LECTURE 9: THE RICCI AND THE SECTIONAL CURVATURE

1. The Ricci and the sectional curvature

¶ The Ricci curvature of a Riemannian manifold.

We start with some simple algebra. Let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form defined on a vector space V. Then we can assign to it a quadratic form

$$Q : V \to \mathbb{R}, \quad Q(v) := B(v, v).$$

Conversely we can recover the symmetric bilinear form B from its quadratic form Q via the polarization formula

$$2B(u, v) = Q(u + v) - Q(u) - Q(v).$$

There is also a more succinct way to recover B from Q is via

$$2B(u, v) = [Q(u + tv)]'(0),$$

where the $'$ refers to t-derivative.

Recall that the Ricci curvature tensor *Ric* is the contraction of the Riemann curvature tensor *Rm*,

$$Rc(X, Y) = c(Rm)(X, Y) = \mathrm{Tr}(Z \mapsto \sharp Rm(Z, X, \cdot, Y)).$$

It is a symmetric $(0, 2)$-tensor field on M. In local coordinates one has

$$Rc_{ij} = g^{pq} R_{ipjq}$$

Applying the previous trick to Rc, we see that to study the (0,2)-tensor Rc, it is enough to study the real-valued function $Rc(X, X)$ defined on TM, which is easier to handle. In view of the fact $Rc(\lambda X, \lambda X) = \lambda^2 Rc(X, X)$, we may simplify a bit further by studying $Rc(X, X)$ only for unit-length vector fields $X \in \Gamma^\infty(SM)$:

Definition 1.1. For any unit tangent vector $X_p \in S_pM \subset T_pM$, we call

$$Ric(X_p) = Rc(X_p, X_p)$$

the *Ricci curvature* of M at p in the direction of X_p.

So the Ricci curvature function *Ric* is not a function on M, but a function on the unit sphere bundle $SM \subset TM$ (one can think of the Ricci curvature as a function defined on one-dimensional subspaces of T_pM). It encodes all information of the tensor Rc via

$$Rc(X_p, Y_p) = \frac{1}{2}\left[\|X_p + Y_p\|^2 Ric(\widehat{X_p + Y_p}) - \|X_p\|^2 Ric(\widehat{X}_p) - \|Y_p\|^2 Ric(\widehat{Y}_p)\right],$$

where $Y_p \neq -X_p$, and we denoted $\widehat{X} = X/\|X\|$.

¶ Reduce a curvature-like tensor to its bi-quadratic form.

Now we move from symmetric (0,2)-tensor to "very symmetric" (0,4)-tensors that we studied last time, namely, curvature-like tensors T. We let

$$Q(X,Y) := T(X,Y,X,Y)$$

be the bi-quadratic form associated to T. It turns out that one can recover T from Q in the same spirit above:

Lemma 1.2. *Let T be a curvature-like tensor, and let*

$$f_{X,Y,Z,W}(t) = Q(X+tZ, Y+tW) - t^2(Q(X,W)+Q(Z,Y)).$$

Then $(f_{X,Y,Z,W} - f_{Y,X,Z,W})''(0) = 12T(X,Y,Z,W)$.

Proof. Obviously $f_{X,Y,Z,W}$ is a polynomial of degree 4 in t, whose quadratic coefficients equals

$$T(Z,W,X,Y) + T(Z,Y,X,W) + T(X,W,Z,Y) + T(X,Y,Z,W).$$

which, by using the symmetries for curvature-like tensors, equals

$$2T(X,Y,Z,W) + 2T(Z,Y,X,W).$$

Similarly the quadratic coefficient of $f_{Y,X,Z,W}$ equals

$$2T(Y,X,Z,W) + 2T(Z,X,Y,W).$$

So the quadratic coefficient of $f_{X,Y,Z,W}(t) - f_{Y,X,Z,W}(t)$ is

$$2T(X,Y,Z,W) + 2T(Z,Y,X,W) - 2T(Y,X,Z,W) - 2T(Z,X,Y,W),$$

which, after applying the first Bianchi identity, equals $6T(X,Y,Z,W)$. □

Remark. One may explicitly write down a "pure algebraic polarization formula" of $T(X,Y,Z,W)$ in terms of the bi-quadratic form Q, which is quite lengthy.

¶ The sectional curvature.

As a result, to study the curvature tensor Rm of a Riemannian manifold (M,g), it is enough to study the associated bi-quadratic form, namely, $Rm(X,Y,X,Y)$.

Again, by using some simple algebra, we can simplify a bit further.

Lemma 1.3. *For any $T \in \otimes^2(\wedge^2 V^*)$ and any $X,Y \in V$, if we denote $X' = aX + bY, Y' = cX + dY$, then*

$$T(X',Y',X',Y') = (ad-bc)^2 T(X,Y,X,Y).$$

Proof. This follows from a very simple computation:

$$\begin{aligned} T(X',Y',X',Y') &= T(aX+bY, cX+dY, aX+bY, cX+dY) \\ &= (ad-bc)T(X,Y,aX+bY,cX+dY) \\ &= (ad-bc)^2 T(X,Y,X,Y). \end{aligned}$$

□

Now suppose (M, g) is a Riemannian manifold. Recall that $\frac{1}{2}g \owedge g$ is a curvature-like tensor, such that

$$\frac{1}{2}g \owedge g(X_p, Y_p, X_p, Y_p) = \langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2,$$

which is nothing else but the square of the area of the parallelogram with sides X_p, Y_p. Applying the previous lemma to Rm and $\frac{1}{2}g \owedge g$, we immediately get

Proposition 1.4. *The quantity*

$$K(X_p, Y_p) := \frac{Rm(X_p, Y_p, X_p, Y_p)}{\langle X_p, X_p\rangle\langle Y_p, Y_p\rangle - \langle X_p, Y_p\rangle^2}$$

depends only on the two dimensional plane $\Pi_p = \mathrm{span}(X_p, Y_p) \subset T_pM$*, i.e. it is independent of the choices of basis* $\{X_p, Y_p\}$ *of* Π_p.

Definition 1.5. We will call

$$K(\Pi_p) = K(X_p, Y_p)$$

the *sectional curvature* of (M, g) at p with respect to the plane Π_p.

Note that the Ricci curvature that we studied above are closely related to the sectional curvatures: If X_p is a unit vector in T_pM, we may extend X_p to an orthonormal basis $\{e_1 = X_p, e_2, \cdots, e_m\}$ of T_pM. As a result,

$$Ric(X_p) = \sum_{i\geq 2} Rm(e_i, e_1, e_i, e_1) = \sum_{i\geq 2} K(e_i, e_1). \tag{1}$$

In other words, the Ricci curvature $Ric(X_p)$ is "the sum of sectional curvatures" for a specially chosen set of $m-1$ pairwise orthogonal planes containing X_p.

Example. Here are three basic examples:

(1) For the Euclidean space $(\mathbb{R}^m, g_0)$, one has $Rm = 0$, so
$$K(\Pi_p) \equiv 0 \quad \text{and} \quad Ric(X_p) \equiv 0.$$
(2) For the unit sphere (S^m, g_{round}), one has $Rm = \frac{1}{2}g \owedge g$, so
$$K(\Pi_p) \equiv 1 \quad \text{and} \quad Ric(X_p) \equiv m-1.$$
One may also prove the conclusion by calculating the Christoffel symbols.

(3) For the hyperbolic space $(H^m, g_{hyperbolic})$, one can prove (exercise)
$$K(\Pi_p) \equiv -1 \quad \text{and} \quad Ric(X_p) \equiv -(m-1).$$

Remark. One may give a conceptional proof of the fact that these three spaces have constant sectional curvature: For $(\mathbb{R}^m, g_0)$, the isometry group $E(m) = O(m) \ltimes \mathbb{R}^m$ acts transitively on the set $Gr_2(T\mathbb{R}^m) = \{\Pi_p \mid p \in \mathbb{R}^m, \Pi_p \subset T_pM \text{ is a plane}\}$. Since the sectional curvature is invariant under the action of the isometric group, we must have $K(\Pi_p) = K(\Pi'_q)$, i.e. the sectional curvature is a constant. The same phenomena occurs for (S^m, g_{round}) (with isometry group $O(m+1)$) and for $(H^m, g_{hyperbolic})$ (with isometry group $O^+(m, 1)$).

Remark. We may compare the sectional curvature K with the curvature operator $\mathcal{R}_p : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$. By definition we have

$$K(X_p, Y_p) = \frac{\langle \mathcal{R}_p(X_p \wedge Y_p), X_p \wedge Y_p \rangle}{\langle X_p, X_p \rangle \langle Y_p, Y_p \rangle - \langle X_p, Y_p \rangle^2}.$$

As a consequence, the curvature operator $\mathcal{R}$ determines the sectional curvature K.

2. Basic properties of sectional curvatures

¶ Sectional curvature for low dimensional manifolds.

Let (M, g) be a Riemannian manifold. By definition, the sectional curvature K is NOT a function on M, but instead a function on the Grassmannian bundle

$$Gr_2(TM) = \{(p, \Pi_p) \mid p \in M, \Pi_p \subset T_pM \text{ is 2-dimensional}\}.$$

(1) If M has dimension $m = 1$, obviously $R \equiv 0$ and it makes no sense to talk about sectional curvature (so again as we have seen in Lecture 3, essentially there is no Riemannian geometry in dimensional one).
(2) If M has dimension $m = 2$, i.e. is a surface, then one can regard the sectional curvature K as a function defined on M in the natural way. Moreover, in view of the equation (1), for any direction X_p we have $Ric(X_p) = K(p)$. In other words, for surfaces the sectional curvature, the Ricci curvature and the scalar curvature are all the same. One can show that in this case K is really the Gauss curvature in undergraduate differential geometry course.
(3) If M has dimension $m \geq 3$, by using the exponential map that we will learn later, for each 2-dimensional plane $\Pi_p \in T_pM$, locally one gets a 2-dimensional submanifold S_p near p in M whose tangent space at p is Π_p, and the sectional curvature $K(\Pi_p)$ is nothing else but the Gauss curvature of S_p (endowed with the subspace Riemannian metric) at p. Thus the sectional curvature is a natural generalization of the Gauss curvature to higher dimensional Riemannian manifolds.
(4) In general the sectional curvatures encodes more information than the Ricci curvatures. However, if M has dimension $m = 3$, then according to the equation (1), for an orthonormal basis e_1, e_2, e_3,

$$\begin{aligned} Ric(e_1) &= K(e_1, e_2) + K(e_1, e_3), \\ Ric(e_2) &= K(e_1, e_2) + K(e_2, e_3), \\ Ric(e_3) &= K(e_1, e_3) + K(e_2, e_3). \end{aligned}$$

As a result, in this case the Ricci curvatures determine the sectional curvatures: For each plane Π_p, one just start with an orthonormal basis $\{e_1, e_2\}$ of Π_p, extend it to an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM and then solve the above system of equations to get

$$2K(\Pi_p) = Ric(e_1) + Ric(e_2) - Ric(e_3).$$

¶ Riemannian manifolds with curvature bounds.

Unlike algebraic quantities like curvature tensors, the sectional/Ricci/scalar curvatures are real-valued functions. Since we may compare real nubmers, we can define

Definition 2.1. Let (M, g) be a Riemannian manifold. We say (M, g) has

(1) *constant sectional curvature* c if $K(\Pi_p) = c$ for all p and all planes $\Pi_p \subset T_pM$.
(2) *constant Ricci curvature* c if $Ric(X_p) = c$ for all p and all vectors $X_p \in S_pM$.
(3) *positive sectional curvature* if $K(\Pi_p) > 0$ for all p and all planes $\Pi_p \subset T_pM$.

Similarly we may define

- (M, g) has negative/nonpositive/nonnegative sectional curvature if $K(\Pi_p)$ is negative/nonpositive/nonnegative for all p and all planes $\Pi_p \subset T_pM$.
- (M, g) has positive/negative/nonpositive/nonnegative Ricci curvature if $Ric(X_p)$ is positive/negative/nonpositive/nonnegative for all p and all $X_p \in S_pM$.
- (M, g) has sectional curvature $K \geq c$ or $K \leq c$ if $K(\Pi_p) \geq c$ or $K(\Pi_p) \leq c$ for all p and all planes $\Pi_p \subset T_pM$.
- (M, g) has Ricci curvature $Ric \geq c$ or $Ric \leq c$ if $Ric(X_p) \geq c$ or $Ric(X_p) \leq c$ for all p and all vectors $X_p \in S_pM$.

More generally, given two Riemannian metrics g_1 and g_2 on any smooth manifold M, with sectional curvature functions K_{g_1}, K_{g_2} and Ricci curvature functions Ric_{g_1}, Ric_{g_2}, we may compare K_{g_1} and K_{g_2} as functions on $Gr_2(TM)$, and compare Ric_{g_1} and Ric_{g_2} as functions on SM.

Example. We can write down the change of sectional curvature under scaling of the metric: if we scale a Riemannian metric g to λg, where λ is a positive constant, then

(1) by the Koszul formula, the Levi-Civita connection $\nabla_X Y$ remains unchanged,
(2) it follows that the (1,3)-curvature tensor R remains unchanged,
(3) as a result, the Riemann curvature tensor is changed to $Rm_{\lambda g} = \lambda Rm_g$,
(4) and thus the Ricci curvature tensor remains unchanged: $Rc_{\lambda g} = Rc_g$,
(5) but the two sectional curvatures are related by $K_{\lambda g} = \lambda^{-1} K_g$,
(6) and it follows that $Ric_{\lambda g} = \lambda^{-1} Ric_g$ (no conflict with (4) since unit vectors changed)
(7) and thus $S_{\lambda g} = \lambda^{-1} S_g$.

In particular, for each c one gets a Riemannian manifold with constant sectional curvature c, namely the space $(S^m, \frac{1}{c} g_{round})$ if $c > 0$, and $(H^m, \frac{1}{-c} g_{hyperbolic})$ if $c < 0$.

Manifolds with curvature bounds will be one of the major themes of this course.

Remark. Since the curvature operator $\mathcal{R}_p$ is a symmetric operator on a real vector space, it has real eigenvalues.

Definition 2.2. We say (M, g) is a Riemannian manifold with *positive curvature operator* if all eigenvalues of $\mathcal{R}_p$ are positive.

Obviously if (M, g) has positive curvature operator, then it has positive sectional curvature. However, the converse is not true:

- It was proven by C. Bohm and B. Wilking in 2008 that manifolds with positive curvature operators are space forms, i.e. are complete Riemannian manifolds with constant sectional curvature.
- On the other hands, there exist Riemannian manifolds with positive sectional curvature which are not constant (e.g. the complex projective space $\mathbb{CP}^m$ endowed with the Fubini-Study metric).

So for such Riemannian manifolds the curvature operator is not positive. The secret is: the space $\Lambda^2(T_pM)$ is a vector space that contains elements of the form $u_1 \wedge v_1 + u_2 \wedge v_2$ which do not correspond to any 2-dimensional plane in T_pM. After all, the Grassmannian $Gr_2(T_pM)$ is a smooth manifold of dimension $2(m-2)$, while the space of bi-vectors $\Lambda^2(T_pM)$ has dimension $\binom{m}{2}$.

¶ Riemannian manifolds with isotropic sectional curvature at a point.

Finally study the following question: At a given point, when will the sectional curvature be independent of the choice of $\Pi_p \subset T_pM$?

Proposition 2.3. *Let (M, g) be a Riemannian manifold and $p \in M$. The following are equivalent:*

(1) $K(\Pi_p) = c$ for all $\Pi_p \subset T_pM$.
(2) $Rm_p = \frac{c}{2} g_p \owedge g_p$.
(3) $R_p(X_p, Y_p)Z_p = c(\langle Y_p, Z_p\rangle X_p - \langle X_p, Z_p\rangle Y_p)$ for any $X_p, Y_p, Z_p \in T_pM$.
(4) $\mathcal{R}_p = c\mathrm{Id}$ on $\Lambda^2 T_pM$.
(5) The Weyl curvature tensor $W_p = 0$ and Ricci curvature tensor $Rc_p = (m-1)cg_p$.

Proof. $\boxed{(1) \Longleftrightarrow (2)}$ According to Lemma 1.2, if T is a curvature-like tensor, then

$$T \equiv 0 \Longleftrightarrow T(X, Y, X, Y) = 0, \forall X, Y.$$

Apply this to the curvature-like tensor $T = Rm_p - \frac{c}{2} g_p \owedge g_p$, we see

$$Rm_p = \frac{c}{2} g_p \owedge g_p \Longleftrightarrow K(\Pi_p) = c, \forall\ \Pi_p \subset T_pM.$$

$\boxed{(2) \Longleftrightarrow (3)}$ and $\boxed{(4) \Longrightarrow (1)}$ are obvious.

$\boxed{(2) \Longrightarrow (4)}$ Take an orthonormal basis $\{e_i\}$ of T_pM, then $\{e_i \wedge e_j \mid i < j\}$ is a basis of $\Lambda^2 T_pM$. On this basis,

$$\langle \mathcal{R}_p(e_i \wedge e_j), e_k \wedge e_l\rangle = Rm_p(e_i, e_j, e_k, e_l) = c\delta_{ik}\delta_{jl},$$

where in the last step we used the fact $i < j, k < l$. As a result, we see

$$\mathcal{R}_p(e_i \wedge e_j) = ce_i \wedge e_j, \quad \forall i < j.$$

In other words, $\mathcal{R}_p = c\mathrm{Id}$ for all $e_i \wedge e_j$. Since $\mathcal{R}_p$ is linear, $\mathcal{R}_p = c\mathrm{Id}$ on $\Lambda^2 T_pM$.

$\boxed{(2) \Longrightarrow (5)}$ We start with the unique orthogonal decomposition

$$Rm_p = W_p + \frac{1}{m-2} E_p \owedge g_p + \frac{S(p)}{2m(m-1)} g_p \owedge g_p,$$

where $E_p = Rc_p - \frac{S(p)}{m} g_p$ is the traceless Ricci tensor. If (2) holds, then by the uniqueness of the decomposition,

$$W_p = 0, \quad E_p = 0 \quad \text{and} \quad S(p) = m(m-1)c.$$

As a result,

$$Rc_p = (m-1)cg_p.$$

$\boxed{(5) \Longrightarrow (2)}$ Conversely if $W_p = 0$ and $Rc_p = (m-1)cg_p$, then

$$S(p) = \mathrm{Tr}(Rc_p) = m(m-1)c.$$

So $E_p = 0$ and thus

$$Rm_p = \frac{c}{2} g_p \owedge g_p.$$

This completes the proof. □

Recall from Lecture 7 that a Riemannian manifold is flat if the (1,3)-curvature tensor $R = 0$. As a consequence,

Corollary 2.4. *A Riemannian manifold (M, g) is a flat manifold if and only if its sectional curvatures are identically zero.*

Similarly by using the polarization formula for Ricci curvature tensor, we may also easily get a Ricci curvature version:

Proposition 2.5. *Let (M, g) be a Riemannian manifold and $p \in M$. Then $Ric(X_p) = c$ for all $X_p \in S_pM$ if and only if $Rc_p = cg_p$.*