LECTURE 10: RIEMANNIAN MANIFOLDS WITH CONSTANT CURVATURES

On any smooth manifold there are numerous different Riemannian metrics, most of which are not interesting to us. Today we will briefly discuss some results on very special Riemannian metrics, namely Riemannian metrics with constant curvatures (including sectional, Ricci, scalar and Einstein curvature).

1. Schur's theorem: From fiber constant to constant

As we have seen, the sectional curvature and the Ricci curvature are functions not defined on M itself, but defined on some fiber bundles over M, namely the Grassmannian 2-plane bundle $Gr_2(TM)$ and the sphere bundle SM. Before we study Riemannian manifolds with constant sectional or Ricci curvatures, let's first study an "intermediate" case, namely Riemannian manifolds whose sectional or Ricci curvatures are fiber-wise constant. It turns out that for connected Riemannian manifolds of dimension $m \geq 3$, fiber-wise constant sectional/Ricci curvature will force the Riemannian manifold to have globally constant sectional/Ricci curvature. This result was first established by German mathematician F. Schur in 1886 (for the sectional curvature case).

¶ The contracted Bianchi identity.

The main tool in the proof of Schur's theorem is the second Bianchi identity

$$(\nabla Rm)(U, V, X, Y, Z) + (\nabla Rm)(U, V, Y, Z, X) + (\nabla Rm)(U, V, Z, X, Y) = 0.$$

We will first prove the stronger Ricci curvature version of Schur's theorem, for which what we need is

Proposition 1.1 (The contracted Bianchi identity). *For any Riemannian manifold,*

$$\nabla S = 2c_{1,3}\nabla Rc,$$

where $c_{1,3}$ is the metric contraction in the first and third entry.

Proof. Since the metric contractions commute with[1] ∇, we may apply metric contractions to the Bianchi identity. Contracting the first and the third entries, then

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[1]We note that the metric compatibility implies $\nabla g^* = 0$, where $g^* = g^{ij}\partial_i\partial_j$ is the "dual of g". This in turn implies that the musical isomorphisms commute with the covariant derivative ∇. Now consider the *metric contraction* $c_{i,j}$ that contracts the ith entry with the jth entry of a $(0, k)$ tensor T, where $1 \leq i \neq j \leq k$. Since $c_{i,j}$ can be written as a composition of the standard contraction C^i_j with a musical isomorphism, and since ∇ commutes with C^i_j, one can prove that ∇ also commutes with the metric contraction $c_{i,j}$.

contract the second and fourth entries:
$$0 = \sum_{\circlearrowleft 3,4,5} (\nabla Rm) \implies 0 = c_{2,4}c_{1,3} \sum_{\circlearrowleft 3,4,5} (\nabla Rm).$$

Note that
$$c_{V,Y}c_{U,X}(\nabla Rm)(U,V,X,Y,Z) = \nabla(c_{V,Y}c_{U,X}Rm)(U,V,X,Y,Z) = (\nabla S)(Z),$$

while
$$c_{V,Y}c_{U,X}(\nabla Rm)(U,V,Y,Z,X) = -c_{U,X}(\nabla c_{V,Y}Rm)(U,V,Z,Y,X)$$
$$= -c_{U,X}(\nabla Rc)(U,Z,X)$$

and
$$c_{V,Y}c_{U,X}(\nabla Rm)(U,V,Z,X,Y) = -c_{V,Y}\nabla(c_{U,X}Rm)(U,V,X,Z,Y)$$
$$= -c_{V,Y}(\nabla Rc)(V,Z,Y)$$

So we arrived at
$$\nabla S = 2c_{1,3}\nabla Rc,$$

which completes the proof.   □

In local coordinates, the *contracted Bianchi identity* can be written as
$$\partial_k S = 2g^{ij}Rc_{ik;j}.$$

¶ Schur's theorem.

Now we are ready to prove

Theorem 1.2 (Schur). *Let (M,g) be a connected Riemannian manifold of dimension $m \geq 3$.*

(1) If $Ric(X_p) = f(p)$ depends only on p, then (M,g) has constant Ricci curvature.
(2) If $K(\Pi_p) = f(p)$ depends only on p, then (M,g) has constant sectional curvature.

Proof. (1) Under the assumption we have $Rc_p = f(p)g_p$. It follows
$$S(p) = \text{Tr}(Rc_p) = f(p)\text{Tr}(g_p) = mf(p).$$
So by the contracted Bianchi identity and the fact $\nabla g = 0$ (which implies $g_{ij;k} = 0$),
$$m\partial_k f = \partial_k S = 2g^{ij}Rc_{ik;j} = 2g^{ij}(fg)_{ik;j} = 2g^{ij}(\partial_j f)g_{ik} = 2\partial_k f.$$
It follows that $\partial_k f = 0$ for any k and thus f is a constant.
(2) If $K(\Pi_p) = f(p)$, then
$$Ric(X_p) = (m-1)f(p).$$
So by (1), f is constant.[2]   □

Remark. Obviously the theorem fails in dimension 2, in which case the sectional/Ricci curvature is always a function on M but need not be a constant.

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[2]We will give another direct proof of this fact using moving frames next time.

2. Riemannian manifolds with constant curvatures

¶ Manifolds with constant sectional curvatures.

Now we study Riemannian manifolds with *constant sectional curvatures*, i.e. $K(\Pi_p) = k$ for all $p \in M$ and all $\Pi_p \in T_pM$. According to what we proved last time, (M, g) has constant curvature k if and only if

$$Rm = \frac{k}{2} g \bigwedge g,$$

which is also equivalent to the fact "(M, g) has Weyl curvature tensor $W = 0$ and Ricci curvature tensor $Rc = (m-1)kg$".

We have constructed, for any constant k, a simple Riemannian manifold which has constant sectional curvature k, namely

(a) $(S^m, \frac{1}{k} g_{round})$ if $k > 0$,
(b) $(\mathbb{R}^m, g_0)$ if $k = 0$,
(c) $(H^m, -\frac{1}{k} g_{hyperbolic})$ if $k < 0$.

Of course there are many other constant sectional curvature manifolds, e.g.

- Any open subset in a Riemannian manifold of constant sectional curvature is again a Riemannian manifold of constant sectional curvature. To make our lives easier, we will exclude such examples[3] by studying only <u>connected</u> <u>complete</u> [i.e. when endowed with the Riemannian distance d, (M, d) is complete as a metric space] Riemannian manifolds of <u>constant sectional curvature</u>, which are known as *space forms*.
- If (M, g) has constant sectional curvature, $\pi : M \to N$ is a smooth <u>normal</u> [i.e. the Deck transformation group acts freely on each fiber] covering map and g is invariant under all its Deck transformations, then $(N, \pi_* g)$ has constant sectional curvature. Since universal cover is always normal, by this way we can easily construct
  - a constant positive sectional curvature metric on the real projective space $\mathbb{RP}^m = S^m/\mathbb{Z}_2$ and on the Lens space $L(p, q)$,
  - a flat metric (constant curvature zero metric) on the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, [in local coordinates $\theta^1, \cdots, \theta^m$ on $\mathbb{T}^m = S^1 \times \cdots \times S^1$, the flat metric has the form $g = d\theta^1 \otimes d\theta^1 + \cdots + d\theta^m \otimes d\theta^m$.]
  - a constant negative sectional curvature metric on any closed orientable surface Σ_g of genus $g \geq 2$.

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[3]Unfortunately it is not true that any constant sectional curvature Riemannian manifold is an open submanifold of a complete constant sectional curvature Riemannian manifold. For example, one may start with $S^2 \setminus \{N, S\}$ be the standard sphere with the north/south poles removed, and consider its universal covering (which is topologically $\mathbb{R}^2$ with pull-back Riemannian metric). According to the Killing-Hopf theorem below, the metric can't be complete. Incomplete Riemannian manifolds are far from well-understood.

- Conversely if (M, g) has constant sectional curvature, and $\pi : \widetilde{M} \to M$ is a smooth covering, then $(\widetilde{M}, \pi^* g)$ has constant sectional curvature.

So it is reasonable to focus first on simply connected complete Riemannian manifolds of constant sectional curvature, and the examples (a), (b), (c) above are all simply connected. It turns out that they are the only ones, both locally (without completeness assumption) and globally (under the assumption of completeness):

Theorem 2.1 (Riemann). *Let (M, g) be a Riemannian manifold with constant sectional curvature k, then any point $p \in M$ has a neighborhood that is isometric to an open subset of (a) or (b) or (c).*

Theorem 2.2 (Killing-Hopf). *Let (M, g) be a complete Riemannian manifold of constant sectional curvature k, then the Riemannian universal cover of (M, g) is either (a) or (b) or (c) above (depending on the sign of k).*

We will postpone the proofs of both theorems to later.

According to Killing-Hopf theorem, any complete Riemannian manifold of constant sectional curvature is the quotient of one of the three canonical examples above by a group (which is a subgroup of the corresponding isometry group) that acts freely and properly discontinuously. As a result, only very few smooth manifolds can admit a constant sectional curvature metric. For example, there is no constant sectional curvature metric on $S^2 \times S^1$ since its universal cover is $S^2 \times \mathbb{R}$, which is not one of the above three.

In fact, we have seen all complete even dimensional Riemannian manifolds which has constant positive sectional curvature:

Corollary 2.3. *If (M, g) is a compact Riemannian manifolds of even dimension $m = 2k$, and g has constant sectional curvature 1, then (M, g) is isometric to either $(\mathbb{S}^m, g_{round})$ or its quotient $(\mathbb{RP}^m, g)$.*

Proof. Since M is compact, (M, g) is complete. By Killing-Hopf theorem, (M, g) is the quotient of (S^m, g_{round}) by a subgroup

$$\Gamma \subset \mathrm{Iso}(S^m, g_{round}) = O(m + 1)$$

which acts on S^m freely and properly discontinuously.

Now let $\gamma \in \Gamma$. If γ has an eigenvalue 1, then γ fixes a point in S^m (which is the unit eigenvector of γ) and thus by freeness of the action, $\gamma = \mathrm{Id}$.

As a consequence, for $\mathrm{Id} \neq \gamma \in \Gamma$, 1 is not an eigenvalue of γ. We may consider its square matrix γ^2, which is again an element in Γ. Since m is even, $\gamma^2 \in SO(m+1)$ must has eigenvalue 1 and thus $\gamma^2 = \mathrm{Id}$. It follows that all eigenvalues of γ are -1, and thus $\gamma = -\mathrm{Id}$.

So we must have $\Gamma = \{Id\}$ or $\Gamma = \{\pm \mathrm{Id}\}$, and the conclusion follows. $\square$

Remark. The result fails in odd dimension, since as we have mentioned, all lens spaces admit constant sectional curvature Riemannian metric.

¶ Spaces with constant Ricci Curvatures: Einstein manifolds.

Now let's turn to Riemannian manifolds with constant Ricci curvature, i.e. satisfying $Ric(X_p) = k$ for any $p \in M$ and any $X_p \in S_pM$. It turns out that such manifolds play important roles in Einstein's general theory of relativity(in a slightly different framework, i.e. pseudo-Riemannian geometry): The Einstein field equation (which, together with the geodesic equation that we will discuss later, form the core of the mathematical formulation of general relativity) has the form

$$Rc - \frac{1}{2}Sg + \Lambda g = \kappa T,$$

where S is the scalar curvature function, Λ is known as the cosmological constant[4], κ is the Einstein gravitational constant, and T is the so-called stress-energy tensor. In the case of vacuum where $T = 0$, the equation becomes

$$Rc = (\frac{S}{2} - \Lambda)g.$$

According to Schur's theorem, the function $\frac{S}{2} - \Lambda$ must be a constant, and thus (M, g) has constant Ricci curvature.

Definition 2.4. We say a Riemannian manifold (M, g) is an *Einstein manifold* if there exists a constant λ such that

$$Rc = \lambda g.$$

Einstein manifolds with $\lambda = 0$ are known as *Ricci-flat* manifolds.

Obviously, if (M, g) has constant sectional curvature k, then (M, g) is an Einstein manifold since

$$Rc = c(Rm) = (m - 1)kg.$$

Since $\text{Tr}(Rc) = S$ (the scalar curvature) and $\text{Tr}(g) = m$ (the dimension of M), we conclude that the constant λ for an Einstein manifold must be

$$\lambda = \frac{S}{m}.$$

Since the traceless Ricci tensor $E = Rc - \frac{S}{m}g$, we conclude

Corollary 2.5. (M, g) *is an Einstein manifold if and only if* $E = 0$.

In particular if (M, g) is an Einstein manifold and $W = 0$, then (M, g) has constant sectional curvature. Since $W = 0$ in dimension 3, it follows

Proposition 2.6. *For* $m = 2$ *or* 3, (M, g) *is Einstein if and only if* (M, g) *has constant sectional curvature.*

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[4]According to Gamow, Einstein regard the introduction of the cosmological term as "the biggest blunder of his life". So we name these manifolds as Einsteins manifolds as a punishment (joke).

So to find an Einstein manifold that is not of constant sectional curvature, one must look at manifolds of dimension at least 4. To discover "which manifold admits an Einstein metric and which does not" is still a very active research topic today. Here is an example:

Example. Let $M = S^m \times S^m$ [or more generally the product of two m-dimensional Riemannian manifolds that have the same constant sectional curvature], equipped with the product metric

$$g = \pi_1^* g_{S^m} + \pi_2^* g_{S^m}.$$

Note that S^m has constant curvature 1, so that it is Einstein and

$$Rc(g_{S^m}) = (m-1)g_{S^m}.$$

It follows

$$Rc(g) = \pi_1^* Rc(g_{S^m}) + \pi_2^* Rc(g_{S^m}) = (m-1)\pi_1^* g_{S^m} + (m-1)\pi_2^* g_{S^m} = (m-1)g.$$

In other words, (M, g) is an Einstein manifold.

On the other hand, (M, g) is not of constant sectional curvature for $m > 1$. This can be proved by using Killing-Hopf theorem, or by direct computation: for $(p, q) \in S^m \times S^m$, if we let e_1, e_2 be linearly independent vectors in $T_p S^m$ and let $e_3 \in T_q S^m$, then

$$K(d\iota_q^1(e_1), d\iota_q^1(e_2)) = 1, \qquad K(d\iota_q^1(e_1), d\iota_p^1(e_3)) = 0,$$

where $\iota_q : S^m \to S^m \times S^m$ is the embedding that maps p to (p, q), while $\iota_p : S^m \to S^m \times S^m$ is the embedding that maps q to (p, q).

Remark. Again there are many topological restrictions for a smooth manifold to admit an Einstein metric. For example, by using the famous Chern-Gauss-Bonnet theorem, which has the following form for orientable closed 4-manifolds,

$$\chi(M) = \frac{1}{32\pi^2} \int_M (|Rm|^2 - 4|Rc|^2 + S^2)dv,$$

where $\chi(M)$ is the Euler characteristic of M, one can easily prove:

Theorem (Berger)**.** *If M is an orientable closed 4-manifold and M admits an Einstein metric g, then $\chi(M) \geq 0$, and the equality holds if and only if g is flat.*

In particular, since $\chi(S^3 \times S^1) = 0$ and $S^3 \times S^1$ admits no flat metric (by Killing-Hopf), we conclude that $S^3 \times S^1$ admits no Einstein metric.

The above result was further strengthened by Thorpe and Hitchin as follows:

Theorem (Hitchin–Thorpe inequality)**.** *If M is an orientable closed 4-manifold and M admits an Einstein metric g, then*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|,$$

where $\tau(M)$ is the signature of M. Moreover, if the equality holds, then the Einstein metric is a Ricci flat metric.

¶ Riemannian manifolds with constant scalar curvature.

Now we turn to Riemannian manifolds with constant scalar curvature. There are many such examples, e.g.

- any Einstein manifold has constant scalar curvature,
- the product of two Riemannian manifolds with constant scalar curvatures is again a Riemannian manifold with constant scalar curvature since[exercise]

Proposition 2.7. *Let S_i $(i = 1, 2)$ be the scalar curvature of (M_i, g_i), and S the scalar curvature of $(M_1 \times M_2, \pi_1^* g_1 + \pi_2^* g_2)$. Then $S(p, q) = S_1(p) + S_2(q)$.*

For simplicity we only consider compact manifolds. It turns out that there always exist lots of constant scalar curvature metrics on any compact manifold M.

(1) Let's start with the case of dimension 2, i.e. S is a compact surface. Note that in this case all curvatures are the same. As we have seen, since the universal covering of S is either S^2 or $\mathbb{R}^2$ or D(the unit disc), M admits a constant curvature metric. However, even in this case, one can say a lot more: Suppose M is orientable (so that it has a Riemann surface structure), then by lifting to the universal covering and using the famous <u>uniformization theorem</u> in complex analysis, one can prove

Theorem. *For any Riemannian metric g on an compact orientable surface S, there is $u \in C^\infty(S)$ so that $(M, e^{2u}g)$ has constant curvature.*

[Note that according to the Gauss-Bonnet theorem, in this case the sign of the constant curvature depends on the topology of M.]

(2) In dimension $m \geq 3$, unlike the sectional curvature or the Ricci curvature, the scalar curvature encodes relatively few information. For example, Kazdan and Warner solved the <u>prescribed scalar curvature problem</u> and get

Theorem (Kazdan-Warner)**.** *For any compact smooth manifold M of dimension $m \geq 3$, exactly one of the following will happen:*
(a) *For any $f \in C^\infty(M)$, there exists a Riemannian metric g on M whose scalar curvature function is f.*
(b) *A function $f \in C^\infty(M)$ is the scalar curvature of some Riemannian metric on M if and only if either $f \equiv 0$, or f is negative somewhere.*
(c) *A function $f \in C^\infty(M)$ is the scalar curvature of some Riemannian metric on M if and only if f is negative somewhere.*

As consequences we immediately get
- any compact manifold of dimension $m \geq 3$ admits a Riemannian metric whose scalar curvature is any given negative constant.
- For any compact manifold M, by Proposition 2.7 the manifold $M \times S^m$(where $m \geq 2$) admits a positive scalar curvature metric, and thus any function on $M \times S^m$ can be realized as its scalar curvature.

On the other hand, Atiyah-Singer index theorem gives topological obstructions for the existence of positive scalar curvature metric. It is also known that any torus $\mathbb{T}^m$ admits no positive scalar curvature metric [Schoen-Yau for $3 \le m \le 7$(related to the positive mass theorem), Gromov-Lawson for general m].

(3) Since for surfaces one can always find a constant scalar curvature metric in any given conformal class, it is natural to ask whether the same result holds in higher dimension [a classical problem in geometric analysis]:

The Yamabe prolbem. *Given a compact*[5] *Riemannian manifold* (M, g) *of dimension* $m \ge 3$, *is there* $u \in C^\infty(M)$ *so that* $(M, e^{2u}g)$ *has constant scalar curvature?*

By computing the scalar curvature under conformal change, the problem is reduced to finding a positive solution to the partial differential equation

$$\frac{4(m-1)}{m-2}\Delta\varphi + S\varphi = \lambda\varphi^{\frac{m+2}{m-2}}.$$

Although there is a gap in Yamabe's origin solution (as pointed out by Trudinger), by combining the works of Yamabe(1960), Trudinger(1968), Aubin(1976) and Schoen(1984), the answer is **YES**. In particular, for any compact manifold of dimension $m \ge 3$, there exist lots of constant scalar curvature metrics.

¶ Riemannian manifolds with $W = 0$ (locally conformally flat manifolds).

Finally we return to constant sectional curvature metrics, but in the framework of conformal geometry. Of course globally there are topological restrictions for the existence (by Killing-Hopf theorem), so we restrict ourselves to the local setting, i.e. study Riemannian manifolds (M, g) so that near each point p, there is an open neighborhood U and a smooth function $u \in C^\infty(U)$ so that the metric $(U, e^{2u}g)$ has constant sectional curvature. According to Theorem 2.1, if $(U, e^{2u}g)$ has constant sectional curvature, then by shrinking U if needed, the manifold $(U, e^{2u}g)$ is isometric to an open subset of one of the three canonical constant sectional curvature spaces, namely the Euclidean space or the sphere or the hyperbolic space (with scaled metric). Let's take a closer look of there three metrics. For $\mathbb{R}^m$ and H^m we have

$$g_0 = dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m \qquad \text{for } \mathbb{R}^m$$

$$g_{hyperbolic} = \frac{1}{(x^m)^2}\big(dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m\big) \qquad \text{for } H^m$$

while for S^m, we may use stereographic coordinates $(t^1, \cdots, t^m)$ [so that any point which is not the north pole on S^m can be written as $(\frac{2t^1}{1+|t|^2}, \cdots, \frac{2t^m}{1+|t|^2}, \frac{1-|t|^2}{1+|t|^2})$] to get [exercise]

$$g_{round} = \frac{4}{(1+|t|^2)^2}\big(dt^1 \otimes dt^1 + \cdots + dt^m \otimes dt^m\big) \qquad \text{for } S^m.$$

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[5]One may pose the same problem for complete noncompact manifolds. The answer is no in general, and counterexamples were constructed by Jin (1988). The problem of finding conditions under which the problem has a solution is still a topic of research today.

So in all these three cases, after multiplying a conformal factor one gets a flat metric. In other words, "locally conformally constant sectional curvature" is the same as "locally conformally flat":

Definition 2.8. We say a Riemannian manifold (M, g) is *locally conformally flat* if for any $p \in M$, there is a neighborhood U of p and a smooth function $u \in C^\infty(U)$ so that the metric $\bar{g} = e^{2u}g$ is flat in U.

Example. We give a couple examples:

- Any constant sectional curvature space is locally conformally flat(as just explained).
- Any surface (with any real analytic Riemannian metric) is locally conformally flat:

  Theorem (Gauss, 1822). *On any surface with real analytic Riemannian metric g, there exists coordinates x, y and smooth function $u = u(x, y)$ so that*
  $$g = e^{2u}(dx \otimes dx + dy \otimes dy).$$
  [*Such coordinates are known as* isothermal coordinates.]

To find out necessary conditions for a metric g to be locally conformally flat, we need to find local quantities that are invariant under a conformal change of metric.

Lemma 2.9. *If $\bar{g} = e^{2u}g$, then $\overline{W} = e^{2u}W$.*

Proof. In PSet 2 we will see $\overline{Rm} = e^{2u}(Rm - g \bigwedge T)$ for some symmetric 2-tensor T. Decomposing both $\overline{Rm}$ and Rm intwo Weyl and non-Weyl parts we get
$$\overline{W} - e^{2u}W = g \bigwedge \widetilde{T},$$
for some symmetric 2-tensor $\widetilde{T}$. So $\overline{W} - e^{2u}W \in \ker(c) \cap \mathrm{Im}(\Psi)$. By Corollary 2.10 in Lecture 8, we must have $\overline{W} - e^{2u}W = 0$. $\qquad \square$

Since any flat metric has Weyl curvature tensor $W = 0$, we conclude

Corollary 2.10. *If (M, g) is locally conformally flat, then $W = 0$.*

It turns out that the condition is also necessary for $m \geq 4$:

Theorem (Weyl-Schouten). *Let (M, g) be a Riemannian manifold.*

(1) *For $m \geq 4$, (M, g) is locally conformally flat if and only if $W = 0$.*
(2) *For $m = 3$, (M, g) is locally conformally flat if and only if [A is the Schouten tenser]*
$$(\nabla_X A)(Y, Z) - (\nabla_Y A)(X, Z) = 0, \quad \forall X, Y, Z \in \Gamma^\infty(TM).$$

[The proof will be left as part of one possible final project.]

For example, by calculating the Weyl curvature tensor one can easily prove that when endowed with the standard product metric, both $S^2 \times S^2$ and $S^2 \times \mathbb{R}^2$ are not locally conformally flat, while $S^{m_1} \times H^{m_2}$ is locally conformally flat.