LECTURE 13: GEODESICS ON RIEMANNIAN MANIFOLDS

After defining geodesics as "self-parallel curves" on any smooth manifold with linear connection, today we will put the Riemannian metric structure into this picture and study what do we gain with this new structure (for the geodesics as self-parallel curves and as integral curves, for the exponential map, and for the normal coordinates etc).

1. Geodesics as integral curves

¶ "Speed" of a geodesics.

Let (M, g) be a Riemannian manifold, and $\gamma : [a, b] \to M$ a smooth curve in M. Recall that γ is a geodesics if and only if it is self-parallel, i.e. $\nabla_{\dot\gamma}\dot\gamma = 0$. By metric compatibility,

$$\frac{d}{dt}\langle \dot\gamma, \dot\gamma\rangle = \nabla_{\dot\gamma}\langle \dot\gamma, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle + \langle \dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = 0.$$

As a result, we get

Proposition 1.1. *If γ is a geodesic on a Riemannian manifold, then $|\dot\gamma|$ must be a constant for all t.*

Note that this also implies that a re-parametrization of a geodesic is again a geodesic if and only if the re-parametrization is a linear re-parametrization.

In particular, on a Riemannian manifold one can always re-parameterize a geodesic so that its "speed" is 1:

Definition 1.2. We will call a geodesics γ on a Riemannian manifold satisfying $|\dot\gamma(t)| = 1$ a *normal geodesics.*

Of course given any geodesic, the corresponding normal geodesic is nothing else but the arc-length re-parametrization of the given geodesic.

¶ Geodesics as integral curves at the presence of metric.

Last time by introducing $y^i = \dot x^i$ we converted the system of second order ODEs for a geodesic to a system of first order ODEs

$$\begin{cases} \dot x^k = y^k, \\ \dot y^k = -\Gamma^k{}_{ij} y^i y^j, \end{cases} \qquad 1 \le k \le m$$

using which we get the existence, smooth dependence and uniqueness of geodesics. In other words, the problem of finding a local geodesic is equivalent to finding the integral curve of the vector field

$$\widetilde{X} = y^k \frac{\partial}{\partial x^k} - \Gamma^k{}_{ij} y^i y^j \frac{\partial}{\partial y^k}.$$

Although one can show that the vector field $\widetilde{X}$ defined above is really globally defined (i.e. independent of the choice of coordinates), its geometric meaning is not that obvious.

It turns out that if one transfer from the tangent bundle to the cotangent bundle, then there is a geometrically important vector field whose integral curves give geodesics on M. Recall that given any coordinate chart $(U, x^1, \cdots, x^m)$ on M, any 1-form ω can be expressed locally on U as $\omega = \xi_i dx^i$ and as a result, one gets a coordinate chart $(T^*U, x^1, \cdots, x^m, \xi_1, \cdots, \xi_m)$ for the cotangent bundle T^*M.

Now given a Riemannian metric g on M, i.e. an inner product on each tangent space, one gets a dual inner product on each cotangent space. Consider the smooth function defined on $T^*M \setminus \{0\}$ by

$$f(x, \xi) = \frac{1}{2}|\xi|_x^2 = \frac{1}{2} g^{ij}(x)\xi_i \xi_j.$$

Definition 1.3. The *Hamiltonian vector field* of f is

$$H_f = \sum \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial \xi_i}.$$

It is a vector field on $T^*M \setminus \{0\}$ which preserves f (and thus preserves $|\xi|_x$),

$$H_f(f) = 0.$$

As a consequence, it defines a vector field on each level set of f, and in particular on the cosphere bundle

$$S^*M = \{(x, \xi) \mid \|\xi\|_x = 1\}.$$

By definition the integral curves of H_f are the curves $\Gamma = \Gamma(t)$ such that

$$\dot{\Gamma}(t) = H_f(\Gamma(t)).$$

More precisely, if we denote

$$\Gamma(t) = (x^1(t), \cdots, x^m(t), \xi_1(t), \cdots, \xi_m(t)),$$

then any integral curve of H_f satisfies the following *Hamilton equations*

$$\begin{cases} \dot{x}^k &= \frac{\partial f}{\partial \xi_k}, \\ \dot{\xi}_k &= -\frac{\partial f}{\partial x^k}. \end{cases}$$

The flow generated by H_f on S^*M is called the *geodesic flow* of (M, g), which is very important in studying Riemannian manifolds. Now we prove

Theorem 1.4. *Any integral curve of H_f on S^*M, when projected onto M, is a normal geodesic in M. Conversely, any normal geodesic in M arises in this way.*

Proof. Let $\Gamma(t) = (x^1(t), \cdots, x^m(t), \xi_1(t), \cdots, \xi_m(t))$ be an integral curve of H_f, then the Hamilton equations become

$$\dot{x}^k = \frac{\partial f}{\partial \xi_k} = \frac{1}{2} g^{ij}\delta_{ik}\xi_j + \frac{1}{2} g^{ij}\xi_i\delta_{jk} = g^{kj}\xi_j$$
$$\dot{\xi}_k = -\frac{\partial f}{\partial x^k} = -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}\xi_i\xi_j$$

From the first equation we get $\xi_k = g_{lk}\dot{x}^l$. Put this into the second equation, we have

$$\frac{\partial g_{lk}}{\partial x^i}\dot{x}^i\dot{x}^l + g_{lk}\ddot{x}^l = -\frac{1}{2}\frac{\partial g^{ij}}{\partial x^k} g_{li}\dot{x}^l g_{nj}\dot{x}^n.$$

Note that

$$-\frac{\partial g^{ij}}{\partial x_k} g_{li} g_{nj} = g^{ij}\frac{\partial g_{li}}{\partial x^k} g_{nj} = \frac{\partial g_{nl}}{\partial x^k},$$

the equation becomes

$$g_{lk}\ddot{x}^l = -\frac{\partial g_{lk}}{\partial x^i}\dot{x}^i\dot{x}^l + \frac{1}{2}\frac{\partial g_{nl}}{\partial x^k}\dot{x}^l\dot{x}^n = -\frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\dot{x}^j + \frac{1}{2}\frac{\partial g_{ji}}{\partial x^k}\dot{x}^i\dot{x}^j.$$

In other words,

$$\ddot{x}^l = g^{kl}\left(-\frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\dot{x}^j + \frac{1}{2}\frac{\partial g_{ji}}{\partial x^k}\dot{x}^i\dot{x}^j\right) = -\frac{1}{2} g^{kl}\left(\frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\dot{x}^j + \frac{\partial g_{ik}}{\partial x^j}\dot{x}^j\dot{x}^i - \frac{\partial g_{ji}}{\partial x^k}\dot{x}^i\dot{x}^j\right),$$

which is exactly the geodesic equation since

$$\Gamma^l_{ij} = \frac{1}{2} g^{kl}(\partial_j g_{ki} + \partial_i g_{jk} - \partial_k g_{ij}).$$

So the projected curve $\gamma(t) = (x^1(t), \cdots, x^m(t))$ is a geodesic on M. It is normal since

$$g_{kl}\dot{x}^k\dot{x}^l = g_{kl} g^{kj} g^{li}\xi_j\xi_i = g^{ij}\xi_j\xi_i = 1.$$

Conversely, for any geodesic $\gamma(t) = (x^1(t), \cdots, x^m(t))$, we let $\xi_k = g_{lk}\dot{x}^l$. Then the above computations shows that $\Gamma(t) = (x^1(t), \cdots, x^m(t), \xi_1(t), \cdots, \xi_m(t))$ is an integral curve of H_f in S^*M. □

Remark. The function $|\xi|^2$ is the *symbol* of the Laplace-Beltrami operator Δ_g. So the geodesic flow is also closely related to spectral geometry.

Remark. As a consequence, (M, g) is geodesically complete if and only if the vector field H_f on S^*M is complete. Note that if M is compact, then S^*M is compact, and thus any smooth vector field on S^*M is complete. As a result, any compact Riemannian manifold is geodesically complete.

2. The exponential map at the presence of metric

¶ The injectivity radius.

Now let's turn to the exponential map and figure out what do we gain with g. For a Riemannian manifold, by definition the point $\exp_p(X_p)$ is the end point of the geodesic segment that starts at p in the direction of X_p whose length equals $|X_p|$.

In general the map $\exp_p : \mathcal{E}_p \cap T_pM \to M$ is not a global diffeomorphism, even if it may be defined everywhere in T_pM. For example, on the round sphere S^m, $\exp_p$ is a diffeomorphism from any ball $B_r(0) \subset T_pM$ of radius $r < \pi$ to an open region in S^m, but it fails to be injective on the ball $B_r(0)$ with $r > \pi$.

Definition 2.1. The *injectivity radius* of Riemannian manifold (M, g) at $p \in M$ is

$$\mathrm{inj}_p(M, g) := \sup\{r \mid \exp_p \text{ is a diffeomorphism on } B_r(0) \subset T_pM\},$$

and the *injectivity radius* of (M, g) is

$$\mathrm{inj}(M, g) := \inf\{\mathrm{inj}_p(M, g) \mid p \in M\}.$$

Example. $\mathrm{inj}(S^m, g_{S^m}) = \pi$.

Remark. If M is compact, then of course

$$0 < \mathrm{inj}(M, g) \le \mathrm{diam}(M, g),$$

where $\mathrm{diam}(M, g) = \sup_{p,q\in M} d(p, q)$ is the diameter of (M, g). But for noncompact manifolds M, we may have $\mathrm{inj}(M, g) = 0$ or $+\infty$. [But for any p, we always have $\mathrm{inj}_p(M, g) > 0$.]

For any $\rho < \mathrm{inj}_p(M, g)$, we have $B_\rho(0) \subset T_pM \cap \mathcal{E}$, where $B_\rho(0)$ is the ball of radius ρ in (T_pM, g_p) centered at 0.

Definition 2.2. We will call $B(p, \rho) = \exp_p(B_\rho(0))$ the *geodesic ball* of radius ρ centered at p in M, and its boundary $S(p, \rho) = \partial B(p, \rho)$ the *geodesic sphere* of radius ρ centered at p in M.

Now let γ be any normal geodesic starting at p. Then for $\rho < \mathrm{inj}_p(M, g)$, we have $\gamma((0, \rho)) \subset B(p, \rho)$ and $\exp_p^{-1}(\gamma((0, \rho)))$ is the line segment in $B_\rho(0) \subset T_pM$ starting at 0 in the direction $\dot\gamma$ whose length is ρ. As a consequence, the geodesics starting at p of lengths less than $\mathrm{inj}_p(M, g)$ are exactly the images under $\exp_p$ of line segments starting at 0 of lengths no more than $\mathrm{inj}_p(M, g)$. In particular,

Corollary 2.3. *Suppose $p \in M$ and $\rho < \mathrm{inj}_p(M, g)$. Then for any $q = \exp_p(X_p) \in B(p, \rho)$, the curve $\gamma(t) = \exp_p(tX_p)$ is the unique normal geodesic connecting p to q whose length is less than ρ.*

Remark. No matter how close p and q are to each other, one might be able to find other geodesics connecting p to q whose length is longer. To see this, one can look at cylinders or torus, in which case one can always find infinitely many geodesics connecting two arbitrary given points p and q.

¶ Gauss Lemma.

Last time we showed that the exponential $(d\exp_p)_0 = \mathrm{Id}$. Now let $(p, X_p) \in \mathcal{E}$. By definition, $\exp_p$ maps the point $X_p \in T_pM$ to the point $\exp_p(X_p) \in M$. In general, the differential $d\exp_p$ at X_p is no longer the identity map Id [In fact, if $(d\exp_p)_{X_p} = \mathrm{Id}$ for all p and X_p, then $\exp_p$ is an isometry from (T_pM, g_p) to (M, g) and thus (M, g) is flat.]. However, we can prove that $\exp_p$ is always a "radial isometry":

Lemma 2.4 (Gauss lemma). *Let (M, g) be a Riemannian manifold and $(p, X_p) \in \mathcal{E}$. Then for any $Y_p \in T_pM = T_{X_p}(T_pM)$, we have*

$$\langle (d\exp_p)_{X_p} X_p, (d\exp_p)_{X_p} Y_p \rangle_{\exp_p(X_p)} = \langle X_p, Y_p \rangle_p.$$

Proof. Without loss of generality, we may assume $X_p, Y_p \neq 0$. By linearity, it's enough to check the lemma for $Y_p = X_p$ and $Y_p \perp X_p$.

Case 1: $Y_p = X_p$. If we denote $\gamma(t) = \exp(tX_p)$, then $X_p = \dot\gamma(0)$ and

$$(d\exp_p)_{X_p} X_p = \frac{d}{dt}\Big|_{t=1} \exp_p(tX_p) = \dot\gamma(1).$$

Since geodesics are always of constant speed, we conclude

$$\langle (d\exp_p)_{X_p} X_p, (d\exp_p)_{X_p} X_p \rangle = \langle \dot\gamma(1), \dot\gamma(1) \rangle = \langle \dot\gamma(0), \dot\gamma(0) \rangle = \langle X_p, X_p \rangle.$$

Case 2: $Y_p \perp X_p$. Under this condition one can find a curve $\gamma_1(s)$ in the sphere of radius $|X_p|$ in T_pM with $\gamma_1(0) = X_p$ and $\dot\gamma_1(0) = Y_p$. Since $(p, X_p) \in \mathcal{E}$, we see that there exists $\varepsilon > 0$ so that for all $0 < t < 1$ and $-\varepsilon < s < \varepsilon$,

$$(p, t\gamma_1(s)) \in \mathcal{E}.$$

Let $A = \{(t, s) \mid 0 < t < 1, -\varepsilon < s < \varepsilon\}$ and consider the smooth map

$$f : A \to M, \quad (t, s) \mapsto f(t, s) := \exp_p(t\gamma_1(s)).$$

As usual we denote $f_t = df(\frac{d}{dt})$ and $f_s = df(\frac{d}{ds})$. The by definition

$$f_t(1, 0) = \frac{d}{dt}\Big|_{t=1} \exp_p(tX_p) = (d\exp_p)_{X_p} X_p,$$

$$f_s(1, 0) = \frac{d}{ds}\Big|_{s=0} \exp_p(\gamma_1(s)) = (d\exp_p)_{X_p} Y_p$$

and thus

$$\langle (d\exp_p)_{X_p} X_p, (d\exp_p)_{X_p} Y_p \rangle = \langle f_t(1, 0), f_s(1, 0) \rangle.$$

On the other hand, we have

- for each fixed s_0, $f(t, s_0)$ is a geodesic with tangent vector field f_t. So
$$\nabla_{f_t} f_t = 0.$$
- since ∇ is torsion free, $\nabla_{f_s} f_t - \nabla_{f_t} f_s = [f_s, f_t] = df([\partial_s, \partial_t]) = 0$ and thus
$$\nabla_{f_s} f_t = \nabla_{f_t} f_s.$$

- Since γ_1 lies in the sphere of radius $|X_p|$, the length
$$|f_t| = |\gamma_1(s)| = |X_p|$$
is a constant.

As a consequence of these three facts,
$$\frac{\partial}{\partial t}\langle f_s, f_t\rangle = \langle \nabla_{f_t} f_s, f_t\rangle + \langle f_s, \nabla_{f_t} f_t\rangle = \langle \nabla_{f_s} f_t, f_t\rangle = \frac{1}{2}\nabla_{f_s}\langle f_t, f_t\rangle = 0,$$
i.e. $\langle f_t, f_s\rangle$ is independent of t. Since
$$\lim_{h\to 0} f_s(h,0) = \lim_{h\to 0} \frac{d}{ds}\Big|_{s=0} \exp_p(h\gamma_1(s)) = \lim_{t\to 0} d(\exp_p)_{hX_p}(hY_p) = 0,$$
we conclude $\langle f_t(1,0), f_s(1,0)\rangle = 0$, which proves the lemma. $\square$

Geometrically, Gauss lemma implies

Corollary 2.5 (The Geometric Gauss Lemma)**.** *For any $\rho < \mathrm{inj}_p(M,g)$ and any $q \in S(p,\rho)$, the shortest geodesic connecting p to q is orthogonal to $S(p,\rho)$.*

¶ Local shortest curves are geodesics.

As a consequence of Gauss lemma, we may strengthen Corollary 2.3 to

Theorem 2.6. *Suppose $p \in M$ and $\delta < \mathrm{inj}_p(M,g)$. Then for any $q = \exp_p(X_p) \in B(p,\delta)$, the geodesic $\gamma(t) = \exp_p(tX_p)(0 \le t \le 1)$ is the only piecewise smooth curve connecting p and q with length $d(p,q)$.*

Proof. Let $\sigma : [0,1] \to M$ be any piecewise smooth curve with $\sigma(0) = p, \sigma(1) = q$, and parameterized with constant speed. We want to show $L(\sigma) \ge d(p,q)$, with equality holds if and only if $\sigma = \gamma$.

Without loss of generality, we may assume $p \notin \sigma((0,1])$ [otherwise we may take $t_0 = \sup\{t|\sigma(t) = p\}$ and consider the curve $\sigma|_{[t_0,1]}$ instead] and assume $\sigma((0,1)) \subset B(p,\delta)$ [otherwise we may take $t_1 = \inf\{t|\sigma(t) \in S(p,\delta)\}$ and consider the curve $\sigma|_{[0,t_1]}$ instead]. As a result, there exits unit vectors $w(t) \in S_pM$ and real numbers $r(t) \in (0,\delta]$ such that
$$\sigma(t) = \exp_p(r(t)w(t)).$$
It follows
$$\dot\sigma(t) = (d\exp_p)_{r(t)w(t)}(r'(t)w(t) + r(t)\dot w(t)).$$
Note that $w(t) \in S_pM$ for all t implies $w(t) \perp \dot w(t)$. So by Gauss lemma,
$$(d\exp_p)_{r(t)w(t)}(r'(t)w(t)) \perp (d\exp_p)_{r(t)w(t)}(r(t)\dot w(t))$$
and thus
$$|\dot\sigma(t)|^2 \ge \langle (d\exp_p)_{r(t)w(t)}(r'(t)w(t), (d\exp_p)_{r(t)w(t)}(r'(t)w(t)\rangle = |r'(t)|^2$$

So if we denote $b = \text{Length}(\sigma)$, then $b = |\dot\sigma(t)|$ at all smooth points t of σ and thus

$$\begin{aligned} b = \text{Length}(\sigma) = \int_0^1 |\dot\sigma(t)| dt &= \frac{1}{b}\int_0^1 |\dot\sigma(t)|^2 dt \\ &\ge \frac{1}{b}\int_0^1 |r'(t)|^2 dt \\ &\ge \frac{1}{b}\left(\int_0^1 |r'(t)| dt\right)^2 \ge \frac{1}{b}\left(\int_0^1 r'(t) dt\right)^2 \ge \frac{\delta^2}{b}, \end{aligned}$$

where we used Cauchy-Schwartz inequality and the fact $r(1) \le \delta$. It follows that $b \ge \delta$ as desired. Moreover, if the equality holds, then $\dot w = 0$ and $|r'(t)|$ is constant, which implies that σ is precisely the geodesic $\gamma(t) = \exp_p(tX_p)$. $\square$

¶ Riemannian metric tensor in Riemannian normal coordinate system.

Now we turn to normal coordinate systems for Riemannian manifolds. Since the Levi-Civita connection is torsion-free, we have seen that with respect to any normal coordinate system centered at p,

$$\Gamma^k{}_{ij}(p) = 0, \qquad 1 \le i, j, k \le m.$$

So what do we gain from the metric? Recall that behind a normal coordinate system $(\exp_p^{-1}, U, V)$ there hides an identification between $\widetilde V = \exp_p^{-1}(U) \subset T_pM$ and $V \subset \mathbb{R}^m$, which is realized after a choice of a basis e_i of T_pM. For a Riemannian manifold (M, g), we will always identify $\widetilde V = \exp_p^{-1}(U) \subset T_pM$ and an open subset $V \subset \mathbb{R}^m$ by choosing an orthonormal basis $\{e_1, \cdots, e_m\}$ of T_pM, and call the resulting normal coordinate system a *Riemannian normal coordinate system* at p.

With a Riemannian normal coordinate system at hand, we can prove the following stronger result[c.f. formula (10) in Lecture 6]:

Lemma 2.7. *Let (M, g) be a Riemannian manifold, and $\{U; x^1, \cdots, x^m\}$ be a Riemannian normal coordinate system centered at p. Then*

(1) For all $1 \le i, j \le m$, $g_{ij}(p) = \delta_{ij}$.
(2) For all $1 \le i, j, k \le m$, $\partial_k g_{ij}(p) = 0$.
(3) $G(p) = 1$ and $\partial_i G(p) = 0$ for all $1 \le i \le m$, where $G = \det(g_{ij})$.

Proof. (1) By definition of Riemannian normal coordinate system we have $\partial_i|_p = d(\exp_p)_0 e_i = e_i$, which implies $g_{ij}(p) = \delta_{ij}$ since $\{e_i\}$ is chosen to be orthonormal.

(2) By metric compatibility we have

$$\partial_k g_{ij}(p) = \langle \nabla_{\partial_k}\partial_i, \partial_j\rangle(p) + \langle \partial_i, \nabla_{\partial_k}\partial_j\rangle(p) = \Gamma^l{}_{ki}(p) g_{lj}(p) + \Gamma^l{}_{kj}(p) g_{li}(p)$$

and thus the conclusion follows from the fact $\Gamma^k{}_{ij}(p) = 0$.

(3) This is a direct consequence of (1), (2) and the definition of determinant. $\square$

Remark. As a result, in a Riemannian normal coordinate centered at p, we have

$$g_{ij} = \delta_{ij} + O(|x|^2) \quad \text{and} \quad \det(g_{ij}) = 1 + O(|x|^2)$$

near p. In fact, as we will see later, what hides in $O(x^2)$ are the curvature information of (M, g) at p: the Riemannian curvature for g_{ij}, and the Ricci curvature for $\det(g_{ij})$.

In Riemannian normal coordinate system centered at p, many differential operators have very simple expressions at p. As a result, it can simplify computations a lot. For example, given any smooth vector field $X = X^i\partial_i$, we have defined its divergence to be $divX = \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}X^i)$. By Lemma 2.7 (3) we have $\partial_i(\sqrt{G})(p) = 0$. So it follows that in a given Riemannian normal coordinate system centered at p,

$$divX(p) = \sum_i \partial_i X^i(p).$$

As a result, the Laplacian Δf at p also has a very simple expression,

$$\Delta f(p) = -div\nabla f(p) = -\partial_i^2 f(p).$$

Similarly the Hessian $\nabla^2 f$ of f, in the Riemannian normal coordinates, becomes

$$(\nabla^2 f)(\partial_i, \partial_j)(p) = \partial_j\partial_i f(p) - (\nabla_{\partial_j}\partial_i)f(p) = \partial_j\partial_i f(p).$$

In particular, we see that at each p,

$$\operatorname{tr}(\nabla^2 f)(p) = g^{ij}(p)(\nabla^2 f)(\partial_i, \partial_j)(p) = g^{ij}(p)\partial_i\partial_j f(p) = \partial_i^2 f(p).$$

So we proved

Proposition 2.8. *For any $f \in C^\infty(M)$, $\Delta f = -\operatorname{tr}(\nabla^2 f)$.*

This formula can be viewed as a second definition of the Laplace operator Δ.

¶ Strongly convex neighborhood.

Finally we take a look at Whitehead's theorem for Riemannian manifolds. We may carefully check the proof of Whitehead's theorem last time: in step 2 we choose the convex normal neighborhood U carefully so that in the normal coordinate system, $\exp_p(U)$ is a ball in $\mathbb{R}^m$. In current setting if we use Riemannian normal coordinate system, then that means U is a small geodesic ball centered at p. Also in step 1 we may choose $\widetilde{U}_1$ carefully so that each V_q is a ball in (T_qM, g_q) instead of only a star-like subset in T_qM, which means each U_q is a geodesic ball in the construction. In view of Theorem 2.6, we conclude that for such a geodesic ball U,

> any two points $q_1, q_2 \in U$ can be connected by a unique geodesic γ of length $d(q_1, q_2)$, and this minimizing geodesic γ lies in U

Such a neighborhood is called *strongly convex* or *geodesically convex*. So we get

Theorem 2.9 (Whitehead)**.** *Let (M, g) be a Riemannian manifold, then for any $p \in M$ there exists $\rho > 0$ so that the geodesic ball $B(p, \rho)$ is strongly convex.*