LECTURE 11: THE METHOD OF MOVING FRAMES

In Riemannian geometry, one frequently encounters with heavy computations (especially for those problems related to curvatures). There are three different methods to do these calculations: the invariant method via global vector fields and tensor fields, the local method via carefully chosen coordinate charts (under the help of Einstein summation convention), and E. Cartan's method of moving frames via calculus of differential forms. Today we will give a brief introduction to the method of moving frames where the use of differential forms is emphasized[when compared with tensor fields, differential forms have the advantage that they can be pulled-back via smooth maps, and we have the powerful tool of exterior derivative].

1. Cartan's method of moving frames

¶ The connection 1-forms for a linear connection in a local frame.

Let M be a smooth manifold and ∇ a linear connection on M. We can regard ∇ (acting on vector fields) as a linear map

$$\nabla : \Gamma^\infty(TM) \to \Gamma^\infty(TM \otimes T^*M).$$

So if $\{e_1, \cdots, e_m\}$ is a *local frame*[i.e. for each $p \in U$, $e_1(p), \cdots, e_m(p)$ form a basis of T_pM] of TM defined on an open set $U \subset M$, then one can find a set of one forms $\{\theta_i^j\}_{1\le i,j\le m}$ defined on U so that $\nabla_X e_i = \theta_i^j(X)e_j$ for all $X \in \Gamma^\infty(TM)$, i.e.

$$\nabla e_i = e_j \otimes \theta_i^j. \tag{1}$$

These θ_i^j's are known as *connection 1-forms* of ∇ with respect to the local frame $\{e_i\}$, which are only locally defined.

Moreover, if we choose another local frame $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ on $\widetilde{U}$, and $\tilde{e}_i = f_i^j e_j$ on $U \cap \widetilde{U}$, then $e_j = (f^{-1})_j^i \tilde{e}_k$ (where f^{-1} is the inverse of the matrix $f = (f_i^j)$) and thus

$$\begin{aligned}\tilde{e}_l \otimes \tilde{\theta}_i^l = \nabla \tilde{e}_i = \nabla(f_i^j e_j) &= f_i^j \nabla e_j + e_j \otimes df_i^j \\ &= f_i^j e_k \otimes \theta_j^k + (f^{-1})_j^l \tilde{e}_l \otimes df_i^j \\ &= \tilde{e}_l \otimes (f^{-1})_k^l \theta_j^k f_i^j + (f^{-1})_j^l df_i^j,\end{aligned}$$

so we end up with

$$\tilde{\theta}_i^l = (f^{-1})_k^l \theta_j^k f_i^j + (f^{-1})_j^l df_i^j,$$

on $U \cap \widetilde{U}$, which can be written in brief as

$$\tilde{\theta} = f^{-1}\theta f + f^{-1}df, \tag{2}$$

where $\tilde{\theta}$ and θ are understood as $m \times m$ matrices whose entries are 1-forms, while f and f^{-1} are invertible $m \times m$ matrices[1] whose entries are functions (and thus one can not exchange their positions in the product above).

To develop Riemannian geometry via differential forms only, let's first derive the dual formula for covariant derivative of differential forms via these connection 1-forms. We denote by $\{\omega^1, \cdots, \omega^m\}$ the local *dual co-frame*[i.e. $\omega^i(e_j) = \delta^i_j$ for all i, j] of T^*M defined on U to the given local frame $\{e_1, \cdots, e_m\}$. Then we have

$$(\nabla_X \omega^i)(e_j) = X(\omega^i(e_j)) - \omega^i(\nabla_X e_j) = -\omega^i(\theta^k_j(X)e_k) = -\theta^i_j(X).$$

It follows that the linear connection ∇ acting on one forms, viewed as a map

$$\nabla : \Gamma^\infty(T^*M) \to \Gamma^\infty(T^*M \otimes T^*M),$$

can be expressed in terms of the co-frame and the connection 1-forms as

$$\nabla \omega^i = -\omega^j \otimes \theta^i_j. \tag{3}$$

¶ The connection 1-forms: torsion freeness and metric compatibility.

Now suppose the linear connection ∇ is torsion free. Then

$$\begin{aligned} d\omega^i(X, Y) &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i([X, Y]) \\ &= X(\omega^i(Y)) - Y(\omega^i(X)) - \omega^i(\nabla_X Y - \nabla_Y X) \\ &= (\nabla_X \omega^i)(Y) - (\nabla_Y \omega^i)(X) \\ &= -\omega^j(Y)\theta^i_j(X) + \omega^j(X)\theta^i_j(Y). \end{aligned}$$

So the torsion free condition for a linear connection can be written, in terms of the dual co-frame and the connection 1-forms, as

$$d\omega^i = \omega^j \otimes \theta^i_j - \theta^i_j \otimes \omega^j = \omega^j \wedge \theta^i_j. \tag{4}$$

which can be written in brief as $d\omega = -\theta \wedge \omega$.

Next suppose there is a Riemannian metric g on M, and the connection ∇ is metric compatible. To encode the information of the metric into our consideration, it is reasonable to choose an orthonormal frame $\{e_1, \cdots, e_m\}$ instead of a general frame. Then

$$0 = \langle \nabla e_i, e_j \rangle + \langle e_i, \nabla e_j \rangle = \langle e_k \otimes \theta^k_i, e_j \rangle + \langle e_i, e_k \otimes \theta^k_j \rangle = \theta^j_i + \theta^j_i.$$

So the metric compatibility of ∇ becomes: for any orthonormal frame, the connection 1-forms satisfy

$$\theta^j_i + \theta^j_i = 0, \tag{5}$$

i.e. the matrix of connection 1-forms is anti-symmetric.

[1]So one may regard f as a map from $U \cap \widetilde{U}$ to the general linear group $GL(m)$. If we are in the setting of Riemannian manifold and we are only using local orthonormal frames, then the group encountered is $O(m)$ instead. The method of moving frame works in a more general setting, and there is always such a Lie group behind the theory that plays an important role.

¶ Cartan's formulation of Riemannian geometry.

It turns out that one can develop Riemannian geometry starting with local frames and connection 1-forms (i.e. via the differential 1-forms ω^i, θ^i_j) instead of the Riemannian metric g and its Levi-Civita connection[since one can recover the Riemannian metric g from the local orthonormal co-frame $\{\omega^i\}$, and then recover the Levi-Civita connection ∇ from its connection 1-forms θ^j_i]. We start with a simple lemma:

Lemma 1.1. *Suppose $\omega^1, \cdots, \omega^s \in \Lambda^1 V^*$ ($s \le m = \dim V$) are linearly independent.*

(1) *If $\eta^1, \cdots, \eta^s \in \Lambda^1 V^*$ and $\sum \eta^i \wedge \omega^i = 0$, then there exist uniquely determined real numbers A^i_j ($1 \le i, j \le s$) with $A^i_j = A^j_i$ such that $\eta^i = A^i_j \omega^j$.*

(2) *If $s = m$, and a collection of linear 1-forms $\theta^i_j \in \Lambda^1 V^*$ ($1 \le i, j \le m$) satisfy*

$$\omega^j \wedge \theta^i_j = 0 \quad \text{and} \quad \theta^i_j + \theta^j_i = 0,$$

then $\theta^i_j = 0$.

Proof. (1) Obviously $\eta^i \in \operatorname{span}\{\omega^1, \cdots, \omega^s\}$. Write $\eta^i = A^i_j \omega^j$. Then

$$\sum \eta^i \wedge \omega^i = \sum_{i<j} (A^i_j - A^j_i) \omega^i \wedge \omega^j$$

and the conclusion follows.

(2) Write $\theta^i_j = a^i_{jk} \omega^k$. Then the two conditions becomes

$$a^i_{jk} - a^i_{kj} = 0 \quad \text{and} \quad a^i_{jk} + a^j_{ik} = 0.$$

Thus

$$a^i_{jk} = a^i_{kj} = -a^k_{ij} = -a^k_{ji} = a^j_{ki} = a^j_{ik} = -a^i_{jk}$$

and the conclusion follows. □

Now we state the fundamental theorem of Riemannian geometry [i.e. the existence and uniqueness of Levi-Civita connection] in the language of connection 1-forms:

Theorem 1.2 (E. Cartan). *Let $\omega^1, \cdots, \omega^m \in \Omega^1(U)$ be a collection of 1-forms on an open set $U \subset M$ that are linearly independent at each point. Then there exists a unique collection of 1-forms, $\theta^i_j \in \Omega^1(U)$ ($1 \le i, j \le m$), so that*

$$d\omega^i = \omega^j \wedge \theta^i_j \quad \text{and} \quad \theta^i_j + \theta^j_i = 0.$$

[*These equations are known as* Cartan's structural equations.]

Proof. Uniqueness follows from Lemma 1.1 (2). For the existence, one just start with the Riemannian metric $g = \sum \omega^i \otimes \omega^i$ (so that the dual frame $\{e_i\}$ of $\{\omega^i\}$ is an orthonormal basis for each point in U) and take θ^i_j to be the connection 1-forms for the Levi-Civita connection of this metric. □

Remark. How to get from local to global? To glue, one need the connection 1-forms to satisfy the change of frame formula (2) for any orthogonal transformation f.

¶ The curvature 2-form.

We start with any linear connection on a smooth manifold M. Suppose we are given a local co-frame $\{\omega^i\}$ and the corresponding connection 1-forms θ^i_j. We may express the curvature using differential forms (in terms of the connection 1-forms) as follows. By definition

$$\begin{aligned} R(X,Y)e_i &= \nabla_X\nabla_Y e_i - \nabla_Y\nabla_X e_i - \nabla_{[X,Y]}e_i \\ &= \nabla_X(\theta^j_i(Y)e_j) - \nabla_Y(\theta^j_i(X)e_j) - \theta^j_i([X,Y])e_j \\ &= X(\theta^j_i(Y))e_j + \theta^j_i(Y)\theta^k_j(X)e_k - Y(\theta^j_i(X))e_j - \theta^j_i(X)\theta^k_j(Y)e_k - \theta^j_i([X,Y])e_j \\ &= (d\theta^j_i)(X,Y)e_j + \theta^j_k\wedge\theta^k_i(X,Y)e_j. \end{aligned}$$

As a consequence, if we denote $R(e_k,e_l)e_i = R_{kli}{}^j e_j$, then we get

$$d\theta^j_i + \theta^j_k\wedge\theta^k_i = R_{kli}{}^j\omega^k\otimes\omega^l = \frac{1}{2}R_{kli}{}^j\omega^k\wedge\omega^l. \tag{6}$$

We shall denote

$$\Omega^j_i = \frac{1}{2}R_{kli}{}^j\omega^k\wedge\omega^l,$$

and call it the *curvature 2-form*, which can be expressed in terms of θ^j_i's as

$$\Omega^j_i = d\theta^j_i + \theta^j_k\wedge\theta^k_i. \tag{7}$$

The formula can be taken as definition of curvature (for given connection 1-forms) and is usually written in brief as

$$\Omega = d\theta + \theta\wedge\theta,$$

where Ω is regarded as an $m\times m$ matrix whose entries are 2-forms.

Unlike the connection 1-forms, given a linear connection, the curvature 2-form is independent of the choice of co-frame and thus is globally defined. To see this, we use the frame transformation formula for connection 1-forms above to get

$$\begin{aligned} \widetilde{\Omega} = d\tilde\theta + \tilde\theta\wedge\tilde\theta =& (df^{-1})\wedge\theta f + f^{-1}(d\theta)f - f^{-1}\theta\wedge df + (df^{-1})\wedge df \\ &+ f^{-1}\theta\wedge\theta f + f^{-1}\theta\wedge df + f^{-1}df\wedge f^{-1}\theta f + f^{-1}df\wedge f^{-1}df. \end{aligned}$$

In view of the fact $df^{-1} = -f^{-1}(df)f^{-1}$, we get

$$\widetilde{\Omega} = f^{-1}(d\theta+\theta\wedge\theta)f = f^{-1}\Omega f,$$

which is equivalent to say Ω is independent of the choice of frames.

Now suppose (M,g) is a Riemannian manifold. Then we may start with orthonormal co-frame $\{\omega^i\}$, and we have Cartan's structural equations, which implies

$$\Omega^j_i = -\Omega^i_j.$$

We may also express the curvature 2-form Ω^j_i using $R_{ijkl} := Rm(e_i,e_j,e_k,e_l)$ as

$$\Omega^i_j = \frac{1}{2}R_{klj}{}^i\omega^k\wedge\omega^l = -\frac{1}{2}R_{klji}\omega^k\wedge\omega^l = \frac{1}{2}R_{ijkl}\omega^k\wedge\omega^l.$$

Remark. More generally, one can develop the theory of linear connections on vector bundles (or principal bundles) via moving frames, as follows. Let E be a rank r vector bundle over M, and $\{e_1, \cdots, e_r\}$ a local frame of E. Then one can either define a linear connection

$$\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E \otimes T^*M)$$

via axioms that we mentioned earlier, or via connection 1-forms $\theta_i^j (1 \le i, j \le r)$ that are locally defined such that

$$\nabla e_i = e_j \otimes \theta_i^j.$$

As we calculated above, the matrix θ transform under change of basis as

$$\tilde{\theta} = f^{-1}\theta f + f^{-1}df.$$

One can further define the curvature 2-form to be

$$\Omega = d\theta + \theta \wedge \theta.$$

2. Applications to Riemannian geometry

¶ Calculating curvatures.

As the first application, we use moving frames to calculate the curvature of a Riemannian manifold (M, g). Let $\{e_1, \cdots, e_m\}$ be a local orthonormal frame of (M, g). By definition the sectional curvature of the plane spanned by $\{e_i, e_j\}$ is

$$K(e_i, e_j) = Rm(e_i, e_j, e_i, e_j) = R_{ijij} = \Omega_i^j(e_j, e_i).$$

Theorem 2.1. *(M, g) has constant sectional curvature c at $p \in M$ if and only if for any local orthonormal frame $\{e_i\}$, at p we have*

$$\Omega_j^i = c\omega^i \wedge \omega^j. \tag{8}$$

Proof. Suppose (8) holds at p for any orthonormal frame. Let Π_p be any two dimensional plane in T_pM. Choose an orthonormal basis $\{e_1, e_2\}$ of Π_p, extend it to an orthonormal frame and denote by $\omega^1, \cdots, \omega^m$ the dual co-frame. Then

$$K(\Pi_p) = K(e_1, e_2) = c\Omega_1^2(e_2, e_1) = c\omega^2 \wedge \omega^1(e_2, e_1) = c.$$

Conversely suppose (M, g) has constant sectional curvature c at p, then with respect to any orthonormal frame,

$$R_{ijkl} = \frac{c}{2} g \owedge g(e_i, e_j, e_k, e_l) = c(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})$$

at p and thus the conclusion follows. □

Example. Consider the upper half space $\mathbb{H}^m$ with the hyperbolic metric

$$g_{hyperbolic} = \frac{1}{(x^m)^2}(dx^1 \otimes dx^1 + \cdots + dx^m \otimes dx^m).$$

With the orthonormal frame $\{e_i = x^m \partial_i\}$ and its dual co-frame $\{\omega^i = \frac{1}{x^m} dx^i\}$,

$$\omega^j \wedge \theta_j^i = d\omega^i = -\frac{1}{(x^m)^2} dx^m \wedge dx^i = -\omega^m \wedge \omega^i.$$

Observe that for the given co-frame $\{\omega^1, \cdots, \omega^m\}$,

$$\theta_j^i = 0, (i, j < m) \quad \text{and} \quad \theta_m^i = -\theta_i^m = -\omega^i \, (i < m)$$

is a solution and thus has to be the unique solution. So we get, for $i, j < m$,

$$\Omega_j^i = d\theta_j^i + \theta_k^i \wedge \theta_j^k = \theta_m^i \wedge \theta_j^m = -\omega^i \wedge \omega^j$$

and for $i < m$

$$\Omega_m^i = d\theta_m^i + \theta_k^i \wedge \theta_m^k = -d\omega^i = -\omega^i \wedge \omega^m.$$

It follows from Theorem 2.1 that the hyperbolic space has constant curvature -1.

¶ Proving the Bianchi identities.

We may also prove the Bianchi identities via moving frame. For the first Bianchi identity, we just take exterior derivative:

$$\begin{aligned}
0 = d^2\omega^i = d\omega^j \wedge \theta_j^i - \omega^j \wedge d\theta_j^i &= \omega^k \wedge \theta_k^j \wedge \theta_j^i - \omega^j \wedge (\Omega_j^i - \theta_k^i \wedge \theta_j^k) \\
&= -\omega^j \wedge \Omega_j^i \\
&= -\frac{1}{2} R_{klj}{}^i \omega^j \wedge \omega^k \wedge \omega^l, \\
&= -\frac{1}{2} \sum_{j<k<l} \left(R_{klj}{}^i + R_{ljk}{}^i + R_{jkl}{}^i \right) \omega^j \wedge \omega^k \wedge \omega^l.
\end{aligned}$$

As a consequence, we get for distinct k, l, j's,

$$R_{klj}{}^i + R_{ljk}{}^i + R_{jkl}{}^i = 0.$$

If two or three of k, l, j's are the same, then the first Bianchi identity trivial.

Similarly by taking exterior derivative of $\Omega = d\theta + \theta \wedge \theta$ we get

$$d\Omega = d\theta \wedge \theta - \theta \wedge d\theta = \Omega \wedge \theta - \theta \wedge \Omega. \tag{9}$$

One can prove that in local frames, together with the first Bianchi identity, the expression above is equivalent to the second Bianchi identity. In fact we can give a very quick proof of the sectional curvature version of Schur's theorem via (9):

Alternative proof of Theorem 1.2(2) in Lecture 10.
Suppose (M, g) has sectional curvature $K(\Pi_p) = f(p)$ for some $f \in C^\infty(M)$. By Theorem 2.1, $\Omega_j^i = f(p)\omega^i \wedge \omega^j$. So

$$\begin{aligned}
df \wedge \omega^i \wedge \omega^j + f d\omega^i \wedge \omega^j - f\omega^i \wedge d\omega^j = d\Omega_j^i &= \Omega_k^i \wedge \theta_j^k - \theta_k^i \wedge \Omega_j^k \\
&= -f\omega^i \wedge \omega^k \wedge \theta_k^j - f\theta_k^i \wedge \omega^k \wedge \omega^j \\
&= -f\omega^i \wedge d\omega^j + f d\omega^i \wedge \omega^j.
\end{aligned}$$

It follows $df \wedge \omega^i \wedge \omega^j = 0$ for all i, j, and, since $m \geq 3$, $df = 0$, i.e. f is consant. □

¶ **Reading: Geometry of Riemannian submanifolds via moving frame.**

Let $(\overline{M}, \bar{g})$ be a Riemannian manifold of dimension m, and $\iota : S \hookrightarrow \overline{M}$ a smooth submanifold of dimension s endowed with the submanifold metric $g = \iota^*\bar{g}$. For simplicity make the following index convention:

- $1 \le A, B, \cdots \le m$,
- $1 \le i, j, \cdots \le s$,
- $s + 1 \le \alpha, \beta, \cdots \le m$

As usual we denote by NS the normal bundle of S in M.

We have three different ways to develop the Riemannian geometry of S. Here we take the moving frame approach. So let's start with a special local orthonormal frame $\{\bar{e}_1, \cdots, \bar{e}_m\}$ of $(\overline{M}, \bar{g})$ with the property that $\bar{e}_i = d\iota(e_i)$ on S for $1 \le i \le s$ and $\{e_1, \cdots, e_s\}$ form a local orthonormal frame of S. Denote by $\{\bar{\omega}^1, \cdots, \bar{\omega}^m\}$ the dual co-frame of $\{\bar{e}_1, \cdots, \bar{e}_m\}$. Then by definition,

$$\iota^*\bar{\omega}^\alpha = 0. \tag{10}$$

Let $\bar{\theta}^A_B$ the connection 1-forms of $(\overline{M}, \bar{g})$ corresponding to the local frame $\{\bar{e}_A\}$. Then Cartan's structural equations of $\overline{M}$ reads

$$\bar{\theta}^A_B + \bar{\theta}^B_A = 0 \quad \text{and} \quad d\bar{\omega}^A = \bar{\omega}^B \wedge \bar{\theta}^A_B.$$

It follows that as 1-forms on S, $\omega^i := \iota^*\bar{\omega}^i$ and $\theta^i_j := \iota^*\bar{\theta}^i_j$ satisfy (here we used (10))

$$\theta^i_j + \theta^j_i = 0 \quad \text{and} \quad d\omega^i = \omega^j \wedge \theta^i_j.$$

By uniqueness in Theorem 1.2, θ^i_j's are the connection 1-forms on S associate with the co-frame $\{\omega^1, \cdots, \omega^m\}$. [This proves the remark on page 9 of Lecture 6.]

We may also study connection 1-forms with indices α's. Using (10) twice we get

$$0 = d\iota^*\bar{\omega}^\alpha = \iota^*\bar{\omega}^A \wedge \iota^*\bar{\theta}^\alpha_A = \omega^i \wedge \iota^*\bar{\theta}^\alpha_i.$$

Thus by Lemma 1.1(1), there exist uniquely determined functions h^α_{ij} such that

$$h^\alpha_{ij} = h^\alpha_{ji} \quad \text{and} \quad \iota^*\bar{\theta}^\alpha_i = h^\alpha_{ij}\omega^j.$$

Definition 2.2. We call the map $\mathrm{II} : \Gamma^\infty(TS) \times \Gamma^\infty(TS) \to \Gamma^\infty(NS)$ defined by

$$\mathrm{II}(X, Y) = h^\alpha_{ij}\omega^i(X)\omega^j(Y)\bar{e}_\alpha$$

the *second fundamental form* of (S, g) as a Riemannian submanifold of $(\overline{M}, \bar{g})$.

Note that the fact $h^\alpha_{ij} = h^\alpha_{ji}$ implies $\mathrm{II}(X, Y) = \mathrm{II}(Y, X)$. We may write

$$\mathrm{II} = h^\alpha_{ij}\omega^i \otimes \omega^j \otimes \bar{e}_\alpha.$$

To see the formula above is independent of the choices of frames, let's reveal the true face of $\mathrm{II}(X, Y)$ by expressing it in the invariant formulation. We will use $\overline{\nabla}$ and ∇ to denote the Levi-Civita connections for $(\overline{M}, \bar{g})$ and (S, g) respectively. For

$X, Y \in \Gamma^\infty(TS)$, we denote $\overline{X} = d\iota(X)$ and $\overline{Y} = d\iota(Y)$. Note that if $X = X^i e_i$, then $\overline{X} = X^i \bar{e}_i$. So on S we have

$$\begin{aligned}\overline{\nabla}_{\overline{Y}}\overline{X} - d\iota(\nabla_Y X) &= \overline{Y}(X^i)\bar{e}_i + X^i \bar{\theta}_i^A(\overline{Y})\bar{e}_A - d\iota\big(Y(X^i)e_i - X^i\theta_i^j(Y)e_j\big)\\ &= X^i\bar{\theta}_i^\alpha(\overline{Y})\bar{e}_\alpha\\ &= \omega^i(X)\iota^*\bar{\theta}_i^\alpha(Y)\bar{e}_\alpha\\ &= h_{ij}^\alpha \omega^i(X)\omega^j(Y)\bar{e}_\alpha.\end{aligned}$$

In other words, for any vector field X, Y tangent to S, we have

$$\mathrm{II}(X,Y) = \overline{\nabla}_{\overline{Y}}\overline{X} - d\iota(\nabla_Y X).$$

In view of the fact $\nabla_Y X$ is the tangential component of $\overline{\nabla}_{\overline{Y}}\overline{X}$, we conclude that $\mathrm{II}(X,Y)$ is really the normal component of $\overline{\nabla}_{\overline{Y}}\overline{X}$.

Example. According to the example on page 9-10 in Lecture 6, for the unit sphere S^m viewed as a Riemannian submanifold of $\mathbb{R}^{m+1}$, we have

$$\mathrm{II}(X,Y) = -\langle X, Y\rangle \vec{n}.$$

The second fundamental form is closely related to the curvature 2-form of (S,g): If we pull back $\overline{\Omega}_j^i = d\bar{\theta}_j^i + \bar{\theta}_A^i \wedge \bar{\theta}_j^A$ to S and compare with $\Omega_j^i = d\theta_j^i + \theta_k^i \wedge \theta_j^k$, we get

$$\Omega_j^i = \iota^*\overline{\Omega}_j^i - \iota^*\bar{\theta}_\alpha^i \wedge \iota^*\bar{\theta}_j^\alpha = \iota^*\overline{\Omega}_j^i + \sum_\alpha \iota^*\bar{\theta}_i^\alpha \wedge \iota^*\bar{\theta}_j^\alpha = \iota^*\overline{\Omega}_j^i + \sum_\alpha h_{ik}^\alpha h_{jl}^\alpha \omega^k \wedge \omega^l.$$

As a consequence,

$$R_{ijkl} = \overline{R}_{ijkl} + (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

which is known as *Gauss equation.*

Example. In the case S is a hypersurface in (M,g), i.e. has co-dimension 1, then one may pair the second fundamental form with $\bar{e}_m$ and thus for each $p \in S$, regard II_p as a symmetric quadratic form on T_pS. With the help of the Riemannian metric, one can convert this symmetric quadratic form into a symmetric operator on T_pM, which is known as the *shape operator*. The eigenvalues of the shape operator are known as the *principal curvatures* of S at p. Its trace and the determinant are known as the *mean curvature* and the *Gauss curvature* of S at p.

In particular, if S is a 2-dimensional surface isometrically embedded in $\mathbb{R}^3$, the only sectional curvature is $R_{1212} = h_{11}h_{22} - h_{12}^2$, which is exactly the Gauss curvature of S. As a consequence, we get Gauss Theorem Egregium: The Gauss curvature [which is defined by the second fundamental form which is extrinsic] is in fact intrinsic [since the sectional curvature depends only on the Riemannian metric and thus is intrinsic].

We say S is a *totally geodesic submanifold* if $\mathrm{II} = 0$, i.e. $h_{ij}^\alpha = 0$ for all i, j, α. From Gauss equation one gets [there is still an issue here that we will explain later]

Theorem 2.3. *Let S be a totally geodesic 2-dimensional submanifold of M with $T_pS = \Pi_p$. Then the sectional curvature $K(\Pi_p)$ of M is the Gauss curvature of S.*