otimes^{*,*}TM)$. Moreover, $\mathcal{L}_X$ satisfies (when acting on any tensor field)

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X.$$

In other words, the linear map "$X \mapsto \mathcal{L}_X$" is a Lie algebra homomorphism from "the Lie algebra of all smooth vector fields on M" to "the Lie algebra of all derivations on $\Gamma^\infty(\otimes^{*,*}TM)$".

Now consider the linear map

$$\Phi : \Gamma^\infty(TM) \to \mathcal{D}, \quad X \mapsto \nabla_X.$$

One may ask: Is Φ a Lie algebra homomorphism? In other words, do we have

$$\nabla_{[X,Y]} = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X?$$

Unfortunately the answer is no in general[1] (as we will see soon). So we are naturally led to study the map $R(X,Y): \Gamma^\infty(\otimes^{k,l}TM) \to \Gamma^\infty(\otimes^{k,l}TM)$ defined by

$$R(X,Y)T = \nabla_X\nabla_Y T - \nabla_Y\nabla_X T - \nabla_{[X,Y]}T.$$

Let's start with two simple cases:

- First for $k = l = 0$, i.e. $T = f \in C^\infty(M)$, the map $R(X,Y)$ is zero, since

$$R(X,Y)f = \nabla_X\nabla_Y f - \nabla_Y\nabla_X f - \nabla_{[X,Y]}f = XYf - YXf - [X,Y]f = 0.$$

- Next we study the case $k = 1, l = 0$, i.e. $T = \omega$ is a smooth 1-form. It turns out that one can convert $R(X,Y)$ on 1-forms to $R(X,Y)$ on vector fields:

  Lemma 1.1. *For any 1-form $\omega \in \Omega^1(M)$,*

$$(R(X,Y)\omega)(Z) = -\omega(R(X,Y)Z).$$

  Proof. Compute by definition. Details left as an exercise. □

In view of the fact that the graded tensor algebra $\Gamma^\infty(\otimes^{*,*}TM)$ is generated by smooth functions, vector fields and 1-forms, together with the fact that $R(X,Y)$ is again a derivation on $\Gamma^\infty(\otimes^{*,*}TM)$, we conclude that to study $R(X,Y)$ on all tensor fields, it is enough to study $R(X,Y)$ on vector fields!

¶ The curvature tensor of a linear connection.

We define

Definition 1.2. Let M be a smooth manifold and ∇ a linear connection on M. We call the map $R: \Gamma^\infty(TM) \times \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{1}$$

the *curvature tensor* of ∇.

As we explained above, R measures to what extend the map Φ fails to be a Lie algebra homomorphism.

Remark. In many books, the definition of curvature tensor is different from the above formula by a negative sign. Both definitions have their own advantages. So when you open a new book on Riemannian geometry, you should first glance at its definition of the curvature tensor.

Example. For the standard linear connection $\overline{\nabla}$ on $\mathbb{R}^m$, we have

$$\overline{\nabla}_{X^i\partial_i}(Y^j\partial_j) = X^i\partial_i(Y^j)\partial_j.$$

which implies

$$\overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z = X^i\partial_i(Y^j\partial_j Z^k)\partial_k - Y^j\partial_j(X^i\partial_i Z^k)\partial_k = \overline{\nabla}_{[X,Y]}Z$$

and thus its curvature tensor $\overline{R} \equiv 0$.

[1]Maybe we should say *fortunately* the answer is no, otherwise there will be no Riemannian geometry, and the world will be boring.

Example. Consider $M = S^m$. Last time we have seen that

$$\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle \vec{n}$$

is the Levi-Civita connection on S^m, where $\overline{\nabla}$ is the standard connection on $\mathbb{R}^{m+1}$. It follows

$$\begin{aligned}\nabla_X \nabla_Y Z &= \overline{\nabla}_X \nabla_Y Z + \langle X, \nabla_Y Z\rangle \vec{n}\\ &= \overline{\nabla}_X (\overline{\nabla}_Y Z + \langle Y, Z\rangle \vec{n}) + Y\langle X, Z\rangle \vec{n} - \langle \nabla_Y X, Z\rangle \vec{n}\\ &= \overline{\nabla}_X \overline{\nabla}_Y Z + X(\langle Y, Z\rangle)\vec{n} + \langle Y, Z\rangle X + Y(\langle X, Z\rangle)\vec{n} - \langle \nabla_Y X, Z\rangle \vec{n}.\end{aligned}$$

In view of the fact $\overline{R} = 0$, we get

$$\begin{aligned}R(X, Y)Z =& X(\langle Y, Z\rangle)\vec{n} + \langle Y, Z\rangle X + Y(\langle X, Z\rangle)\vec{n} - \langle \nabla_Y X, Z\rangle \vec{n}\\ &- Y(\langle X, Z\rangle)\vec{n} - \langle X, Z\rangle Y - X(\langle Y, Z\rangle)\vec{n} + \langle \nabla_X Y, Z\rangle \vec{n} - \langle [X, Y], Z\rangle \vec{n}\\ =& \langle Y, Z\rangle X - \langle X, Z\rangle Y.\end{aligned}$$

¶ The curvature tensor is a tensor.

Now we prove that R is a tensor of type (1,3), in the sense

$$\widetilde{R}(\omega, X, Y, Z) := \omega(R(X, Y)Z))$$

is really an element in $\Gamma^\infty(\otimes^{1,3}TM)$:

Proposition 1.3. *The curvature tensor R is a $(1, 3)$-tensor.*

Proof. We need to prove

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z.$$

Here we only check one of them:

$$\begin{aligned}R(fX, Y)Z &= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{(fX)Y - Y(fX)} Z\\ &= f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) - (Yf)\nabla_X Z + (Yf)\nabla_X Z\\ &= fR(X, Y)Z.\end{aligned}$$

The others are similar and are left as happy exercises. □

Locally, we write the tensor R (or $\widetilde{R}$) as[2]

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l,$$

i.e. if we denote

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}{}^l \partial_l,$$

[2]Note that here we are using a "non-standard" order: for the local expression of the $(1, 3)$-tensor R, we write the "1"-part (i.e. the vector ∂_l) after the "3"-parts (i.e. the co-vectors). In some books people use different orders like $R(\partial_i, \partial_j)\partial_k = R^l{}_{kij}\partial_l$. In other words, in writing local expressions of $\widetilde{R}(\omega, X, Y, Z)$, we always want to put the index for ω next to the index for Z. The reason will be clear in next lecture.

then the coefficients $R_{ijk}{}^l$ are related to the Christoffel symbols by

$$R_{ijk}{}^l = \partial_i \Gamma^l{}_{jk} - \partial_j \Gamma^l{}_{ik} + \Gamma^s{}_{jk}\Gamma^l{}_{is} - \Gamma^s{}_{ik}\Gamma^l{}_{js},$$

which is a consequence of the fact

$$R_{ijk}{}^l \partial_l = R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}(\Gamma^s{}_{jk}\partial_s) - \nabla_{\partial_j}(\Gamma^s{}_{ik}\partial_s).$$

Note that this also implies that the curvature tensor for the standard connection on $\mathbb{R}^m$ is identically zero, since its Christoffel symbols are all zero.

¶ The commutator of two coordinate covariant derivatives of a surface.

Consider an embedded 2-dimensional parametric surface in M,

$$\varphi : U \subset \mathbb{R}^2 \to S = \varphi(U) \subset M.$$

Denote the parameters for U by s and t. We let ∂_s and ∂_t be the two coordinate vector fields

$$\partial_s := d\varphi(\frac{\partial}{\partial s}) \quad \text{and} \quad \partial_t := d\varphi(\frac{\partial}{\partial t})$$

on the 2-dimensional surface S. Note that by locality, for any smooth vector field Z on surface S, the expression $\nabla_{\partial_s}\nabla_{\partial_t} Z$ make sense and will be thought of as the iterated second order covariant derivative of Z with respect to ∂_s, ∂_t. It turns out that R measures the non-commutativity of such iterated covariant derivatives:

Proposition 1.4. *For any smooth vector field Z (defined on the surface S),*

$$\nabla_{\partial_s}\nabla_{\partial_t} Z - \nabla_{\partial_t}\nabla_{\partial_s} Z = R(\partial_s, \partial_t)Z.$$

Proof. Take a coordinate chart of M near S so that S is defined by $x^3 = \cdots = x^m = 0$. Then $(s, t, x^3, \cdots, x^m)$ is a local coordinate system on M in a tubular neighborhood of S. Now the conclusion follows from the fact as coordinate vector fields, $[\partial_t, \partial_s] = 0$. □

Remark. Geometrically, R is closely related to "the holonomy along an infinitesimal path which is the boundary of $\varphi((0,\varepsilon) \times (0,\varepsilon))$". Details are left as a term project.

¶ Flat connection.

We are interested in linear connections with vanishing curvature tensor, i.e. with $R = 0$. For example, the standard connection on $\mathbb{R}^m$ satisfies $R = 0$. The nice point for $\mathbb{R}^m$ is that the coordinate vector fields are parallel along any vector fields.

Definition 1.5. Let M be a smooth manifolds with a connection ∇. We say (M, ∇) admits a *local flat frame* everywhere if near any point p, there is a set of vector fields $X_1, \cdots, X_m$ on a neighborhood U of p such that

(1) **[frame]** $\{X_i(q) \mid 1 \le i \le m\}$ form a basis of T_qM for every $q \in U$,
(2) **[flatness]** $\nabla_Y X_i = 0$ for all i and for all vector field Y.

It is easy to see that if (M, ∇) admits a local flat frame everywhere, then $R(X, Y)X_i = 0$ for all X, Y, and thus $R \equiv 0$ since R is a tensor. Conversely,

Proposition 1.6. *Let M be a smooth manifold, ∇ be a linear connection. Then $R = 0$ if and only if (M, ∇) admits a local flat frame everywhere.*

Proof. It remains to prove the "only if" part. Without loss of generality, we may take U to be a coordinate neighborhood and $p = (0, \cdots, 0)$ the origin. We start with any basis $\{v_1, \cdots, v_m\}$ of T_pM and let $X_i(p) = v_i$. We extend X^i to the "line" $\{(a, 0, \cdots, 0)\}$ by parallel transporting the vector $X_i(p)$ along the curve $\gamma_0(t) = (t, 0, \cdots, 0)$. Then we extend further to the "plane" $\{(a, b, 0, \cdots, 0)\}$ by parallel transporting each $X_i(\gamma_0(a))$ along $\gamma_a(t) = (a, t, 0, \cdots, 0)$. Repeating this procedure, we get a set of smooth (why?) vector fields $\{X_1, \cdots, X_m\}$ on the whole of U. By construction, they are a frame. It remains to prove the flatness.

First by construction, we have

- $\nabla_{\partial_1} X_i = 0$ at any point on the line $(a, 0, \cdots, 0)$.
- $\nabla_{\partial_2} X_i = 0$ at any point on the plane $(a, b, 0, \cdots, 0)$.

Moreover, since $R = 0$ and $[\partial_1, \partial_2] = 0$, we get

$$\nabla_{\partial_2}\nabla_{\partial_1} X_i = \nabla_{\partial_1}\nabla_{\partial_2} X_i = 0$$

on the plane $(a, b, 0, \cdots, 0)$. As a consequence, $\nabla_{\partial_1} X_i$ is parallel along each line $\gamma_a(t) = (a, t, 0, \cdots, 0)$, with initial condition $(\nabla_{\partial_1} X_i)(a, 0, \cdots, 0) = 0$. By uniqueness, one must have $\nabla_{\partial_1} X_i = 0$ along each γ_a. In other words, we get

- $\nabla_{\partial_1} X_i = 0, \nabla_{\partial_2} X_i = 0$ at any point on the plane $(a, b, 0, \cdots, 0)$.

By the same argument, we get

- $\nabla_{\partial_1} X_i = 0, \nabla_{\partial_2} X_i = 0, \nabla_{\partial_3} X_i = 0$ at any point of the form $(a, b, c, 0, \cdots, 0)$.

Continuing this argument, one can see that $\nabla_{\partial_j} X_i = 0$ for all i, j, at all points in U. As a consequence, $X_1, \cdots, X_m$ form a local flat frame. □

As a consequence,

Corollary 1.7. *If (M, g) is a Riemannian manifold for which the curvature of the Levi-Civita connection vanishes, then (M, g) is locally isometric to $(\mathbb{R}^m, g_0)$.*

Proof. In the proof above, we take $\{v_1, \cdots, v_m\}$ to be an orthonormal basis. Then after parallel transport, the vector fields $X_1, \cdots, X_m$ are orthonormal everywhere. Since ∇ is Levi-Civita connection, it is torsion free. It follows that for any i, j,

$$0 = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = -[X_i, X_j].$$

By Frobenius theorem, there exists a local coordinate chart with $X_i = \partial_i$ for all i. In this chart, we have $g_{ij} = \langle \partial_i, \partial_j \rangle = \delta_{ij}$, and thus locally $g = \sum dx^i \otimes dx^i$. □

Definition 1.8. A linear connection ∇ on M is called *flat* if $R = 0$. A Riemannian manifold is called *flat* if the Levi-Civita connection is flat.

2. Symmetries of the curvature tensor for torsion free connection

¶ Curvature tensor as commutator of ∇^2 (for torsion free connection).

Regard ∇ as a map of the form

$$\nabla : \Gamma^\infty(\otimes^{k,l}TM) \to \Gamma^\infty(\otimes^{k,l+1}TM).$$

We have studied the composition $\nabla^2 : \Gamma^\infty(\otimes^{0,0}TM) \to \Gamma^\infty(\otimes^{0,2}TM)$ which maps any $f \in C^\infty(M)$ to its Hessian

$$\nabla^2 f : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to C^\infty(M), \quad \nabla^2 f(X,Y) = \nabla_Y \nabla_X f - \nabla_{\nabla_Y X} f.$$

In general, for any $T \in \Gamma^\infty(\otimes^{k,l}TM)$, the second covariant derivative map ∇^2 sends T to the $(k, l+2)$-tensor $\nabla^2 T$, which, by definition, equals

$$\nabla^2 T(\cdots, X, Y) = (\nabla_Y \nabla_X T)(\cdots) - (\nabla_{\nabla_Y X} T)(\cdots).$$

For simplicity will denote

$$\nabla^2_{X,Y} : \Gamma^\infty(\otimes^{k,l}TM) \to \Gamma^\infty(\otimes^{k,l}TM), \quad T \mapsto \nabla^2 T(\cdots, X, Y).$$

One should be aware of the difference between the second covariant derivative and the iterated covariant derivative.

From now on suppose ∇ is torsion free, which as we have seen, is equivalent to the fact that $\nabla^2_{X,Y} f$ is symmetric with respect to the entries X and Y. A natural question is: if ∇ is torsion free, is $\nabla^2_{X,Y} T$ symmetric with respect to X and Y for all T? The answer is no. For example, if $T = Z$ is a vector field,

$$\begin{aligned}\nabla^2_{Y,X} Z - \nabla^2_{X,Y} Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X,Y)Z,\end{aligned}$$

where in the last step we used the fact ∇ is torsion free, i.e. $\nabla_X Y - \nabla_Y X = [X,Y]$. More generally, by exactly the same computation one has

Lemma 2.1 (Ricci identity)**.** *Let ∇ be any torsion free connection, then*

$$\nabla^2_{Y,X}(T) - \nabla^2_{X,Y}(T) = R(X,Y)T, \quad \forall T \in \Gamma^\infty(\otimes^{k,l}TM).$$

In conclusion, for a torsion free connection, the operator $R(X,Y)$ (on any tensors) measures the non-commutativity of second covariant derivatives.

Remark. One may ask: What about higher order covariant derivatives? It turns out that under the torsion free assumption, the operator $R(X,Y)$ also measures the non-commutativity of higher covariant derivatives. For example,

$$\begin{aligned}\nabla^3_{Y,Z,X} T - \nabla^3_{Z,Y,X} T &= (\nabla_X \nabla^2 T)(Y,Z) - (\nabla_X \nabla^2 T)(Z,Y) \\ &= \nabla_X \nabla^2_{Y,Z} T - \nabla^2_{\nabla_X Y, Z} T - \nabla^2_{Y, \nabla_X Z} T - \nabla_X \nabla^2_{Z,Y} T + \nabla^2_{\nabla_X Z, Y} T + \nabla^2_{Z, \nabla_X Y} T \\ &= -\nabla_X (R(Y,Z)T) + R(\nabla_X Y, Z)T + R(Y, \nabla_X Z)T.\end{aligned}$$

¶ The First/Algebraic Bianchi identity.

Now we study symmetries of the curvature tensor R. By definition one immediately see that for any linear connection ∇, R admits the following anti-symmetry:

$$R(X,Y)Z = -R(Y,X)Z \tag{2}$$

In local coordinates it can be written as

$$R_{ijk}{}^l = -R_{jik}{}^l.$$

It turns out that for torsion free connections, R admits more symmetries.

Proposition 2.2 (The First Bianchi identity)**.** *If ∇ is a torsion-free, then*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0. \tag{3}$$

Proof. Since for torsion free connection, we have $\nabla_X Y - \nabla_Y X - [X,Y] = 0$, so

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X \\
&\qquad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y \\
&= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
&= 0,
\end{aligned}$$

where in the last step we used the Jacobi identity for vector fields. □

In local coordinates the first Bianchi identity can be written as

$$R_{ijk}{}^l + R_{jki}{}^l + R_{kij}{}^l = 0.$$

¶ The Second/Differential Bianchi identity.

It turns out that not only R has the above cyclic symmetry, but also its covariant derivative ∇R has a similar cyclic symmetry. To understand ∇R, let's go back to the $(1,3)$-tensor $\widetilde{R}$ defined by

$$\widetilde{R}(\omega, X, Y, Z) = \omega(R(X,Y)Z).$$

It follows by definition that $\nabla\widetilde{R}$ is the (1,4)-tensor given by

$$\begin{aligned}
(\nabla\widetilde{R})(\omega,X,Y,Z,W) &= (\nabla_W\widetilde{R})(\omega,X,Y,Z) \\
&= \nabla_W(\omega(R(X,Y)Z)) - (\nabla_W\omega)(R(X,Y)Z) - \omega(R(\nabla_W X,Y)Z) \\
&\quad - \omega(R(X,\nabla_W Y)Z) - \omega(R(X,Y)\nabla_W Z) \\
&= \omega\big(\nabla_W(R(X,Y)Z) - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z\big).
\end{aligned}$$

So it is reasonable to define $\nabla_W R$ as

$$(\nabla_W R)(X,Y,Z) := \nabla_W(R(X,Y)Z) - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)\nabla_W Z.$$

Proposition 2.3 (The Second Bianchi Identity)**.** *Suppose ∇ is torsion free, then*

$$(\nabla_X R)(Y,Z,W)+(\nabla_Y R)(Z,X,W)+(\nabla_Z R)(X,Y,W)=0.$$

Proof. By definition,

$$\begin{aligned}&(\nabla_X R)(Y,Z,W)+(\nabla_Y R)(Z,X,W)+(\nabla_Z R)(X,Y,W)\\ =&\nabla_X(R(Y,Z)W)-R(\nabla_X Y,Z)W-R(Y,\nabla_X Z)W-R(Y,Z)\nabla_X W+\\ &\nabla_Y(R(Z,X)W)-R(\nabla_Y Z,X)W-R(Z,\nabla_Y X)W-R(Z,X)\nabla_Y W+\\ &\nabla_Z(R(X,Y)W)-R(\nabla_Z X,Y)W-R(X,\nabla_Z Y)W-R(X,Y)\nabla_Z W.\end{aligned}$$

Using the torsion-freeness and (2), one can simplify the middle two columns to

$$R([X,Z],Y)W+R([Y,X],Z)W+R([Y,X],Z)W.$$

Now expand each R using its definition, the whole expression becomes a summation of 27 terms, the first 9 terms being

$$\begin{aligned}&+\nabla_X\nabla_Y\nabla_Z W-\nabla_X\nabla_Z\nabla_Y W-\nabla_X\nabla_{[Y,Z]}W\\ &+\nabla_{[X,Z]}\nabla_Y W-\nabla_Y\nabla_{[X,Z]}W-\nabla_{[[X,Z],Y]}W\\ &-\nabla_Y\nabla_Z\nabla_X W+\nabla_Z\nabla_Y\nabla_X W+\nabla_{[Y,Z]}\nabla_X W,\end{aligned}$$

the second and third 9 terms are similar to the first 9 terms above: one just replace X,Y,Z by Y,Z,X and Z,X,Y respectively. It is not hard to check that all those expressions containing three ∇'s (12 terms in total) cancel out trivially, all those expressions containing two ∇'s (also 12 terms in total) cancel out by using the fact $[X,Y]=-[Y,X]$, and the remaining three terms

$$\nabla_{[[X,Z],Y]}W+\nabla_{[[Y,X],Z]}W+\nabla_{[[Z,Y],X]}W=0$$

in view of the Jacobi identity. □

In local coordinates we can write $\nabla_{\partial_n}R=R_{ijk}{}^{l}{}_{;n}dx^i\otimes dx^j\otimes dx^k\otimes\partial_l$, Then the second Bianchi identity can be written as

$$R_{ijk}{}^{l}{}_{;n}+R_{jnk}{}^{l}{}_{;i}+R_{nik}{}^{l}{}_{;j}=0.$$

Remark. If we denote $\sum_{\bigcirc X,Y,Z}$ to be the cyclic sum over X,Y,Z, then the first and second Bianchi identities can be written as

$$\sum_{\bigcirc X,Y,Z}R(X,Y)Z=0\qquad\text{and}\qquad\sum_{\bigcirc X,Y,Z}(\nabla_X R)(Y,Z,W)=0.$$

More generally, if ∇ is not torsion free, then in terms of the torsion tensor $\mathcal{T}$,

$$\sum_{\bigcirc X,Y,Z}R(X,Y)Z=\sum_{\bigcirc X,Y,Z}\big((\nabla_X\mathcal{T})(Y,Z)+\mathcal{T}(\mathcal{T}(X,Y),Z)\big)$$

and

$$\sum_{\bigcirc X,Y,Z}(\nabla_X R)(Y,Z,W)+\sum_{\bigcirc X,Y,Z}R(\mathcal{T}(X,Y),Z)W=0.$$