LECTURE 19: CONJUGATE POINT AND APPLICATIONS

1. Conjugate points

¶ The index form.

We want to understand the mechanism for geodesics to fail to be length minimizing among nearby curves with the same endpoints. So we go back to the second variation formula for a proper variation $f(t,s)=\gamma_s(t)$ of a geodesic $\gamma:[a,b]\to M$ with variation field X,

$$\frac{d^2}{ds^2}\Big|_{s=0} E(\gamma_s) = \int_a^b \big(\langle R(\dot\gamma,X)\dot\gamma,X\rangle + \langle \nabla_{\dot\gamma}X,\nabla_{\dot\gamma}X\rangle\big)dt.$$

If the second variation is positive, then γ is length minimizing among these γ_s's for s small. Since any vector field can be realized as a variation field[See PSet 3], we are led to study the quadratic form

$$I(X,X) := \int_a^b \big(\langle R(\dot\gamma,X)\dot\gamma,X\rangle + \langle \nabla_{\dot\gamma}X,\nabla_{\dot\gamma}X\rangle\big)dt$$

for any vector field X along γ, and thus are led to study its polarization

$$\begin{aligned} I(X,Y) &:= \int_a^b \big(\langle R(\dot\gamma,X)\dot\gamma,Y\rangle + \langle \nabla_{\dot\gamma}X,\nabla_{\dot\gamma}Y\rangle\big)dt \\ &= \int_a^b \langle R(\dot\gamma,X)\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}X, Y\rangle dt + \langle \nabla_{\dot\gamma}X,Y\rangle\big|_a^b. \end{aligned}$$

In many applications one need to consider a variation whose variation field is only continuous and piecewise smooth. So in this most general case, the *index form* $I=I^\gamma$ of the geodesic γ is a symmetric bilinear form defined on

$$\mathcal{V}=\mathcal{V}_\gamma=\{X \text{ is a continuous piecewise smooth vector field along } \gamma\}$$

by the formula

$$\begin{aligned} I(X,Y) &= \int_a^b \big(\langle R(\dot\gamma,X)\dot\gamma,Y\rangle + \langle \nabla_{\dot\gamma}X,\nabla_{\dot\gamma}Y\rangle\big)dt \\ &= \int_a^b \langle R(\dot\gamma,X)\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}X, Y\rangle dt + \langle \nabla_{\dot\gamma}X,Y\rangle\big|_a^b - \sum_{j=1}^k \langle \nabla_{\dot\gamma(t_j^+)}X - \nabla_{\dot\gamma(t_j^-)}X, Y\rangle, \end{aligned} \tag{1}$$

where $a<t_1<\cdots<t_k<b$ are those points where X is not smooth, and $\nabla_{\dot\gamma(t_j^+)}X$ means $\lim_{t\to t_j+}\nabla_{\dot\gamma(t)}X$.

¶ Index form v.s. Jacobi fields.

Now we relate the index form I with Jacobi fields along $\gamma : [a,b] \to M$. Denote

$$\mathcal{V}^0 = \mathcal{V}^0_\gamma := \{X \in \mathcal{V} \mid X(a) = 0, X(b) = 0\}.$$

We have

Proposition 1.1. *Let $V \in \mathcal{V}$. Then V is a Jacobi field along γ if and only if for any $X \in \mathcal{V}^0$, $I(V,X) = 0$.*

Proof. ($\Longrightarrow$) According to (1), if $X \in \mathcal{V}^0$ and V is a Jacobi field (which has to be smooth) along γ, then $I(V,X) = 0$.

($\Longleftarrow$) Conversely assume $V \in \mathcal{V}$ satisfies $I(V,X) = 0$ for all $X \in \mathcal{V}^0$, and

$$a = t_0 < t_1 < \cdots < t_k < t_{k+1} = b$$

is a subdivision of $[a,b]$ so that V is smooth on each $[t_j, t_{j+1}]$.

- First take a smooth function $f : [a,b] \to \mathbb{R}$ with $f(t_i) = 0$ for all i and $f(t) > 0$ for all $t \notin \{t_0, t_1, \cdots, t_{k+1}\}$, and define
$$X = f(t)(R(\dot\gamma, V)\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}V).$$
Then $X \in \mathcal{V}^0$ and so
$$0 = I(V,X) = \int_a^b f(t)|R(\dot\gamma, V)\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}V|^2 dt.$$
It follows that V is a Jacobi field on each (t_j, t_{j+1}).
- Next let's choose any $X' \in \mathcal{V}^0$ with
$$X'(t_i) = \nabla_{\dot\gamma(t_i^+)}V - \nabla_{\dot\gamma(t_i^-)}V.$$
Then
$$0 = I(V,X') = -\sum_{i=1}^k |\nabla_{\dot\gamma(t_i^+)}V - \nabla_{\dot\gamma(t_i^-)}V|^2.$$
It follows that V is of class C^1 at each t_i.

By uniqueness, V is smooth. So V is a Jacobi field. □

¶ Conjugate points.

In particular, if there is a nonzero Jacobi field V along γ with $V(t_1) = V(t_2) = 0$, then $I^{\bar\gamma}(V,V) = 0$, where $\bar\gamma = \gamma|_{[t_1,t_2]}$, and thus $I^{\bar\gamma}$ is not positive definite. As a result, $\bar\gamma$ may fail to be length minimizing for the variation with variation field V.

Definition 1.2. Let (M,g) be a Riemannian manifold, $\gamma : [a,b] \to M$ a geodesic, and $t_1 \neq t_2 \in [a,b]$. If there exists a Jacobi field V along γ which is not identically zero, such that $V(t_1) = V(t_2) = 0$, then we say $\gamma(t_2)$ is *conjugate* to $\gamma(t_1)$ along γ.

Note that according to Corollary 2.4 in Lecture 17, any Jacobi field V along γ satisfying $V(t_1) = 0$ and $V(t_2) = 0$ (where $t_1 \neq t_2$) must be a normal Jacobi field. So if $q = \gamma(t_2)$ is a conjugate point of $p = \gamma(t_1)$ along γ, then

$$\mathcal{J}_{\gamma,t_1,t_2} = \{V \mid V \text{ is a Jacobi field along } \gamma \text{ with } V(t_1) = V(t_2) = 0\}$$

is a vector subspace of the space $\mathcal{J}_\gamma^\perp$ of normal Jacobi fields along γ.

Definition 1.3. If $q = \gamma(t_2)$ is a conjugate point of $p = \gamma(t_1)$ along γ, we call

$$n_{\gamma,t_1}(t_2) := \dim \mathcal{J}_{\gamma,t_1,t_2}$$

the *multiplicity* of the conjugate point q to p along γ.

By definition, if q is conjugate to p along a geodesic γ, then p is conjugate to q along the geodesic $-\gamma$, with the same multiplicity. We have

Lemma 1.4. *Suppose* $\dim M = m$, *then* $n_{\gamma,t_1}(t_2) \leq m - 1$.

Proof. As we have seen in Lecture 17, a Jacobi field V is uniquely determined by $V(t_1)$ and $\nabla_{\dot\gamma(t_1)}V$. Moreover, V is normal implies $\nabla_{\dot\gamma(t_1)}V \in (\dot\gamma(t_1))^\perp$. So $\mathcal{J}_{\gamma,t_1,t_2}$ is isomorphic to a subspace of

$$\{(0, v) \mid v \in (\dot\gamma(t_1))^\perp\} \subset \{(u, v) \mid u, v \in T_{\gamma(t_1)}M\}$$

and the conclusion follows. □

Example. Consider the round sphere (S^m, g_{round}) whose sectional curvature is 1. Let $\gamma : [0, l] \to M$ be a normal geodesic starting from any p. Then in Lecture 17 we have seen that any normal Jacobi field along γ with $V(0) = 0$ must be of the form

$$V(t) = \sum_{i=2}^{m} c^i \sin(t) e_i(t),$$

where $\{e_i(t)\}$ is a parallel orthonormal frame along γ, with $e_1(t) = \dot\gamma(t)$. It follows

- if γ has length less than π, then there is no conjugate point of p,
- if the length of γ is between π and 2π, then the antipodal point $\gamma(\pi) = -p$ is the only conjugate point to the north pole along any geodesic starting at p, and its multiplicity equals $m - 1$.

We may also repeat the same argument for $(\mathbb{R}^m, g_0)$ and $(\mathbb{H}^m, g_{hyperbolic})$, and arrive at the conclusion that there is no conjugate point at all. In fact the same result holds for any Riemannian manifold whose sectional curvatures are non-positive:

Proposition 1.5. *Let (M, g) be a Riemannian manifold whose sectional curvature is non-positive. Then any $p \in M$ has no conjugate point along any geodesic.*

Proof. Let γ be any geodesic from $\gamma(0) = p$ and V any nonzero normal Jacobi field along γ with $V(0) = 0$. Let $f(t) = \langle V(t), V(t)\rangle$. Then

$$f'(t) = 2\langle \nabla_{\dot\gamma(t)}V, V\rangle$$

and thus, in view of $R(\dot\gamma, V, \dot\gamma, V) = -K(\dot\gamma, V)|\dot\gamma|^2|V|^2 \ge 0$,

$$f''(t) = 2\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V, V\rangle + 2|\nabla_{\dot\gamma}V|^2 = 2R(\dot\gamma, V, \dot\gamma, V) + 2|\nabla_{\dot\gamma}V|^2 \ge 0.$$

Since $f(0) = 0, f'(0) = 0$ and $f''(0) > 0$, we conclude $f(t) > 0$ for $t > 0$. In other words, V has no other zeroes along γ. So p has no conjugate point along γ. □

Remark. In the proof we also get $f''(t) \ge -K(t)|\dot\gamma|^2 f(t)$, where $K(t) = K(\dot\gamma(t), V(t))$, from which one may derive better lower bounds of f.

2. Conjugate points via critical points of the exponential map

¶ Conjugate points v.s. the exponential map.

It turns out that conjugate points are exactly those points where the exponential map fails to be diffeomorphism. In fact, suppose $\gamma : [0, l] \to M$ is a geodesic, and V is a Jacobi field along γ with $V(0) = 0$, then by Corollary 2.6 in Lecture 17,

$$V(t) = (d\exp_p)_{t\dot\gamma(0)}(t\nabla_{\dot\gamma(0)}V).$$

It follows

$$\begin{aligned} & q = \gamma(t_0) \text{ is conjugate to } p = \gamma(0) \\ \Longleftrightarrow & \text{there is a nonzero Jacobi field } V \text{ along } \gamma \text{ so that } V(0) = 0 \text{ and } V(t_0) = 0 \\ \overset{(*)}{\Longleftrightarrow} & \nabla_{\dot\gamma(0)}V \ne 0 \text{ and } 0 = V(t_0) = (d\exp_p)_{t_0\dot\gamma(0)}(t_0\nabla_{\dot\gamma(0)}V) \\ \Longleftrightarrow & \ker(d\exp_p)_{t_0\dot\gamma(0)} \ne 0. \end{aligned}$$

Moreover, $(*)$ also implies that $\mathcal{J}_{\gamma,p,q}$ is isomorphic to $\ker(d\exp_p)_{t_0\dot\gamma(0)}$. So we get the following useful characterization of conjugate point:

Theorem 2.1. *Let $\gamma : [0, l] \to M$ be a geodesic. Then $q = \gamma(t_0)$ is a conjugate point of $p = \gamma(0)$ if and only if $\exp_p$ is singular at $t_0\dot\gamma(0)$. Moreover, the multiplicity*

$$n_{\gamma,p}(q) = \dim\ker(d\exp_p)_{t_0\dot\gamma(0)}.$$

¶ The Cartan-Hadamard theorem.

As an immediate application, we prove

Theorem 2.2 (Cartan-Hadamard)**.** *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature, then*

(1) for any $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a covering map.
(2) if M is also simply connected, then $\exp_p$ is a diffeomorphism.

Proof. According to Proposition 1.5 and Theorem 2.1,

$$\exp_p : T_pM \to M$$

is a local diffeomorphism everywhere. So (1) follows from Corollary 2.4 in Lecture 15. If M is simply connected, then any covering map to M must be a homeomorphism. Since $\exp_p$ is also a local diffeomorphism, it must be diffeomorphism. □

Definition 2.3. A complete simply-connected Riemannian manifold with non-positive curvature is called a *Cartan-Hadamard manifold*, or an *Hadamard manifold*.

Remark. Let (M,g) be a complete Riemannian manifold. We say $p \in M$ is a *pole* of (M,g) if $\exp_p : T_pM \to M$ is non-singular everywhere, i.e. p has no conjugate point along any geodesic. Repeating the proof of Cartan-Hadamard theorem word by word, we can prove

Theorem 2.4. *If (M,g) has a pole p, then $\exp_p : T_pM \to M$ is a smooth covering.*

¶ The Killing-Hopf theorem.

As another application we prove

Theorem 2.5 (Killing-Hopf)**.** *Let (M,g) be a complete Riemannian manifold of constant sectional curvature k, then the Riemannian universal cover of (M,g) is*

(a) $(S^m, \frac{1}{k}g_{round})$ *if* $k>0$,
(b) $(\mathbb{R}^m, g_0)$ *if* $k=0$,
(c) $(H^m, -\frac{1}{k}g_{hyperbolic})$ *if* $k<0$.

Proof. It is enough to work on the Riemannian universal covering of (M,g) directly, i.e. prove that if (M,g) is also simply connected, then (M,g) is isometric to one of these model spaces above. It is also enough to prove the theorem for $k = 0, \pm 1$.

Case 1: $k=-1$ or $k=0$ Write $(\mathbb{S}^m, \bar{g}) = (H^m, g_{hyperbolic})$ for $k=-1$, and $(\mathbb{S}^m, \bar{g}) = (\mathbb{R}^m, g_0)$ for $k=0$. Choose any point $\tilde{p} \in \mathbb{S}^m$ and fix any linear isometry

$$L : (T_{\tilde{p}}\mathbb{S}^m, g_{\tilde{p}}) \to (T_pM, g_p)$$

and consider

$$F = \exp_p \circ L \circ (\exp_{\tilde{p}})^{-1} : (\mathbb{S}^m, \bar{g}) \to (M,g).$$

By Cartan's local isometry theorem, F is a local isometry. By Cartan-Hadamard theorem, $\exp_p : T_pM \to M$ is a diffeomorphism. So F is a diffeomorphism, and thus an isometry.

Case 2: $k=1$ Again we start with a point $\tilde{p} \in S^m$ and fix any linear isometry

$$L : (T_{\tilde{p}}S^m, g_{\tilde{p}}) \to (T_pM, g_p).$$

Since

$$\exp_{\tilde{p}} : B_\pi(0) \subset T_{\tilde{p}}S^m \to S^m \setminus \{-\tilde{p}\}$$

is a diffeomorphism, by Cartan's local isometry theorem the map

$$F_1 = \exp_p \circ L \circ (\exp_{\tilde{p}})^{-1} : (S^m \setminus \{-\tilde{p}\}, g_{round}) \to (M,g)$$

is a local isometry. Similarly we start with $\tilde{q} \neq \pm\tilde{p}$ and get a local isometry

$$F_2 = \exp_{F_1(\tilde{q})} \circ (dF_1)_{\tilde{q}} \circ (\exp_{\tilde{q}})^{-1} : (S^m \setminus \{-\tilde{q}\}, g_{round}) \to (M,g).$$

Note that by construction,

$$F_2(\tilde{q}) = \exp_{F_1(\tilde{q})} \circ (dF_1)_{\tilde{q}} \circ (\exp_{\tilde{q}})^{-1}(q) = \exp_{F_1(\tilde{q})} \circ (dF_1)_{\tilde{q}}(0) = \exp_{F_1(\tilde{q})}(0) = F_1(\tilde{q})$$

and

$$(dF_2)_{\tilde{q}} = (d\exp_{F_1(\tilde{q})})_0 \circ (dF_1)_{\tilde{q}} \circ (d\exp_{\tilde{q}})_0^{-1} = (dF_1)_{\tilde{q}}.$$

So by Lemma 2.6 below, we have $F_1 = F_2$ on $S^m \setminus \{-\tilde{p}, -\tilde{q}\}$. So we may glue F_1 and F_2 to get a local isometry

$$F : (S^m, g_{round}) \to (M, g).$$

Finally by Ambrose theorem, F is a covering map and thus a diffeomorphism. So F is the desired isometry. □

It remains to prove

Lemma 2.6. *Let M be connected. If $f_i : (M, g) \to (\widetilde{M}, \tilde{g})(i = 1, 2)$ are two local isometries, and if there exists $p \in M$ with*

$$f_1(p_0) = f_2(p_0) \quad \textit{and} \quad (df_1)_{p_0} = (df_2)_{p_0},$$

then $f_1 = f_2$.

Proof. Consider the subset

$$A = \{p \in M \mid f_1(p) = f_2(p) \text{ and } (df_1)_p = (df_2)_p\}$$

of M, and apply the standard connectedness argument.

- By assumption $p_0 \in A$, so A is non-empty.
- Obviously A is closed.
- Take any $p \in A$ and consider normal ball $B(p, r)$ so that both f_1 and f_2 are isometries on $B(p, r)$. The fact $p \in A$ implies that both f_1 and f_2 map the "radial geodesics" in $B(p, r)$ to the "radial geodesics" in $B(f(p), r)$, which implies $f_1 = f_2$ on $B(p, r)$, and as a result, $df_1 = df_2$ on $B(p, r)$. So $B(p, r) \subset A$, i.e. A is also open.

Since M is connected, we conclude $A = M$. □

Note that in the proof of Theorem 2.5, the first step is "start with any $p \in M$ and $\tilde{p}$ in the model space" and "fix any linear isometry L". So we may fix an orthonormal basis of T_pM and an orthonormal basis at $T_{\tilde{p}}\mathbb{S}_k$ and take L to be the linear isometry that maps the first basis to the second basis. As a result, the isometry we get has such L as its differential at p. On the other hand, according to Lemma 2.6, such an isometry is unique. So we get

Proposition 2.7. *Let $(M, g), (\widetilde{M}, \tilde{g})$ be two simply connected Riemannian manifolds of constant sectional curvature k. Then for any $p \in M, \tilde{p} \in \widetilde{M}$, any orthonormal basis $\{e_1, \cdots, e_m\}$ of T_pM, and any orthonormal basis $\{e_1', \cdots, e_m'\}$ of $T_{\tilde{p}}\widetilde{M}$, there is a unique isometry $\varphi : (M, g) \to (\widetilde{M}, \tilde{g})$ such that*

$$\varphi(p) = \tilde{p}, \quad \textit{and} \quad (d\varphi)_p(e_i) = \tilde{e}_i \quad (1 \le i \le m).$$