LECTURE 8: THE RIEMANNIAN CURVATURE

1. The Riemann curvature tensor

¶ The Riemann curvature tensor of type $(0,4)$.

Given any linear connection ∇ on M, one gets a type $(1,3)$ curvature tensor R

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

which measures the non-commutativity of "second order/iterated covariant derivatives". Locally one may write R as

$$R = R_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \otimes \partial_l.$$

Now suppose (M,g) is a Riemannian manifold and ∇ is the Levi-Civita connection. By using the Riemannian metric g (via the musical isomorphism) one can convert the $(1,3)$-tensor R to a $(0,4)$-tensor $Rm \in \Gamma^\infty(\otimes^{0,4}TM)$ defined by

$$Rm(X,Y,Z,W) := -g(R(X,Y)Z,W).$$

Definition 1.1. We call Rm the *Riemann curvature tensor* of (M,g).

Locally if we write

$$Rm = R_{ijkl}dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

then

$$R_{ijkl} = Rm(\partial_i,\partial_j,\partial_k,\partial_l) = -g(R_{ijk}{}^m\partial_m,\partial_l) = -g_{ml}R_{ijk}{}^m.$$

In other words, the Riemannian metric "lower one of the the index".

Example. For S^m (equipped with the standard round metric), we have seen

$$R(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y.$$

Thus the Riemann curvature tensor is

$$Rm(X,Y,Z,W) = -\langle Y,Z\rangle\langle X,W\rangle + \langle X,Z\rangle\langle Y,W\rangle.$$

Now we introduce the *Kulkarni-Nomizu product* $\bigcirc\!\!\!\!\wedge$ that converts 2 symmetric $(0,2)$-tensors T_1 and T_2 into one $(0,4)$-tensor $T_1 \bigcirc\!\!\!\!\wedge T_2$ defined by

$$\begin{aligned}(T_1\bigcirc\!\!\!\!\wedge T_2)(X,Y,Z,W) :=& T_1(X,Z)T_2(Y,W) + T_1(Y,W)T_2(X,Z)\\ &- T_1(X,W)T_2(Y,Z) - T_1(Y,Z)T_2(X,W).\end{aligned}$$

As a result, we get a very brief expression for the Riemann curvature tensor of S^m,

$$Rm = \frac{1}{2} g \bigcirc\!\!\!\!\wedge g.$$

¶ **Symmetries of Rm.**

By definition the $(1,3)$-tensor R admits the anti-symmetry

$$R(X,Y)Z = -R(Y,X)Z.$$

Moreover, if ∇ is torsion free, then the curvature tensor R admits two more cyclic symmetry, namely the first Bianchi identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

and the second Bianchi identity

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0.$$

Obviously one can convert the symmetries of R to symmetries of Rm, namely

$$Rm(X,Y,Z,W) + Rm(Y,X,Z,W) = 0, \tag{1}$$

the first Bianchi identity

$$Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0, \tag{2}$$

and the second Bianchi identity

$$(\nabla_X Rm)(Y,Z,W,V) + (\nabla_Y Rm)(Z,X,W,V) + (\nabla_Z Rm)(X,Y,W,V) = 0, \tag{3}$$

or in local coordinates as

$$\begin{aligned} R_{ijkl} + R_{jikl} &= 0, \\ R_{ijkl} + R_{jkil} + R_{kijl} &= 0, \\ R_{ijkl;n} + R_{jnkl;i} + R_{nikl;j} &= 0. \end{aligned}$$

where we denote $R_{ijkl;n} = (\nabla_{\partial_n} R)(\partial_i, \partial_j, \partial_k, \partial_l)$.

By staring at the Riemann curvature tensor Rm of the standard S^m, we may find more (anti-)symmetries than the ones we have seen, e.g. one can exchange Z with W to get a negative sign, or even exchange X,Y with Z,W. In fact these two (anti-)symmetries are consequences of metric compatibility, and thus hold for any Riemannian manifold:

Proposition 1.2. *The Riemann curvature tensor Rm satisfies*

$$Rm(X,Y,Z,W) = -Rm(X,Y,W,Z), \tag{4}$$

and

$$Rm(X,Y,Z,W) = Rm(Z,W,X,Y). \tag{5}$$

Proof. For simplicity we denote $f = \langle Z, Z\rangle$, then by metric compatibility,

$$\langle \nabla_X Z, Z\rangle = Xf - \langle Z, \nabla_X Z\rangle,$$

in other words,

$$\langle \nabla_X Z, Z\rangle = \frac{1}{2} Xf.$$

It follows

$$\langle \nabla_X \nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle = \frac{1}{2}X(Yf) - \langle \nabla_Y Z, \nabla_X Z\rangle.$$

So

$$\begin{aligned} -Rm(X,Y,Z,Z) = \langle R(X,Y)Z, Z\rangle &= \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z\rangle \\ &= \frac{1}{2}X(Yf) - \frac{1}{2}Y(Xf) - \frac{1}{2}[X,Y]f = 0. \end{aligned}$$

As a consequence, we get

$$\begin{aligned} &Rm(X,Y,Z,W) + Rm(X,Y,W,Z) \\ =&Rm(X,Y,Z+W,Z+W) - Rm(X,Y,Z,Z) - Rm(X,Y,W,W) = 0, \end{aligned}$$

which implies (4).

The equation (5) is a consequence of (4) first one together with (1) and (2). In fact, by the first Bianchi identity (2) one has

$$\begin{aligned} Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0, \\ Rm(Y,Z,W,X) + Rm(Z,W,Y,X) + Rm(W,Y,Z,X) = 0, \\ Rm(Z,W,X,Y) + Rm(W,X,Z,Y) + Rm(X,Z,W,Y) = 0, \\ Rm(W,X,Y,Z) + Rm(X,Y,W,Z) + Rm(Y,W,X,Z) = 0, \end{aligned}$$

Adding these equations and using (1) and (4), we get

$$Rm(Z,X,Y,W) + Rm(W,Y,Z,X) = 0,$$

which is equivalent to (5). □

By using (5) we may rewrite the second Bianchi identity (3) as

$$(3') \qquad (\nabla_U Rm)(Y,Z,V,W) + (\nabla_V Rm)(Y,Z,W,U) + (\nabla_W Rm)(Y,Z,U,V) = 0,$$

In local coordinates, the identities (4), (5) and (3′) become

$$R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} = R_{klij}, \quad \text{and} \quad R_{ijkl;n} + R_{ijln;l} + R_{ijnk;l} = 0.$$

¶ The curvature operator $\mathcal{R}$.

According to (1) and (4), the Riemann curvature tensor Rm can be considered as acting on two bi-vectors $X \wedge Y$ and $Z \wedge W$ instead of acting on four vectors X, Y, Z, W. In other words, we may write Rm as

$$\widetilde{Rm} : \Lambda^2(TM) \times \Lambda^2(TM) \to C^\infty(M).$$

Since the Riemannian metric on M induces an inner product on each $\Lambda^2(T_pM)$, one may convert the tensor Rm into an operator $\mathcal{R} : \Lambda^2(TM) \to \Lambda^2(TM)$ such that

$$\langle \mathcal{R}(X\wedge Y), Z\wedge W\rangle = \widetilde{Rm}(X\wedge Y, Z\wedge W) := Rm(X,Y,Z,W).$$

Moreover, the symmetry equation (5) implies that $\mathcal{R}$ is a self-adjoint operator on each $\Lambda^2(T_pM)$. The operator $\mathcal{R}$ is called the *curvature operator*.

2. Decomposition of the Riemann curvature tensor

¶ Some tensor algebra: symmetric tensors.

Let V be any vector space. Recall that $\wedge^2 V^* \subset \otimes^2 V^*$ represents the space of anti-symmetric 2-tensors on V, while $S^2V^* \subset \otimes^2 V^*$ represents the space of symmetric 2-tensors on V. Any 2-tensor T on V can be decomposed uniquely as the summation of a symmetric 2-tensor and an anti-symmetric 2-tensor as

$$T(u,v) = \frac{T(u,v)+T(v,u)}{2} + \frac{T(u,v)-T(v,u)}{2}.$$

If $\dim V = m$, then we have

$$\dim \wedge^2 V^* = \frac{m(m-1)}{2}, \quad \text{and} \quad \dim S^2V^* = \frac{m(m+1)}{2}.$$

Note that by definition, $S^2(\wedge^2 V^*)$ contains (0,4)-tensors that are symmetric with respect to $(1,2) \leftrightarrow (3,4)$ and anti-symmetric with respect to $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, i.e.

$$T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y).$$

For example, one can easily check that for any two symmetric (0,2)-tensor $S, T \in S^2(V^*)$, their Kulkarni-Nomizu product $S \owedge T \in S^2(\wedge^2 V^*)$. The set $S^2(\wedge^2 V^*)$ is a vector space of dimension

$$\dim S^2(\wedge^2 V^*) = \frac{m(m-1)(m^2-m+2)}{8}. \tag{6}$$

Moreover, the space of 4-forms, $\wedge^4 V^*$, is a subspace of $S^2(\wedge^2 V^*)$ with dimension

$$\dim \wedge^4 V^* = \binom{m}{4}. \tag{7}$$

Let $\alpha, \beta \in \wedge^2 V^*$ be any two linear 2-forms, both viewed as skew-symmetric 2-tensors on V. Define the *symmetric product* of α and β to be the $(0,4)$-tensor

$$(\alpha \odot \beta)(X,Y,Z,W) := \alpha(X,Y)\beta(Z,W) + \alpha(Z,W)\beta(X,Y). \tag{8}$$

In local coordinates one can write

$$(\alpha \odot \beta)_{ijkl} = \alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij}.$$

Obviously each $\alpha \odot \beta$ is in $S^2(\wedge^2 V^*)$. It turns out that these (0,4)-tensors generates the whole space $S^2(\wedge^2 V^*)$:

Lemma 2.1. *Any element in $S^2(\wedge^2 V^*)$ can be written as a linear combination of elements of the form $\alpha \odot \beta$.*

Proof. Let $e^1, \cdots, e^m$ be a basis of V^*, then

$$E^1 = e^1 \wedge e^2, E^2 = e^1 \wedge e^3, \cdots, E^{m(m-1)/2} = e^{m-1} \wedge e^m$$

is a basis of $\Lambda^2 V^*$, and $E^i \odot E^j$ $(i \le j)$ are linearly independent in $S^2(\wedge^2 V^*)$ (check). By dimension counting, we see these elements form a basis of $S^2(\wedge^2 V^*)$. $\square$

¶ Some tensor algebra: Curvature-like tensors.

To explore the cyclic symmetry that arise in the first Bianchi identity, we define

Definition 2.2. The *Bianchi symmetrization* of any $T \in S^2(\wedge^2 V^*)$ is the 4-tensor

$$bT(X,Y,Z,W) = \frac{1}{3}(T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W)). \tag{9}$$

So the first Bianchi identity for Rm now becomes the simple equation $b(Rm) = 0$.

Definition 2.3. If $T \in S^2(\wedge^2 V^*)$ and $b(T) = 0$, we call T a *curvature-like* tensor. The set of all curvature-like tensors is denoted by $\mathscr{C}$.

Example. For any $S, T \in S^2V^*$, we have $b(S \owedge T) = 0$ since

$$\begin{aligned} 3b(S \owedge T)_{ijkl} =& S_{ik}T_{jl} + S_{jl}T_{ik} - S_{il}T_{jk} - S_{jk}T_{il} + S_{jk}T_{li} + S_{li}T_{jk} \\ & - S_{ji}T_{lk} - S_{lk}T_{ji} + S_{lk}T_{ij} + S_{ij}T_{lk} - S_{lj}T_{ik} - S_{ik}T_{lj} \\ =& 0. \end{aligned}$$

As a result, $S \owedge T$ is a curvature-like tensor.

Note that by definition, curvature-like tensors are exactly those tensors satisfying all the algebraic symmetries that Rm admit, namely (1), (4), (5) and (2).

To study the set $\mathscr{C}$ of all curvature-like tensors, we first study the image of the Bianchi symmetrization b. Note that for any $\alpha, \beta \in \Lambda^2(V^*)$,

$$\begin{aligned} 3(b(\alpha \odot \beta))_{ijkl} &= \alpha_{ij}\beta_{kl} + \alpha_{kl}\beta_{ij} + \alpha_{jk}\beta_{il} + \alpha_{il}\beta_{jk} + \alpha_{ki}\beta_{jl} + \alpha_{jl}\beta_{ki} \\ &= \alpha_{ij}\beta_{kl} - \alpha_{ik}\beta_{jl} + \alpha_{il}\beta_{jk} + \alpha_{jk}\beta_{il} - \alpha_{jl}\beta_{ik} + \alpha_{kl}\beta_{ij} \\ &= \frac{4!}{2!2!}\frac{1}{4!}\sum_{\pi \in S_4}\left((\alpha \otimes \beta)^\pi\right)_{ijkl} = (\alpha \wedge \beta)_{ijkl}. \end{aligned}$$

In other words, we have

Lemma 2.4. *For any $\alpha, \beta \in \Lambda^2(V^*)$, $b(\alpha \odot \beta) = \frac{1}{3}\alpha \wedge \beta$.*

In view of Lemma 2.1, we conclude that the map b has image

$$\mathrm{Im}(b) = \wedge^4 V^* \subset \Lambda^2(V^*).$$

In particular, for any $T \in S^2(\wedge^2 V^*)$ we have $bT \in S^2(\wedge^2 V^*)$. By definition it is straightforward to check

Lemma 2.5. *For any $T \in S^2(\wedge^2 V^*)$, one has $b(b(T)) = T$.*

So the Bianchi symmetrization map b, as a linear map $b : S^2(\wedge^2 V^*) \to S^2(\wedge^2 V^*)$, is a projection. It follows from the standard linear algebra that

$$S^2(\wedge^2 V^*) = \mathrm{Ker}(b) \oplus \mathrm{Im}(b) = \mathscr{C} \oplus \wedge^4 V^*.$$

As a consequence, $\mathscr{C}$ is a vector space of dimension

$$\dim \mathscr{C} = \frac{m(m-1)(m^2-m+2)}{8} - \binom{m}{4} = \frac{1}{12}m^2(m^2-1). \tag{10}$$

¶ Some tensor algebra: metric contractions.

Now suppose the vector space V is endowed with an inner product $g(\cdot,\cdot) = \langle\cdot,\cdot\rangle$, so that one can identify V^* with V using the musical isomorphisms $\flat$ and $\sharp$. In particular, for any (0,4)-tensor T and for any vectors X, Y, Z, the linear map

$$T(Z, X, \cdot, Y) : V \to \mathbb{R},$$

is in V^* and thus be identified with a vector $\sharp T(Z, X, \cdot, Y) \in V$.

Definition 2.6. For any (0,4)-tensor T on an inner product space $(V, \langle\cdot,\cdot\rangle)$, the *Ricci contraction* $c(T)$ of T is the following (0,2)-tensor:[1]

$$c(T)(X, Y) := \mathrm{Tr}(Z \mapsto \sharp T(Z, X, \cdot, Y)). \tag{11}$$

For a Riemannian manifold, we call $Rc := c(Rm)$ its *Ricci curvature tensor.*

It turns out that $c(T)$ is symmetric if T is curvature-like:

Lemma 2.7. *If $T \in \mathscr{C}$, then $c(T)(X, Y) = c(T)(Y, X)$.*

Proof. Fix X and Y. Let $K : V \to V$ and $\widetilde{K} : V \to V$ be the maps

$$K(Z) = \sharp T(Z, X, \cdot, Y) \qquad \text{and} \qquad \widetilde{K}(Z) = \sharp T(\cdot, X, Z, Y).$$

Then for any $Z, W \in V$,

$$\langle K(Z), W\rangle = \langle \sharp T(Z, X, \cdot, Y), W\rangle = T(Z, X, W, Y) = \langle Z, \sharp T(\cdot X, W, Y)\rangle = \langle Z, \widetilde{K}(W)\rangle.$$

So $\widetilde{K}$ is the transpose of K, and thus they have the same trace. But by definition $\mathrm{Tr}(K) = c(T)(X, Y)$, while $\mathrm{Tr}(\widetilde{K}) = c(T)(Y, X)$ (since $T \in \mathscr{C} \subset S^2(\wedge^2 V^*)$). □

Similarly one can define the trace of a (0,2)-tensor T using the metric: the map

$$T(X, \cdot) : V \to \mathbb{R}$$

is an element in V^*, and thus can be identified with a vector $\sharp T(X, \cdot)$ in V.

Definition 2.8. The *trace* of a (0,2)-tensor T on $(V, \langle\cdot,\cdot\rangle)$ is

$$\mathrm{Tr}(T) := \mathrm{trace}(X \mapsto \sharp T(X, \cdot)). \tag{12}$$

For a Riemannian manifold (M, g), we call $S(g) = \mathrm{Tr}(Rc)$ the *scalar curvature.*

Example. For the metric tensor g, by definition $\sharp g(X, \cdot) = X$ and thus $\mathrm{Tr}(g) = m$.

Locally if $v^1, \cdots, v^n$ be a basis of V^*, and if we denote $g^{pq} = g^*(v^p, v^q)$ (where g^* is the dual metric on V^*), then one can check [exercise]

$$c(T)_{ij} = g^{pq} T_{ipjq} \qquad \text{and} \qquad \mathrm{Tr}(T) = g^{ij} T_{ij}.$$

Note that $\langle T, g\rangle = T_{ij} g_{kl} g^{ik} g^{jl} = T_{ij} \delta^i_l g^{jl} = T_{ij} g^{ij}$, one has

$$\mathrm{Tr}(T) = \langle T, g\rangle. \tag{13}$$

[1] One can define metric contractions between other pairs of indices. However, for curvature-like tensors, what one can get are either Ricci contraction, or zero.

¶ The dual of the Ricci contraction.

Now let's study the Ricci contraction map

$$c : \mathscr{C} \to S^2V^*, \quad S \mapsto c(S)$$

which maps a curvature-like tensor to a symmetric 2-tensor. We also have a map

$$\Psi : S^2V^* \to \mathscr{C}, \quad T \mapsto T \owedge g. \tag{14}$$

that maps any symmetric 2-tensor to a curvature-like tensor. It turns out with respect to the induced metrics on the spaces of tensors that we learned in lecture 2, these two maps are almost adjoint to each other:

Lemma 2.9. *For any $S \in \mathscr{C}$ and any $T \in S^2V^*$, one has*

$$\langle S, \Psi(T)\rangle = 4\langle c(S), T\rangle.$$

Proof. We calculate using an orthonormal basis of V, so that $g_{ij} = g^{ij} = \delta_{ij}$:

$$\begin{aligned}\langle S, T \owedge g\rangle &= \sum S_{ijkl}(T_{ik}g_{jl} + T_{jl}g_{ik} - T_{jk}g_{il} - T_{il}g_{jk}) \\ &= \sum (S_{ijkj}T_{ik} + S_{ijik}T_{jk} - S_{ijki}T_{jk} - S_{ijjk}T_{ik}) \\ &= \sum 4S_{ijkj}T_{ik} \\ &= 4\sum (c(S))_{ik}T_{ik} \\ &= 4\langle c(S), T\rangle.\end{aligned}$$

where each summation is over all indices that appeared. □

In particular, if $S_1 = \Psi(T_1) \in \mathrm{Im}(\Psi), S_2 \in \ker(c)$, then

$$\langle S_1, S_2\rangle = \langle \Psi(T_1), S_2\rangle = 4\langle T_1, c(S_2)\rangle = 0.$$

So we get

Corollary 2.10. $\mathrm{Im}(\Psi) \perp \ker(c)$.

Similarly, we may calculate $c \circ \Psi$ via an orthonormal basis: for any $T \in S^2V^*$,

$$\begin{aligned}c(T \owedge g)_{ij} &= g^{pq}(T_{ij}g_{pq} - T_{pj}g_{iq} - T_{iq}g_{pj} + T_{pq}g_{ij}) \\ &= mT_{ij} - T_{ij} - T_{ij} + \mathrm{Tr}(T)g_{ij} \\ &= (m-2)T_{ij} + \mathrm{Tr}(T)g_{ij}.\end{aligned}$$

In other words, we have

Lemma 2.11. *For any symmetric 2-tensor $T \in S^2V^*$,*

$$c(\Psi(T)) = (m-2)T + \mathrm{Tr}(T)g.$$

Together with (13), we get

$$|\Psi(T)|^2 = \langle \Psi(T), \Psi(T)\rangle = 4\langle c(\Psi(T)), T\rangle = 4(m-2)|T|^2 + 4(\mathrm{Tr}(T))^2. \tag{15}$$

¶ Decomposition of a curvature-like tensor via the metric: Step 1.

Now suppose $m > 2$. We first prove

Proposition 2.12. *The map Ψ is injective for $m > 2$, and is bijective for $m = 3$.*

Proof. Suppose $m \geq 2$ and $\Psi(T) = 0$. Then by (15), $T = 0$, so Ψ is injective. For $m = 3$ it is bijective since $\dim S^2V^* = \dim \mathscr{C} = 6$. □

As a result, the Ricci contraction map $c : \mathscr{C} \to S^2V^*$ is surjective, and thus

$$\dim \mathrm{Im}(\Psi) + \dim \ker(c) = \dim S^2V^* + \dim \ker(c) = \dim \mathrm{Im}(c) + \dim \ker(c) = \dim \mathscr{C}.$$

So by Corollary 2.10, we really have an orthogonal decomposition

$$\mathscr{C} = \ker(c) \oplus \mathrm{Im}(\Psi).$$

In particular, for any curvature-like tensor $T \in \mathscr{C}$, there is a curvature-like tensor $W \in \ker(c)$ and a symmetric 2-tensor $A \in S^2V^*$ so that

$$T = W + A \owedge g,$$

and the decomposition is orthogonal. To find out A, we apply c to both sides to get

$$c(T) = c(\Psi(A)) = (m-2)A + \mathrm{Tr}(A)g.$$

To find out $\mathrm{Tr}(A)$, we continue to take $\mathrm{Tr}(A)$ for both sides,

$$\mathrm{Tr}(c(T)) = (m-2)\mathrm{Tr}(A) + m\mathrm{Tr}(A) = 2(m-1)\mathrm{Tr}(A).$$

So we get

$$A = \frac{1}{m-2}\left(c(T) - \frac{\mathrm{Tr}(c(T))}{2(m-1)}g\right). \tag{16}$$

For a Riemannian manifold (M, g), we have

$$c(Rm) = Rc \quad \text{and} \quad \mathrm{Tr}(c(Rm)) = S.$$

In this case the tensors W and A have their own names:

Definition 2.13. For a Riemannian manifold (M, g), we call

$$A = \frac{1}{m-2}\left(Rc - \frac{S}{2(m-1)}g\right)$$

the *Schouten tensor* of (M, g), and call

$$W := Rm - A \owedge g$$

the *Weyl curvature tensor* (or conformal curvature tensor) of (M, g).

Both Weyl curvature tensor and the Schouten tensor play very important roles in conformal geometry. For example, we will show that the Weyl tensor is invariant under conformal transformations. Note that by Corollary 2.10 and Proposition 2.12, the Weyl curvature tensor vanishes for $m = 3$:

Proposition 2.14. *If $m = 3$, then $W = 0$.*

¶ Decomposition of a curvature-like tensor via the metric: Step 2.

We may continue to decompose any $R \in S^2V^*$ into orthogonal ones via the map $\mathrm{Tr} : S^2V^* \to \mathbb{R}$. Again we need the dual of Tr, which, in view of (13), is simply

$$\mathrm{Tr}^* : \mathbb{R} \to S^2V^*, \quad t \mapsto tg.$$

By repeating the same arguments, again we get an orthogonal decomposition

$$S^2V^* = \ker(\mathrm{Tr}) \oplus \mathrm{Im}(\mathrm{Tr}^*).$$

So there exist $E \in \ker(\mathrm{Tr})$ and $t \in \mathbb{R}$ so that R can be decomposed orthogonally to

$$R = E + tg,$$

Applying the map Tr to both sides we get $\mathrm{Tr}(R) = t\mathrm{Tr}(g) = mt$, and thus

$$t = \frac{\mathrm{Tr}(R)}{m}.$$

Note that if $R = Rc$ is the Ricci curvature tensor of (M, g), then $t = S/m$.

Definition 2.15. For a Riemannian manifold (M, g), we call

$$E := Rc - \frac{S}{m}g$$

the *traceless Ricci tensor* of (M, g).

We point out that the two decompositions are compatible, in the sense that

$$\mathrm{Im}(\Psi) = \Psi(S^2V^*) = \Psi(\ker(\mathrm{Tr})) \oplus \Psi(\mathrm{Im}(\mathrm{Tr}^*))$$

is an orthogonal decomposition of $\mathrm{Im}(\Psi)$, since for $t \in \mathbb{R}$ and any $E \in \ker(\mathrm{Tr})$,

$$\langle \Psi(tg), \Psi(E) \rangle = \langle c(\Psi(tg)), E \rangle = \langle (2m-2)(tg), E \rangle = 2(m-2)t\mathrm{Tr}(E) = 0.$$

Thus we end up with an orthogonal decomposition

$$\mathscr{C} = \ker(c) \oplus \Psi(\ker(\mathrm{Tr})) \oplus \Psi(\mathrm{Im}(\mathrm{Tr}^*)),$$

so that any $T \in \mathscr{C}$ can be decomposed into three curvature-like tensors which are orthogonal to each other: the "Weyl part" W that lies in $\ker(c)$, $E \owedge g$ for a trace-less symmetric 2-tensor E, and a multiple of $g \owedge g$.

Remark. A more algebraic way to understand the two decompositions: Consider the natural action of $O(m)$ on V that preserves $\langle \cdot, \cdot \rangle$, which induces natural actions of $O(m)$ on S^2V^* and on $\mathscr{C}$ that preserve the induced inner products.

- For the case of S^2V^*, since g (and thus the 1-dimensional space $\mathbb{R}g$) is invariant under the $O(m)$-action, one may decompose $S^2V^* = \mathbb{R}g \oplus (\mathbb{R}g)^\perp$. This decomposition can be explained via the $O(m)$-invariant map Tr as above, and thus $(\mathbb{R}g)^\perp = \ker(\mathrm{Tr})$ consists of all traceless symmetric (0,2)-tensors.
- Similarly since c is $O(m)$-equivariant, it induce a decomposition of $\mathscr{C}$ into $\ker(c) \oplus (\ker(c))^\perp$, and we have seen $(\ker(c))^\perp = \Psi(S^2V^*)$ since c is surjective.

Can we decompose further? The answer is no, since one can prove the $O(m)$-action on $\ker(\mathrm{Tr})$ and on $\ker(c)$ are transitive(i.e. they are irreducible representations of $O(m)$).

¶ Decomposition of the Riemann curvature tensor.

Now suppose (M, g) be a Riemannian manifold. First we have a decomposition

$$Rm = W + A \owedge g.$$

We may continue to decompose the Schouten tensor A: By definition formula (16), the traceless part of A equals the traceless part of $\frac{Rc}{m-2}$, which is $\frac{E}{m-2}$. Since

$$\frac{1}{m-2}\frac{S}{m}g - \frac{1}{m-2}\frac{S}{2(m-1)}g = \frac{S}{2m(m-1)}g,$$

we get the orthogonal decomposition of Schouten tensor:

$$A = \frac{E}{m-2} + \frac{S}{2m(m-1)}g.$$

So we end up with an orthogonal decomposition

$$Rm = W + \frac{1}{m-2}E \owedge g + \frac{S}{2m(m-1)}g \owedge g \tag{17}$$

Our final goal in this lecture is to prove

Theorem 2.16. *For any Riemannian manifold, the (pointwise) norm squares of the Riemann/Ricci/Weyl curvature tensors and the scalar curvature are related by*

$$|Rm|^2 = |W|^2 + \frac{4}{m-2}|Rc|^2 - \frac{2}{(m-1)(m-2)}S^2.$$

Proof. In view of (15) and the fact $\mathrm{Tr}(g) = |g|^2 = m$, we have

$$|E \owedge g|^2 = 4(m-2)|E|^2,$$
$$|g \owedge g|^2 = 4(m-2)m + 4m^2 = 8m(m-1).$$

Since the decomposition (17) is orthogonal, we get

$$|Rm|^2 = |W^2| + \frac{4}{m-2}|E|^2 + \frac{2}{m(m-1)}S^2.$$

Finally we use

$$\begin{aligned}|E|^2 &= \langle Rc - \frac{S}{m}g, Rc - \frac{S}{m}g\rangle \\ &= |Rc|^2 - \frac{2S}{m}\langle Rc, g\rangle + \frac{S^2}{m^2}|g|^2 \\ &= |Rc|^2 - \frac{2S^2}{m} + \frac{S^2}{m} \\ &= |Rc|^2 - \frac{S^2}{m}\end{aligned}$$

to get

$$|Rm|^2 = |W|^2 + \frac{4}{m-2}|Rc|^2 - \frac{2}{(m-1)(m-2)}S^2. \qquad \square$$