LECTURE 18: IMMEDIATE APPLICATIONS OF JACOBI FIELD TO CURVATURE

Last time we studied Jacobi fields along a geodesic γ, which control all possible geodesic variations of γ. Today we give some immediate applications.

1. Geometric interpretations of various curvatures

¶ Taylor's expansion of metric tensor in Riemannian normal coordinates.

Recall that if V, W are Jacobi fields along a geodesic γ with

$$V(0) = 0, \nabla_{\dot\gamma(0)} V = X_p \in T_pM \quad \text{and} \quad W(0) = 0, \nabla_{\dot\gamma(0)} V = Y_p \in T_pM,$$

then the function $f(t) = \langle V(t), W(t)\rangle$ has the Taylor's expansion

$$f(t) = \langle X_p, Y_p\rangle t^2 - \frac{1}{3} Rm(\dot\gamma(0), X_p, \dot\gamma(0), Y_p)t^4 + O(t^5).$$

For the first application, we calculate the next term in the Taylor's expansion of any Riemannian metric tensor in any Riemannian normal coordinate system. Recall that with any Riemannian normal coordinate system centered at p,

$$g_{ij}(p) = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}(p) = 0.$$

We now prove that the next term encodes the curvature information:

Theorem 1.1. *With respect to Riemannian normal coordinates near p, the functions g_{ij}'s admit the following Taylor expansion at $x = 0$,*

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + O(|x^3|). \tag{1}$$

Proof. Let $(U; x^1, \cdots, x^m)$ be a Riemannian normal coordinate system near p. Fix x^i's and let γ be the geodesic starting at p in the direction $X_p = x^i\partial_i$,

$$\gamma(t) = (tx^1, \cdots, tx^m), \quad 0 \le t \le \varepsilon.$$

For each $1 \le i \le m$, consider a geodesic variation

$$f_i(t, s) = (tx^1, \cdots, t(x^i + s), \cdots, tx^m).$$

Its variation field $V_i = t\partial_i$ is thus a Jacobi field along γ, which satisfies

$$V_i(0) = 0, \quad \nabla_{\dot\gamma(0)} V_i = \partial_i.$$

So if we let

$$h(t) = t^2 g_{ij}(tx^1, \cdots, tx^m) = \langle V_i(t), V_j(t)\rangle,$$

then

$$\begin{aligned} g_{ij}(tx^1, \cdots, tx^m) &= \frac{1}{t^2}\langle V_i(t), V_j(t)\rangle \\ &= \frac{1}{t^2}\left(\delta_{ij}t^2 - \frac{1}{3}Rm(X_p, \partial_i, X_p, \partial_j)t^4 + O(t^5)\right) \\ &= \delta_{ij} - \frac{1}{3}Rm(\partial_i, X_p, \partial_j, X_p)t^2 + O(t^3) \\ &= \delta_{ij} - \frac{1}{3}Rm(\partial_i, \partial_k, \partial_j, \partial_l)(tx^k)(tx^l) + O(t^3). \end{aligned}$$

This proves the theorem. □

Remark. One can continue to calculate $\nabla^{(k)}_{\dot\gamma(0)}V_i$'s and get a full expansion of g_{ij} in Riemannian normal coordinates. For example, the next two terms are

$$\frac{1}{6}R_{iklj;r}x^kx^lx^r + (\frac{1}{20}R_{iklj;rs} + \frac{2}{45}R_{kil}{}^{m}R_{rjsm})x^kx^lx^rx^s.$$

Taking derivative of (1), we get

$$\partial_r g_{ij} = -\frac{1}{3}R_{irjl}x^l - \frac{1}{3}R_{ikjr}x^k + O(|x|^2).$$

Taking derivative again and evaluate at p, we get

$$\partial_s\partial_r g_{ij}(0) = -\frac{1}{3}R_{irjs}(p) - \frac{1}{3}R_{isjr}(p).$$

As a consequence, we get Riemann's original definition of the curvature tensor:

Corollary 1.2. *With respect to Riemannian normal coordinates, one has*

$$R_{ijkl}(p) = \frac{1}{2}(\partial_i\partial_l g_{jk} + \partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik})(0).$$

Proof. The right hand side equals

$$\frac{1}{6}(R_{jlik} + R_{jilk} + R_{ikjl} + R_{ijkl} - R_{jkil} - R_{jikl} - R_{iljk} - R_{ijlk})(p),$$

which equals $R_{ijkl}(p)$ by using symmetries of the Riemann curvature tensor. □

¶ Geometric meaning of sectional curvature.

Now we are ready to give geometric interpretations of curvatures. We start with sectional curvature:

Theorem 1.3. *Let $\Pi_p \subset T_pM$ be a 2-dimensional plane. Denote by C_r^0 the circle of radius r in Π_p centered at p, and $C_r = \exp_p(C_r^0)$. Let L_r be the length of C_r. Then*

$$\lim_{r\to 0}\frac{2\pi r - L_r}{r^3} = \frac{\pi}{3}K(\Pi_p).$$

Proof. Take an orthonormal basis $\{e_1, \cdots, e_m\}$ of T_pM so that Π_p is spanned by e_1, e_2, and consider the normal coordinate system with respect to $\{e_i\}$. Then for r small, the circle C_r has equation

$$C_r : x^1(t) = r\cos t,\ x^2(t) = r\sin t,\ x^k(t) = 0\ (k \geq 3).$$

It follows that

$$\begin{aligned}|\dot{C}_r(t)|^2 &= g_{ij}(C_r(t))\dot{x}^i(t)\dot{x}^j(t)\\ &= \left(1-\frac{1}{3}R_{1212}x^2x^2\right)\dot{x}^1\dot{x}^1+\left(1-\frac{1}{3}R_{2121}x^1x^1\right)\dot{x}^2\dot{x}^2-2\frac{1}{3}R_{1221}x^1x^2\dot{x}^1\dot{x}^2+O(r^5)\\ &= r^2 - \frac{r^4}{3}K(\Pi_p) + O(r^5).\end{aligned}$$

So

$$\begin{aligned}L_r = \mathrm{Length}(C_r) = \int_0^{2\pi} |\dot{C}_r|dt &= r\int_0^{2\pi}\sqrt{1-\frac{r^2}{3}K(\Pi_p)+O(r^3)}dt\\ &= 2\pi r - \frac{\pi}{3}K(\Pi_p)r^3 + O(r^4).\end{aligned}$$

This implies the theorem. □

So the sectional curvature measures the deviation of the length of small geodesic circles centered at p to the standard circles of the same radius in Euclidean plane.

¶ Geometric meaning of Ricci curvature.

With the Taylor's expansion of g_{ij} at hand, it is easy to get the Taylor's expansion of $\det(g_{ij})$: According to Theorem 1.1 we get

$$(g_{ij}) = I + \left(-\frac{1}{3}R_{ikjl}(p)x^kx^l + O(|x^3|)\right)$$

which implies [1]

$$\log(g_{ij}) = \left(-\frac{1}{3}R_{ikjl}(p)x^kx^l + O(|x^3|)\right)$$

and thus

$$\begin{aligned}\det(g_{ij}) = \det(e^{\log(g_{ij})}) = e^{\mathrm{tr}(\log(g_{ij}))} &= e^{-\frac{1}{3}Rc_{kl}(p)x^kx^l+O(|x|^3)}\\ &= 1-\frac{1}{3}Rc_{kl}(p)x^kx^l + O(|x|^3).\end{aligned}$$

As an immediate consequence, we get the Taylor's expansion for the volume element:

Corollary 1.4. *In Riemannian normal coordinates centered at p, one has*

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6}Rc_{kl}(p)x^kx^l + O(|x|^3).$$

[1] Here we use $\log(I+A) = A + O(|A|^2)$ and $e^A = 1 + A + O(|A|^2)$ for matrix A.

In particular, we can prove that the Ricci curvature measures the change of the volume element in the given direction:

Corollary 1.5. *Let $u_p \in S_pM$ be a unit tangent vector at p, and let $\gamma(t)$ be the geodesic starting at p with $\dot\gamma(0) = u_p$. Then*

$$\sqrt{\det(g_{ij}(\gamma(t)))} = 1 - \frac{\mathrm{Ric}(u_p)}{6}t^2 + O(t^3).$$

Proof. Take an orthonormal basis of T_pM with $e_1 = u_p$. With respect to the associated Riemannian normal coordinates, the geodesic $\gamma(t) = \exp_p(tu_p)$ is given by

$$\gamma : x^1(t) = t, \quad x^2 = \cdots = x^m = 0.$$

It follows

$$\sqrt{\det(g_{ij}(\gamma(t)))} = 1 - \frac{1}{6}Rc_{11}(p)t^2 + O(t^3) = 1 - \frac{1}{6}\mathrm{Ric}(u_p)t^2 + O(t^3).$$

□

¶ Geometric meaning of scalar curvature.

Finally we study the scalar curvature S. We have

Proposition 1.6. *For r small enough,*

$$\mathrm{Vol}(B(p,r)) = \omega_m r^m \left(1 - \frac{S(p)}{6(m+2)}r^2 + O(|r^3|)\right),$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$.

Proof. By definition

$$\begin{aligned}\mathrm{Vol}(B(p,r))) &= \int_{B_r(0)} \sqrt{\det(g_{ij})}dx^1\cdots dx^m \\ &= \int_{B_r(0)} \left(1 - \frac{1}{6}Rc_{kl}(p)x^kx^l\right) dx^1\cdots dx^m + O(r^3) \\ &= \omega_m r^m - \frac{Rc_{kl}(p)}{6}\int_{B_r(0)} x^kx^l dx^1\cdots dx^m + O(r^3).\end{aligned}$$

An elementary computation shows

$$\int_{B_r(0)} x^kx^ldx^1\cdots dx^m = \frac{\omega_m}{m+2}r^{m+2}\delta^{kl}$$

and the conclusion follows. □

So the scalar curvature measures the deviation of the volume of a geodesic ball to the standard Euclidean ball with the same radius.

Corollary 1.7. *The surface area of geodesic sphere $S(p,r)$ is*

$$\mathrm{Area}(S(p,r)) = m\omega_m r^{m-1} - \frac{S(p)}{6}\omega_m r^{m+1} + O(r^{m+2}).$$

2. Cartan's local isometry theorem

¶ Cartan's local isometry theorem.

In view of Theorem 1.1, one may anticipate that "curvature determines the Riemannian metric". Although this is not true in the most general sense, there are many theorems supporting this philosophy. In what follows we prove a theorem of Cartan in this direction. More precisely, let (M, g) and $(\widetilde{M}, \tilde{g})$ be two Riemannian manifolds, $p \in M$ and $\tilde{p} \in \widetilde{M}$. Let $B(p, r)$ and $B(\tilde{p}, r)$ be normal neighborhoods of p and $\tilde{p}$ respectively. Given a smooth map $\phi : B(p, r) \to B(\tilde{p}, r)$, one may ask: under what condition will ϕ be an isometry? Of course if ϕ is an isometry, then

- $L = d\phi_p : (T_pM, g_p) \to (T_{\tilde{p}}\widetilde{M}, \tilde{g}_{\tilde{p}})$ is a linear isometry,
- "the curvature tensor at corresponding points are the same".

Cartan's theorem claims that the converse is true.

We need more explanation for the second condition above. How to compare the curvature tensor at q and $\varphi(q)$? We need to identify the tangent spaces T_qM and $T_{\phi(q)}\widetilde{M}$ first. How? We already have the map L which identifies T_pM with $T_{\tilde{p}}\widetilde{M}$. For $q \neq p$ we simply parallel transport vectors in T_qM and in $T_{\phi(q)}\widetilde{M}$ along geodesics to get vectors in T_pM and in $T_{\phi(p)}\widetilde{M}$ respectively, and then apply the map L.

Now we start to state Cartan's theorem. Suppose

$$L : (T_pM, g_p) \to (T_{\tilde{p}}\widetilde{M}, \tilde{g}_{\tilde{p}})$$

is a linear isometry. Then one may define a map

$$\phi = \exp_{\tilde{p}} \circ L \circ (\exp_p)^{-1} : B(p, r) \to B(\tilde{p}, r).$$

For any $q \in B(p, r)$, we let $\gamma = \gamma_q : [0, 1] \to M$ be the unique geodesic in $B(p, r)$ with $\gamma(0) = p, \gamma(1) = q$, and let $\tilde{\gamma} = \phi \circ \gamma$. Note that $\tilde{\gamma}$ is the unique geodesic in $B(\tilde{p}, r) \subset \widetilde{M}$ with $\tilde{\gamma}(0) = \tilde{p} = \phi(p)$, $\tilde{\gamma}(1) = \tilde{q} = \phi(q)$ and $\dot{\tilde{\gamma}}(0) = L(\dot{\gamma}(0))$. Define

$$L_q = P^{\tilde{\gamma}} \circ L \circ (P^{\gamma})^{-1} : T_qM \to T_{\phi(q)}\widetilde{M},$$

Theorem 2.1 (Cartan's local isometry theorem)**.** *If for any $q \in B(p, r)$ and any $u, v, w \in T_qM$, one has*

$$L_q(R(u, v)w) = \widetilde{R}(L_q(u), L_q(v))L_q(w),$$

then ϕ is an isometry, and $d\phi_q = L_q$ for all $q \in B(p, r)$.

Note that for constant curvature spaces, the condition holds trivially. So we get

Corollary 2.2. *If both (M, g) and $(\widetilde{M}, \tilde{g})$ has constant sectional curvature k, then for any $p \in M$ and $\tilde{p} \in \widetilde{M}$, one can find a neighborhood $U \ni p$ and $\widetilde{U} \ni \tilde{p}$ so that (U, g) and $(\widetilde{U}, \tilde{g})$ are isometric.*

which implies Riemann's theorem for constant sectional curvature spaces, namely, Theorem 2.1 in Lecture 10.

¶ Proof of Cartan's local isometry theorem.

We first prove $|d\phi_q(v)| = |v|$ for any $v \in T_qM$. By using polarization this implies that $d\phi_q$ preserves inner products. Since ϕ is already a diffeomorphism, we conclude that ϕ is an isometry.

The idea to prove $|d\phi_q(v)| = |v|$ is: realize both v and $d\phi_q(v)$ as Jacobi field vector at endpoints, and compare the length of two Jacobi fields at each point. We first construct a Jacobi field V along γ with $V(0) = 0, V(1) = v$. By Corollary 2.6 in Lecture 17, if V is such a Jacobi field, then

$$V(t) = (d\exp_p)_{t\dot\gamma(0)}(t\nabla_{\dot\gamma(0)}V),$$

which implies $v = (d\exp_p)_{\dot\gamma(0)}(\nabla_{\dot\gamma(0)}V)$. As a result, V is the unique Jacobi field with $V(0) = 0$ and $(\nabla_{\dot\gamma(0)}V = (d\exp_p)^{-1}_{t\dot\gamma(0)}(v)$.

Next we construct the Jacobi field $\widetilde{V}$ along $\tilde\gamma$ with $\widetilde{V}(0) = 0$ and $\widetilde{V}(1) = d\phi_q(v)$. In fact, we may simply take $\widetilde{V}$ to be the Jacobi field along $\tilde\gamma$ with $\widetilde{V}(0) = 0$ and $\widetilde{\nabla}_{\dot{\tilde\gamma}(0)}V = L(\nabla_{\dot\gamma(0)}V)$. Since $\exp_p(\dot\gamma(0)) = \gamma(1) = q$, it follows

$$\begin{aligned} d\phi_q(v) &= (d\exp_{\tilde p})_{L(\dot\gamma(0))} \circ L \circ \big(d\exp_p^{-1})_q(v) \\ &= (d\exp_{\tilde p})_{L(\dot\gamma(0))} \circ L \circ (d\exp_p)^{-1}_{\dot\gamma(0)}(v) = (d\exp_{\tilde p})_{\dot{\tilde\gamma}(0)}\big(L(\nabla_{\dot\gamma(0)}V)\big) = \widetilde{V}(1), \end{aligned}$$

where the last equality follows from Corollary 2.6 in Lecture 17.

To prove $|V(1)| = |\widetilde{V}(1)|$ we apply a standard trick: Let $e_1(t) = \dot\gamma(t)$, $e_2(t), \cdots$, $e_m(t)$ be an orthonormal frame that is parallel along γ. Let $V(t) = V^i(t)e_i(t)$, then

$$|V(1)|^2 = \sum (V^i(1))^2.$$

Moreover, $V^i(t)$ is the solution to the Jacobi equation

$$\ddot V^i(t) - \langle R(\dot\gamma, e_k)\dot\gamma, e_i\rangle V^k = 0,$$

with initial conditions $V^i(0) = 0$ and $\nabla_{\dot\gamma(0)}V = \dot V^i(0)e_i(0)$.

Similarly we let $\tilde e_1(t) = \dot{\tilde\gamma}(t), \tilde e_2(t), \cdots, \tilde e_m(t)$ be the parallel orthonormal frame along $\tilde\gamma$ with $\tilde e_i(0) = L(e_i(0))$. Then $L_{\gamma(t)}(e_j(t)) = \tilde e_j(t)$ for all j, and thus

$$\langle \widetilde{R}(\dot{\tilde\gamma}, \tilde e_k)\dot{\tilde\gamma}, \tilde e_i\rangle = \langle R(\dot\gamma, e_k)\dot\gamma, e_i\rangle.$$

As a result, if $\widetilde{V} = \widetilde{V}^i\tilde e_j(t)$, then $\widetilde{V}^i$'s satisfy exactly the same equations and the same initial conditions as V^i's, and thus $\widetilde{V}^i = V^i$ for all $1 \le i \le m$. It follows

$$|\widetilde{V}(1)|^2 = \sum (\widetilde{V}^i(1))^2 = \sum (V^i(1))^2 = |V(1)|^2.$$

Finally, the fact $\widetilde{V} = V^i\tilde e_i$ that we just proved also implies

$$(d\phi_q)(v) = \widetilde{V}(1) = V^i(1)\tilde e_i(1) = V^i(1)L_q(e_i(1)) = L_q(V(1)) = L_q(v)$$

and thus the proof is completed.