LECTURE 12: GEODESICS AS SELF-PARALLEL CURVES (ON MANIFOLDS WITH CONNECTION)

Now we turn to the next topic in this course: geodesic, which is a generalization of the notion of straight line in the Euclidean space. As we know, a line in $\mathbb{R}^m$ is both a curve "with constant direction", and a curve that "minimize distances between any two points on it". As a result, we will have two ways to define geodesics on Riemannian manifolds, which, as we will see, are equivalent. On the other hand, for the first method (i.e. regard geodesics as curves "with constant directions"), what we need is the existence of a covariant derivative instead of a Riemannian metric structure, and as a result, it works for any smooth manifold with a linear connection. So today we will introduce the first method, i.e., focus on "non-metric properties" of geodesics.

1. Geodesics on manifolds with linear connections

¶ Geodesics for manifolds with linear connections.

Let M be a smooth manifold. To define a geodesic as a "curve with constant direction", what we need is a structure that can be used to compare tangent vectors at different points along a curve, i.e. a parallel transport, or equivalently, a linear connection. So we let ∇ be a linear connection on M. Now suppose $\gamma : [a, b] \to M$ is a smooth curve in M. Then "γ is a geodesic" means that the tangent vector field $\dot{\gamma}$ is "unchanged" along γ(under parallel transport), i.e. is covariantly constant along γ:

Definition 1.1. We say γ is a *geodesic* if $\dot{\gamma}$ is parallel along γ, i.e.

$$\nabla_{\dot{\gamma}(t)}\dot{\gamma} = 0, \quad \forall t.$$

In local coordinates, if we write $\gamma(t) = (x^1(t), \cdots, x^m(t))$, then

$$\dot{\gamma}(t) = d\gamma(\frac{d}{dt}) = \dot{x}^i(t)\partial_i.$$

Now suppose $X = X^i\partial_i$ is a smooth vector field near γ [If X is only defined on γ, then we need to extend it to a smooth vector field in a neighborhood of γ. By locality of ∇, the extension will not affect the computation below]. If we denote $f^i(t) = X^i(\gamma(t))$, then

$$\nabla_{\dot{\gamma}(t)}X^i = \dot{\gamma}(t)X^i = \frac{d}{dt}(X^i \circ \gamma) = \dot{f}^i(t)$$

[i.e. the covariant derivative of any function along γ is its t-derivative] and thus

$$(\nabla_{\dot{\gamma}}X)|_{\gamma(t)} = (\dot{\gamma}(t)X^i)\partial_i + \Gamma^k{}_{ij}\dot{x}^i(t)f^j(t)\partial_k = \dot{f}^k(t)\partial_k + \Gamma^k{}_{ij}\dot{x}^i(t)f^j(t)\partial_k.$$

As a result, the condition $\nabla_{\dot{\gamma}}X = 0$, i.e. "$X$ is parallel along γ" becomes

$$\dot{f}^k(t) + \Gamma^k{}_{ij}(\gamma(t))\dot{x}^i(t)f^j(t) = 0, \quad \forall k.$$

Apply this to the vector field $X = \dot{\gamma}$, we see γ is a geodesic if and only if locally its coordinate functions satisfy the following system of second order ODEs

$$\ddot{x}^k(t) + \dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij} = 0, \quad 1 \le k \le m. \tag{1}$$

Remark. A natural question is:

> **Question**: is a re-parametrization of a geodesics still a geodesic?

Suppose γ is a geodesic and $\dot{\gamma} \neq 0$(otherwise γ is constant), and $\tilde{\gamma}(s) = \gamma(t(s))$ is a regular re-parametrization of γ, then

$$\begin{aligned}\nabla_{\dot{\tilde{\gamma}}(s)}\dot{\tilde{\gamma}}(s) &= \nabla_{\dot{\tilde{\gamma}}(s)}(t'(s)\dot{\gamma}(t(s))) \\ &= \dot{\gamma}(t(s)) + (t'(s))^2\nabla_{\dot{\gamma}(t(s))}\dot{\gamma}(t(s)) = t''(s)\dot{\gamma}(t(s)).\end{aligned}$$

So $\tilde{\gamma}$ is also a geodesic if and only if $t''(s) = 0$, i.e. $t(s) = as + b$ for some constants a and b. So the answer to the above question is:

> **Answer**: A re-parametrization of a geodesics is still a geodesic if and only if the re-parametrization is linear.

¶ Basic examples.

Example. Let $M = \mathbb{R}^m$, equipped with standard linear connection ∇ such that $\nabla_X Y = X(Y^j)\partial_j$, or equivalently, ${\Gamma^k}_{ij} = 0$. Let γ be any curve and X be a vector field. Then for X to be parallel along γ, we need $\dot{f}^k(t) = 0$ for all k, i.e. if and only if X^i's are constants on γ [so X is a constant vector field in $\mathbb{R}^m$ along γ in the usual sense].

In particular, the geodesic equations in $\mathbb{R}^m$ above become

$$\ddot{x}^k(t) = 0, \qquad 1 \le k \le m.$$

The solution to the system are linear functions, i.e. $x^k(t) = a_k t + b_k$ for some constants a_k, b_k. As a consequence, γ is a geodesic if and only if it is the straight line in the direction $\vec{a} = \langle a_1, \cdots, a_m\rangle$ that passes the point $(b_1, \cdots, b_m)$.

Example. Consider $M = S^m$ the m-sphere, equipped with the Levi-Civita connection. For any $p \in S^m$, regarded as a unit vector $p = \vec{u} \in \mathbb{R}^{m+1}$, and for any unit tangent vector $\vec{w} \in T_pS^m$, we let

$$\gamma(t) = (\cos t)\ \vec{u} + (\sin t)\ \vec{w}.$$

be the great circle in S^m passing p in the direction of $\vec{w}$. Since the Levi-Civita connection on S^m is given by $\nabla_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle\vec{n}$, where $\overline{\nabla}$ is the Levi-Civita connection for $\mathbb{R}^{m+1}$, i.e. with ${\overline{\Gamma}^k}_{ij} = 0$. So

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \overline{\nabla}_{\dot{\gamma}}\dot{\gamma} + \langle\dot{\gamma}, \dot{\gamma}\rangle\vec{n} = \ddot{\gamma} + \vec{n}.$$

But at the point $\gamma(t)$, one has $\vec{n} = \gamma(t)$, and $\ddot{\gamma}(t) = -\gamma(t)$. So we get

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

In other words, any great circle on S^m is a geodesic. [By uniqueness below, up to linear re-parametrizations they are essentially the only geodesics on S^m]

¶ The existence, uniqueness and smoothness.

To find a geodesic is equivalent to solve the system of second order ODEs (1). By introducing $y^i = \dot{x}^i$, we may convert it to a system of first order ODEs (with more variables and more equations)

$$\begin{cases} \dot{x}^k = y^k, \\ \dot{y}^k = -\Gamma^k{}_{ij} y^i y^j, \end{cases} \qquad 1 \le k \le m.$$

So suppose we want to find a geodesic with $\gamma(t_0) = p = (p^1, \cdots, p^m)$ and $\dot{\gamma}(t_0) = X_p = X^i \partial_i \in T_pM$, then we need to solve the above system with initial condition $x(t_0) = (x^1(t_0), \cdots, x^m(t_0)) = p$, $y(t_0) = (y^1(t_0), \cdots, y^m(t_0)) = X_p$. According to the fundamental theorem for systems of first order ODEs,

- **Existence**: For any $t_0 \in \mathbb{R}$ and any $(p, X_p) \in TM$, there is an open interval $I \ni t_0$ and open set $\mathcal{U} \ni (p, X_p)$ so that for any $(q, X_q) \in \mathcal{U}$, the system has a smooth solution $\gamma_{q,X_q}(t)$ in $t \in I$ with initial condition $x(t_0) = q, y(t_0) = X_q$.
- **Smooth dependence**: The solution above, viewed as a map $\Upsilon(t, q, X_q) = \gamma_{q,X_q}(t)$, is a smooth map from $I \times \mathcal{U}$ to M.
- **Uniqueness**: If (x_1, y_1) is a solution of the system on an interval $I_1 \ni t_0$, (x_2, y_2) is a solution of the system on an interval $I_2 \ni t_0$, both with the initial condition (p, X_p) at t_0, then $(x_1, y_1) = (x_2, y_2)$ on $I_1 \cap I_2$.

As a consequence, we conclude

Theorem 1.2. *For any $p \in M$ and any $X_p \in T_pM$, there exists an $\varepsilon > 0$ and a unique geodesic $\gamma = \gamma_{p,X_p}$ defined for $|t| < \varepsilon$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. Moreover, the map $\gamma(t; p, X_p) = \gamma_{p,X_p}(t)$ depends smoothly on (t, p, X_p).*

Note that by uniqueness, for any $(p, X_p) \in TM$, there is a maximal interval $J_{p,X_p} \subset \mathbb{R}$ on which a geodesic γ with $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$ exists. Note that by the "linear re-parametrization remark" above,

$$J_{p,tX_p} = \frac{1}{t} J_{p,X_p}.$$

If $J_{p,X_p} = \mathbb{R}$ for all $(p, X_p) \in TM$, then we say (M, ∇) is *geodesically complete.*

Remark. The dependence of the maximal interval J on the initial data (p, X_p) is not continuous: for example, one can consider in the punctured plane $\mathbb{R}^2 - \{(0, 0)\}$. Then the geodesic starting at $(-1, 0)$ in the direction $\langle 1, 0\rangle$ has maximal existence interval $(-\infty, 1)$, while the geodesic starting at $(-1, 0)$ in any other direction has maximal existence interval $\mathbb{R}$.

It is not hard to see that if M is compact, then it must be geodesically complete. We will see later that for Riemannian manifolds, (M, g) is geodesically complete if and only if as a metric space, (M, dist) is complete.

2. The exponential map and normal coordinates

¶ The exponential map.

Let M be a smooth manifold endowed with a linear connection ∇. Consider

$$\mathcal{E} = \{(p, X_p) \mid \gamma_{p,X_p}(t) \text{ is defined on an interval containing } [0,1]\}.$$

[So by definition $\mathcal{E} = TM$ if and only if (M, g) is geodesically complete.]

By existence and smoothness above, for any $(p, X_p) \in TM$ there is $\varepsilon_0 > 0$ and an open neighborhood $\mathcal{U}$ of (p, X_p) so that for any $(q, X_q) \in \mathcal{U}$, the maximal existence interval J_{q,X_q} of γ_{q,X_q} contains the interval $(-\varepsilon_0, \varepsilon_0)$. As a result,

$$J_{q,\varepsilon_0 X_q/2} \supset (-2, 2),$$

So $\mathcal{E}$ contains a neighborhood of the zero section M in TM. Note that $\mathcal{E} \cap T_pM$ is always a star-like subset in T_pM for any p.

Definition 2.1. The *exponential map* is defined to be

$$\exp : \mathcal{E} \to M, \quad (p, X_p) \mapsto \exp_p(X_p) := \gamma_{p,X_p}(1).$$

Example. For $(\mathbb{R}^m, g_0)$, we can identify each $T_p\mathbb{R}^m$ with $\mathbb{R}^m$. Then $\exp_p(X_p) = p + X_p$.

Example. For $(S^1, d\theta \otimes d\theta)$, we can identify T_eS^1 with $\mathbb{R}^1$. Then $\exp_e(X_p) = e^{iX_p}$.

Remark. Let $M = G$ be a Lie group, endowed with the Levi-Civita connection of the bi-invariant metric on G, then $\exp_e$ coincides with the exponential map $\exp : \mathfrak{g} \to G$ in Lie theory. In particular, if G is a matrix Lie group, then

$$\exp_e(A) = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots.$$

The smoothness of $\Upsilon(t; p, X_p)$ implies that the exponential map is smooth. In particular, for each $p \in M$, the map

$$\exp_p : T_pM \cap \mathcal{E} \to M$$

is smooth. By definition $\exp_p$ maps $0 \in T_pM$ to $p \in M$. As in Lie theory we also have the following useful lemma:

Lemma 2.2. *For any $p \in M$, if we identify $T_0(T_pM)$ with T_pM, then*

$$(d\exp_p)_0 = \mathrm{Id}|_{T_pM} : T_pM \to T_pM.$$

Proof. for any $X_p \in T_0(T_pM) = T_pM$,

$$(d\exp_p)_0(X_p) = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tX_p) = \left.\frac{d}{dt}\right|_{t=0} \gamma(1; p, tX_p) = \left.\frac{d}{dt}\right|_{t=0} \gamma(t; p, X_p) = X_p.$$

□

So by the inverse function theorem, we immediately get

Corollary 2.3. *For any $p \in M$, there exists a neighborhood V of 0 in T_pM and a neighborhood U of p in M so that $\exp_p : V \to U$ is a diffeomorphism.*

¶ Normal neighborhoods and normal coordinates.

So for any $p \in M$, there exists a neighborhood $U \subset M$ of p and a neighborhood $\widetilde{V} \subset T_pM$ of 0 so that the exponential map $\exp_p : \widetilde{V} \to U$ is a diffeomorphism. By fixing a basis $\{e_i\}$ of T_pM, we may identify $\widetilde{V}$ with an open subset V of $\mathbb{R}^m$, and as a result, the triple $(\exp_p^{-1}, U, V)$ form a local chart of M near p.

Definition 2.4. If $\widetilde{V}$ is star-like, then we call U a *normal neighborhood* of p, call the local chart $(\exp_p^{-1}, U, V)$ a *normal chart* on M, and call the coordinate system $\{U; x^1, \cdots, x^m\}$ a *normal coordinate system* centered at p.

By definition, the normal coordinate system centered at p has the nice characterizing property that any geodesic starting at p is given in such coordinates by

$$\gamma : x(t) = (tv^1, tv^2, \cdots, tv^m),$$

where $(v^1, \cdots, v^m)$ is the direction of the geodesics. Moreover, we have

Lemma 2.5. *Let $\{U; x^1, \cdots, x^m\}$ be a normal coordinate system centered at p. Then for all $\vec{v} \in \mathbb{R}^m$ and all $1 \le k \le m$, ${\Gamma^k}_{ij}(p)v^iv^j = 0$.* [*In particular, if the linear connection ∇ is torsion free, then ${\Gamma^k}_{ij}(p) = 0$ for all i, j, k.*]

Proof. Put the parametric equation $x(t) = (tv^1, tv^2, \cdots, tv^m)$. of a geodesic into the geodesic equation, we get for $1 \le k \le m$,

$$0 = \ddot{x}^k(t) + \Gamma_{ij}^k(\gamma(t))\dot{x}^i(t)\dot{x}^j(t) = \Gamma_{ij}^k(\gamma(t))v^iv^j.$$

Letting $t = 0$, we get $\Gamma_{ij}^k(p)v^iv^j = 0$ for all $\vec{v}$ and for any $1 \le k \le m$. □

¶ Normal convex neighborhoods.

We may go a lot further.

Theorem 2.6 (Whitehead)**.** *For any smooth manifold M with a linear connection, any p has a neighborhood U such that U is a normal neighborhood for any $q \in U$.*

Let's explain the meaning before we prove the theorem. For any $q, q' \in U$, since U is a normal neighborhood of q, there is a vector $X_{q\to q'} \in T_qM$ so that

$$\gamma_{q,q'}(t) := \exp_q(tX_{q\to q'})$$

is a geodesic from $q = \gamma(0)$ to $q' = \gamma(1)$ that lies entirely in U. Such an open set is called a *convex normal neighborhood* of p. So Whitehead theorem claims that any p admits a normal convex neighborhood. As a consequence, we can prove

Corollary 2.7. *Any smooth manifold M admits a good covering.*

Proof. Endow with M a linear connection ∇. Then by Whitehead theorem, each $p \in M$ admits a normal convex neighborhood U_p. Because each normal convex neighborhood is contractible [since it is diffeomorphic to a star-like subset in a vector space], and because arbitrary intersection of normal convex neighborhoods is still a normal convex neighborhood, they form a good covering of M. □

¶ Proof of Whitehead theorem.

Proof. **Step 1:** There is a neighborhood U of p such that for any $q \in U$, there is a normal chart $(\exp_q^{-1}, U_q, V_q)$ with $U_q \supset U$.

Take a neighborhoods U_1 of $p \in M$ and a neighborhood $\widetilde{\mathcal{U}_1}$ of $(p,0) \in TM$ over U_1 [i.e. $\pi(\widetilde{\mathcal{U}_1}) = U_1$, where $\pi : TM \to M$ is the bundle projection] such that for each $q \in U_1$,

- $\widetilde{\mathcal{U}_1}$ is fiberwise star-like, i.e. $V_q = \widetilde{\mathcal{U}_1} \cap T_qM$ is star-like in T_qM,
- the exponential map $\exp_q : V_q \to U_q$ is a diffeomorphism.

Consider the map

$$\Psi : \widetilde{\mathcal{U}_1} \to M \times M, \quad (q, X_q) \mapsto (q, \exp_q(X_q)).$$

The Jacobian of Ψ at $(p,0)$ is $\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$. So Ψ is a local diffeomorphism, i.e. it maps a smaller neighborhood $\mathcal{U}_1 \subset \widetilde{\mathcal{U}_1}$ diffeomorphically onto a neighborhood of (p,p) in $M \times M$. In particular, one may find a neighborhood U of p in M so that $U \times U \subset \Psi(\mathcal{U}_1)$. By construction, $\Psi^{-1}(U \times U) \cap T_qM \subset \mathcal{U}_1 \cap T_qM \subset V_q$ and thus

$$U \subset \exp_q(\Psi^{-1}(U \times U) \cap T_qM) \subset U_q.$$

Step 2: U can be chosen to be normal with respect to any $q \in U$.

We fix a normal normal chart (φ, U_0, V_0) centered at p, with normal coordinates $x^1, \cdots, x^m$, where for simplicity we denote $\varphi = \exp_p^{-1}$. Apply Lemma 2.5 and shrink U_0 if necessary, we may assume that the matrix $\left(\delta_{ij} - \sum_k \Gamma^k{}_{ij} x^k\right)$ is "positive" at each point in U_0, i.e. such that $\left(\delta_{ij} - \sum_k \Gamma^k{}_{ij} x^k\right) v^i v^j \geq 0$ for all $\vec{v} \in \mathbb{R}^m$ and all $q \in U_0$. We may assume U_q we get in Step 1 are all inside U_0.

Now we endow T_pM with any inner product, and shrink U we get in Step 1 so that $\varphi(U)$ is a ball of radius δ. By Step 1, for any $q, q' \in U$, there is a vector $X_{q \to q'} \in T_qM$ with $\exp_q(X_q^{q'}) = q'$. Since V_q is star-like, the curve $\gamma_{q,q'}(t) := \exp_q(tX_q^{q'})$ is a geodesic from $q = \gamma(0)$ to $q' = \gamma(1)$ that lies in U_q. It remains to prove that $\gamma_{q,q'}(t)(0 \leq t \leq 1)$ lies in U.

Since the geodesic $\gamma_{q,q'}$ lies in U_0, we work on its parametric equations $x^i = x^i(t)$. Consider the function $f(t) = \sum_i (x^i(t))^2$. Then

$$\begin{aligned} \ddot{f}(t) &= 2 \sum_i \left[(\dot{x}^k(t))^2 + \ddot{x}^k(t) x^k(t) \right] \\ &= 2 \sum_k \left[(\dot{x}^k(t))^2 - \Gamma^k{}_{ij} \dot{x}^i(t) \dot{x}^j(t) x^k(t) \right] \\ &= 2 \big[\delta_{ij} - \sum_k \Gamma^k{}_{ij} x^k(t) \big]_{\gamma(t)} \dot{x}^i(t) \dot{x}^j(t) \geq 0. \end{aligned}$$

As a consequence, f is convex and thus $f(t) \leq \max\{f(0), f(1)\}$ for $0 \leq t \leq 1$. Since $q, q' \in U$, we have $f(0), f(1) \leq \delta^2$. So the geodesic $\gamma_{q,q'}$ is inside U. □