LECTURE 17: JACOBI FIELDS

As we have seen, in the second variational formula the curvature term appears. As a result, the formula will play a crucial role in studying the relation between curvature and topology of Riemannian manifolds. Usually the first step will be: start with a geodesic, and take a special variation (e.g. a geodesic variation, sometimes with one endpoint fixed). Thus the variation field of a geodesic variation will be very important for the remaining of this course.

1. Definition of the Jacobi field

¶ The Jacobi field.

Let γ be a geodesic in (M, g). Suppose $f : [a, b] \times (-\varepsilon, \varepsilon) \to M$ is a geodesic variation of γ, i.e. each curve

$$\gamma_s = f(\cdot, s)$$

is a geodesic in M. Then for any s,

$$\widetilde{\nabla}_{\partial/\partial t} f_t = \widetilde{\nabla}_{\partial/\partial t} \dot{\gamma}_s = 0.$$

As a consequence,

$$\widetilde{\nabla}_{\partial/\partial t}\widetilde{\nabla}_{\partial/\partial t} f_s = \widetilde{\nabla}_{\partial/\partial t}\widetilde{\nabla}_{\partial/\partial s} f_t = \widetilde{\nabla}_{\partial/\partial t}\widetilde{\nabla}_{\partial/\partial s} f_t - \widetilde{\nabla}_{\partial/\partial s}\widetilde{\nabla}_{\partial/\partial t} f_t = \widetilde{R}(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}) f_t.$$

Taking $s = 0$, we see that the variation field V of any geodesic variation satisfies

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} V = R(\dot{\gamma}, V)\dot{\gamma}. \tag{1}$$

Definition 1.1. Let X be a smooth vector field X along a geodesic γ. We call X a *Jacobi field* along γ if the equation (1) holds.

Remark. Let γ be a geodesic. There are two trivial Jacobi fields along γ:

- Obviously $X = \dot{\gamma}$ is a Jacobi field. It is the variation field of $f(t, s) = \gamma(t+s)$.
- $X = t\dot{\gamma}$ is a Jacobi field since

  $$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}(t\dot{\gamma}) = \nabla_{\dot{\gamma}}(\dot{\gamma} + t\nabla_{\dot{\gamma}}\dot{\gamma}) = 0$$

  and $R(\dot{\gamma}, t\dot{\gamma})\dot{\gamma} = 0$. It is the variation field of $f(t, s) = \gamma(t + st)$.
- But $X = t^2\dot{\gamma}$ is **NOT** a Jacobi field since

  $$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}(t^2\dot{\gamma}) = \nabla_{\dot{\gamma}}(2t\dot{\gamma}) = 2\dot{\gamma} \neq 0.$$

It is not amazing that $t^2\dot{\gamma}$ is no longer a Jacobi field along γ:

Lemma 1.2. *Let X be a Jacobi field along γ, then $f(t) = \langle X, \dot{\gamma}\rangle$ is a linear function.*

Proof. According to the Jacobi field equation,

$$f''(t) = \frac{d^2}{dt^2}\langle X, \dot\gamma\rangle = \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}X, \dot\gamma\rangle = \langle R(\dot\gamma, X)\dot\gamma, \dot\gamma\rangle = 0.$$

It follows that $\langle X, \dot\gamma\rangle$ is a linear function along γ. □

¶ Existence and uniqueness of Jacobi field.

So the variation field of any geodesic variation is a Jacobi fields. As a result, the second variation formula for a geodesic variation is very simple. We will show that conversely, any Jacobi field on γ can be realized as the variation field of some geodesic variation of γ. Before we prove it, we need some basic properties of Jacobi fields.

Let's take a closer look of the equation for Jacobi fields. Since it is a differential equation, it is enough to study the equation in a coordinate chart. Although one may work on a general frame, to simply the computation one may use a special frame that are parallel along γ [so that the covariant derivatives of the frame are as simple as possible]. So we start with an orthonormal basis $\{e_1, \cdots, e_m\}$ of T_pM, with $e_1 = \dot\gamma(a)$, where $p = \gamma(a)$. Let

$$e_i(t) := \text{ the parallel transport of } e_i \text{ along } \gamma, \quad 1 \le i \le m.$$

According to Proposition 2.1 in Lecture 6,

$$\langle e_i(t), e_j(t)\rangle_{\gamma(t)} = \langle e_i, e_j\rangle_{\gamma(a)} = \delta_{ij}.$$

In other words, we get an orthonormal frame $\{e_1(t), \cdots, e_m(t)\}$ along γ with $e_1(t) = \dot\gamma(t)$, and this frame is parallel along γ, i.e.

$$\nabla_{\dot\gamma(t)}e_k(t) = 0, \quad 1 \le k \le m.$$

Let X be a Jacobi field along γ, then with respect to this orthonormal frame we can write $X = X^i(t)e_i(t)$, and we get

$$\nabla_{\dot\gamma}X = \dot X^i(t)e_i(t) \qquad \text{and} \qquad \nabla_{\dot\gamma}\nabla_{\dot\gamma}X = \ddot X^i(t)e_i(t).$$

It follows that the Jacobi field equation becomes

$$\ddot X^i(t)e_i(t) - X^i(t){R_{1i1}}^j e_j(t) = 0.$$

So we arrived at a system of second order homogeneous ODEs,

$$\ddot X^i(t) - X^j(t){R_{1j1}}^i = 0, \quad 1 \le i \le m,$$

Using the basic theory for second order homogeneous ODEs, we get

Theorem 1.3. *Let $\gamma : [a, b] \to M$ be a geodesic, then for any $X_{\gamma(a)}, Y_{\gamma(a)} \in T_{\gamma(a)}M$, there exists a unique Jacobi field X along γ so that*

$$X(a) = X_{\gamma(a)} \qquad \text{and} \qquad \nabla_{\dot\gamma(a)}X = Y_{\gamma(a)}.$$

Moreover, the set of Jacobi fields along γ is a linear space of dimension $2m$ (which is canonically isomorphic to $T_{\gamma(a)}M \oplus T_{\gamma(a)}M$).

The following consequence is fundamental:

Corollary 1.4. *If $X(t)$ is a Jacobi field along γ, and X is not identically zero, then the zeroes of X are discrete.*

Proof. If X has a sequence of zeroes that converges to t_0, then $X^1(t) = \cdots = x^m(t) = 0$ for a sequence of points t_k converging to $\gamma(t_0)$. It follows that $X^i(t_0) = 0$ and $\dot{X}^i(t_0) = 0$ for all i, i.e. $X(t_0) = 0, \nabla_{\dot\gamma(t_0)}X = 0$. By uniqueness, $X \equiv 0$. □

¶ Jacobi fields as variational fields of geodesic variation.

Now we prove that each Jacobi field X along a geodesic γ can be realized as the variation field of a geodesic variation of γ(So the space of all the Jacobi fields along γ describes all possible ways that γ can vary in "the space of all geodesics" infinitesimally):

Theorem 1.5. *A vector field X along a geodesic γ is a Jacobi field if and only if X is the variation field of some geodesic variation of γ.*

Proof. We have seen that the variation field of any geodesic variation of γ is a Jacobi field. Now we suppose X is a Jacobi field along γ and construct the desired geodesic variation. For simplicity we parameterize γ as $\gamma : [0, 1] \to M$, so $\gamma(t) = \exp_{\gamma(0)}(t\dot\gamma(0))$ is defined for $0 \le t \le 1$. It follows that for any (p, Y_p) in a small neighborhood of $(\gamma(0), \dot\gamma(0))$, the exponential map $\exp_p(tY_p)$ is defined for $0 \le t \le 1$.

Let $\xi : (-\varepsilon, \varepsilon) \to M$ be the geodesic with initial conditions

$$\xi(0) = \gamma(0), \quad \dot\xi(0) = X_{\gamma(0)}.$$

Let $T(s), W(s)$ be parallel vector fields along ξ with

$$T(0) = \dot\gamma(0) \quad \text{and} \quad W(0) = \nabla_{\dot\gamma(0)}X.$$

Define $f : [0, 1] \times (-\varepsilon, \varepsilon) \to M$ by

$$f(t, s) = \exp_{\xi(s)}(t(T(s) + sW(s))).$$

As we mentioned above, for ε small enough, f is well-defined. Moreover, $f(t, 0) = \gamma(t)$, so f is a geodesic variation of γ. Let V be the variation field of f. Since both V and X are Jacobi fields along γ, to show $V = X$, it is enough to show

$$V(0) = X_{\gamma(0)} \quad \text{and} \quad \nabla_{\dot\gamma(0)}V = \nabla_{\dot\gamma(0)}X.$$

The first one follows from

$$V(0) = f_s(0, 0) = \left.\frac{d}{ds}\right|_{s=0} f(0, s) = \left.\frac{d}{ds}\right|_{s=0} \xi(s) = X_{\gamma(0)}.$$

For the second one, we start with the fact $\widetilde\nabla_{\partial/\partial t}f_s = \widetilde\nabla_{\partial/\partial s}f_t$. Evaluate the left hand side at $(0, 0)$ we get

$$\left(\widetilde\nabla_{\partial/\partial t}f_s\right)_{0,0} = \left.\left(\widetilde\nabla_{\partial/\partial t}f_s(t, 0)\right)\right|_{t=0} = \nabla_{\dot\gamma(0)}V,$$

and evaluate the right hand side at $(0,0)$ and use the fact

$$f_t(0,s) = (d\exp_{\xi(s)})_0 \left.\frac{d}{dt}\right|_{t=0} (t(T(s)+sW(s))) = T(s)+sW(s)$$

we get

$$\left(\widetilde{\nabla}_{\partial/\partial s} f_t\right)_{0,0} = \left.\left(\widetilde{\nabla}_{\partial/\partial s} f_t(0,s)\right)\right|_{s=0} = \nabla_{X_{\gamma(0)}}(T(s)+sW(s)) = W(0) = \nabla_{\dot\gamma(0)}X.$$

So we get $\nabla_{\dot\gamma(0)}V = \nabla_{\dot\gamma(0)}X$ and thus completes the proof. □

Note that given any Jacobi field V along a geodesic γ, there exist many geodesic variations of γ whose variation fields are V [analogue: given any vector v at a point p, there exist many curves whose tangent vector at p is v]. In the proof above we give an explicit formula for one such geodesic variations, namely,

$$f(t,s) = \exp_{\xi(s)}(t(T(s)+sW(s))), \tag{2}$$

where ξ is a geodesic with $\xi(0)=\gamma(0)$ and $\dot\xi(0)=V(0)$, and T, W are parallel vector fields along ξ with $T(0)=\dot\gamma(0)$ and $W(0)=\nabla_{\dot\gamma(0)}V$.

2. Jacobi fields with special conditions

¶ Normal Jacobi fields.

The obviously Jacobi fields $\dot\gamma$, $t\dot\gamma$ [and their linear combinations] along γ are both tangent to γ and are not so interesting in applications. Very often we need to rule out them and mainly consider normal Jacobi fields.

Definition 2.1. A Jacobi field along γ is called a *normal Jacobi field* if it is perpendicular to $\dot\gamma$ along γ.

It turns out that for any Jacobi field, the tangential components must be a linear combination of $\dot\gamma$ and $t\dot\gamma$:

Proposition 2.2. *For any Jacobi field X along γ, there exists $c^1, d^1 \in \mathbb{R}$ so that*

$$X^\perp = X - c^1 t\dot\gamma - d^1\dot\gamma$$

is a normal Jacobi field along γ.

Proof. By Lemma 1.2, $\langle X, \dot\gamma\rangle$ is a linear function along γ, i.e.

$$\langle X, \dot\gamma\rangle = c_1 t + d_1$$

for some constant $c_1, d_1 \in \mathbb{R}$. Now we let

$$X^\perp = X - c^1 t\dot\gamma - d^1\dot\gamma$$

with $c^1 = \frac{c_1}{|\dot\gamma|^2}, d^1 = \frac{d_1}{|\dot\gamma|^2}$. Then it is a Jacobi field along γ since it is a linear combination of Jacobi fields along γ, and it is normal since

$$\langle X^\perp, \dot\gamma\rangle = c_1 t + d_1 - c^1 t|\dot\gamma^2| - d^2|\dot\gamma|^2 = 0.$$

□

Note that if $X^\perp$ is a normal Jacobi field along γ, then

$$\langle \nabla_{\dot\gamma} X^\perp, \dot\gamma\rangle = \frac{d}{dt}\langle X^\perp, \dot\gamma\rangle - \langle X^\perp, \nabla_{\dot\gamma}\dot\gamma\rangle = 0$$

and thus $\nabla_{\dot\gamma} X^\perp \perp \dot\gamma$. It follows

Corollary 2.3. *A Jacobi field X along γ is normal if and only if*

$$\langle X(a), \dot\gamma(a)\rangle = \langle \nabla_{\dot\gamma(a)} X, \dot\gamma(a)\rangle = 0.$$

In particular, the set of normal Jacobi fields form a linear space of dimension $2m-2$.

Proof. With $X = X^\perp + c^1 t\dot\gamma + d^1\dot\gamma$, we have

$$\langle X(a), \dot\gamma(a)\rangle = (c^1 a + d^1)|\dot\gamma|^2,$$
$$\langle \nabla_{\dot\gamma(a)} X, \dot\gamma(a)\rangle = \langle \nabla_{\dot\gamma(a)}(c^1 t\dot\gamma + d^1\dot\gamma), \dot\gamma(a)\rangle = c^1|\dot\gamma|^2.$$

The conclusion follows. □

Corollary 2.4. *Let X be a Jacobi field so that*

$$\langle X(t_1), \dot\gamma(t_1)\rangle = \langle X(t_2), \dot\gamma(t_2)\rangle = 0$$

for two distinct numbers t_1, t_2. Then X is a normal Jacobi field.

Proof. This follows from Lemma 1.2, i.e. $\langle X, \dot\gamma\rangle$ is a linear function along γ, and the fact that a linear function has no more than one zero unless it is identically zero. □

¶ Normal Jacobi fields on spaces with constant sectional curvature.

Let (M, g) be a Riemannian manifold with constant sectional curvature k, i.e.

$$R(X,Y)Z = -k(\langle X, Z\rangle Y - \langle Y, Z\rangle X).$$

Let γ be a normal geodesic in M, and X a normal Jacobi field along γ. Then

$$R(\dot\gamma, X)\dot\gamma = -k(\langle \dot\gamma, \dot\gamma\rangle X - \langle X, \dot\gamma\rangle\dot\gamma) = -kX.$$

So the equation for a normal Jacobi field X along γ becomes

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} X + kX = 0.$$

Again we take an orthonormal frame $\{e_i(t)\}$ along γ so that

- $e_1(t) = \dot\gamma(t)$,
- each $e_i(t)$ is parallel along γ,

as we did in the proof of Theorem 1.3, and write

$$X = \sum_{i=2}^m X^i(t) e_i(t),$$

then the equation for the coefficient $X^i(t)$ becomes

$$\ddot{X}^i(t) + kX^i(t) = 0, \qquad 2 \le i \le m.$$

The solution to this equation is

$$X^i(t) = \begin{cases} c^i \frac{\sin(\sqrt{k}t)}{\sqrt{k}} + d^i \cos(\sqrt{k}t), & \text{if } k > 0, \\ c^i t + d^i, & \text{if } k = 0, \\ c^i \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}} + d^i \cosh(\sqrt{-k}t), & \text{if } k < 0, \end{cases}$$

where c^i, d^i are constants, and

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

are the hyperbolic cosine and hyperbolic sine functions. Some times people denote

$$sn_k(t) = \begin{cases} \frac{\sin(\sqrt{k}t)}{\sqrt{k}}, & k > 0 \\ t, & k = 0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}}, & k < 0 \end{cases} \quad \text{and} \quad cn_k(t) = sn_k'(t) = \begin{cases} \cos(\sqrt{k}t), & k > 0 \\ 1, & k = 0 \\ \cosh(\sqrt{-k}t), & k < 0 \end{cases}$$

so that we can write $X^i(t) = c^i sn_k(t) + d^i cn_k(t)$.

¶ Jacobi fields with $V(0) = 0$.

For simplicity let $a = 0$ for the defining interval $[a, b]$ of γ. In many applications we need geodesic variations that fix one end, i.e. with $\gamma_s(0) = \gamma(0)$ for all s. Of course the Jacobi field for such geodesic variations satisfies $V(0) = 0$. Conversely, if V is a Jacobi field along γ with $V(0) = 0$, then in (2) we may take

$$\xi(s) \equiv \gamma(0), \quad T(s) \equiv \dot{\gamma}(0), \quad W(s) \equiv \nabla_{\dot{\gamma}(0)} V$$

and get an explicit geodesic variation with one end fixed, whose variation field is V:

Proposition 2.5. *If V is a Jacobi field along geodesic γ with $V(0) = 0$, then*

$$f(t, s) = \exp_{\gamma(0)}(t(\dot{\gamma}(0) + s\nabla_{\dot{\gamma}(0)} V)).$$

is a geodesic variation of γ with $\gamma_s(0) = \gamma(0)$ and whose variation field is V.

In particular, by calculating the variation field of the above variation via its formula, we get

Corollary 2.6. *If V is a Jacobi field along geodesic γ with $V(0) = 0$, then*

$$V(t) = f_s(t, 0) = (d\exp_{\gamma(0)})_{t\dot{\gamma}(0)}(t\nabla_{\dot{\gamma}} V).$$

¶ Taylor's expansion of the Jacobi field with $V(0) = 0$.

Now let V, W be Jacobi fields along a geodesic γ with

$$V(0) = 0, \nabla_{\dot\gamma(0)} V = X_p \in T_pM \quad \text{and} \quad W(0) = 0, \nabla_{\dot\gamma(0)} V = Y_p \in T_pM.$$

According to Corollary 2.6, we have

$$V(t) = (d\exp_p)_{t\dot\gamma(0)}(tX_p) \quad \text{and} \quad W(t) = (d\exp_p)_{t\dot\gamma(0)}(tY_p).$$

Let $f(t) = \langle V(t), W(t)\rangle$. Then we have

$$f(0) = \langle V(0), W(0)\rangle = 0,$$
$$f'(0) = \langle \nabla_{\dot\gamma(0)} V, W(0)\rangle + \langle V(0), \nabla_{\dot\gamma(0)} W\rangle = 0,$$
$$f''(0) = \langle \nabla_{\dot\gamma(0)}\nabla_{\dot\gamma} V, W(0)\rangle + 2\langle \nabla_{\dot\gamma(0)} V, \nabla_{\dot\gamma(0)} W\rangle + \langle V(0), \nabla_{\dot\gamma(0)}\nabla_{\dot\gamma} W\rangle = 2\langle X_p, Y_p\rangle.$$

To compute more derivatives, we note that in view of $V(0) = 0$,

$$\nabla_{\dot\gamma(0)}\nabla_{\dot\gamma} V = R(\dot\gamma(0), V(0))\dot\gamma(0) = 0,$$

and similarly $\nabla_{\dot\gamma(0)}\nabla_{\dot\gamma} W = 0$. So [We abbreviate the k^{th} composition $\nabla_{\dot\gamma}\cdots\nabla_{\dot\gamma}$ to $\nabla_{\dot\gamma}^{(k)}$]

$$f'''(0) = \sum_{l=0}^{3} \binom{3}{l} \langle \nabla_{\dot\gamma}^{(3-l)} V, \nabla_{\dot\gamma}^{(l)} W\rangle(0) = 0,$$
$$f''''(0) = \sum_{l=0}^{4} \binom{4}{l} \langle \nabla_{\dot\gamma}^{(4-l)} V, \nabla_{\dot\gamma} W\rangle(0) = 4\langle \nabla_{\dot\gamma}^{(3)} V, \nabla_{\dot\gamma} W\rangle(0) + 4\langle \nabla_{\dot\gamma} V, \nabla_{\dot\gamma}^{(3)} W\rangle(0).$$

To calculate the third order derivative, we note that if we take the $(k-2)^{th}$ covariant derivative of the Jacobi field equation for V, then

$$\nabla_{\dot\gamma}^{(k)} V - \sum_{l=0}^{k-2} \binom{k-2}{l} (\nabla_{\dot\gamma}^{(k-2-l)} R)(\dot\gamma, \nabla_{\dot\gamma}^{(l)} V)\dot\gamma = 0,$$

where we used the facts

$$\nabla_W(R(X,Y)Z) = (\nabla_W R)(X,Y)Z + R(\nabla_W X, Y)Z + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z$$

and $\nabla_{\dot\gamma}\dot\gamma = 0$. Taking $k = 3$, we get

$$\nabla_{\dot\gamma}^{(3)} V - (\nabla_{\dot\gamma} R)(\dot\gamma, V)\dot\gamma - R(\dot\gamma, \nabla_{\dot\gamma} V)\dot\gamma = 0.$$

Evaluate at $t = 0$, and use $V(0) = 0$, we get $(\nabla_{\dot\gamma}^{(3)} V)(0) = R(\dot\gamma(0), X_p)\dot\gamma(0)$. Thus

$$f''''(0) = 4\langle R(\dot\gamma(0), X_p)\dot\gamma(0), Y_p\rangle + 4\langle X_p, R(\dot\gamma(0), Y_p)\dot\gamma(0)\rangle = -8Rm(\dot\gamma(0), X_p, \dot\gamma(0), Y_p).$$

So we get

$$\langle V(t), W(t)\rangle = \langle X_p, Y_p\rangle t^2 - \frac{1}{3} Rm(\dot\gamma(0), X_p, \dot\gamma(0), Y_p) t^4 + O(t^5).$$

In particular, if we take $W = V$ and assume $|X_p| = 1$, then

$$|V(t)|^2 = t^2 - \frac{1}{3} Rm(\dot\gamma(0), X_p, \dot\gamma(0), X_p) t^4 + O(t^5).$$