

LECTURE 20: THE INDEX FORM

1. LENGTH MINIMIZING THROUGH INDEX FORM

¶ Index form as Hessian.

Last times we defined the index form $I = I^\gamma$ of the geodesic $\gamma : [0, l] \rightarrow M$,

$$I(X, Y) = \int_a^b (\langle R(\dot{\gamma}, X)\dot{\gamma}, Y \rangle + \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} Y \rangle) dt$$

defined on

$$\mathcal{V} = \mathcal{V}_\gamma = \{V \text{ is a continuous piecewise smooth vector field along } \gamma\}.$$

If $0 < t_1 < \dots < t_k < l$ are those points where V is not smooth, then

$$(1) \quad I(X, Y) = \int_a^b \langle R(\dot{\gamma}, X)\dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, Y \rangle dt + \langle \nabla_{\dot{\gamma}} X, Y \rangle \Big|_a^b - \sum_{j=1}^k \langle \nabla_{\dot{\gamma}(t_j^+)} X - \nabla_{\dot{\gamma}(t_j^-)} X, Y \rangle$$

Recall that geodesics γ with $\gamma(0) = p$ and $\gamma^l = q$ are precisely the critical points of the energy functional defined on the space $\mathcal{C}_{p,q}^{0,l}$ of “piecewise smooth curves with fixed endpoints parametrized on $[0, l]$ ”. The index form $I = I^\gamma$ originates from the second variation, namely for a proper variation $\gamma_s(t)$ of γ ,

$$\left. \frac{d^2}{ds^2} \right|_{s=0} E(\gamma_s) = \int_0^l (\langle R(\dot{\gamma}, X)\dot{\gamma}, X \rangle + \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} X \rangle) dt = I(X, X),$$

where $X \in \mathcal{V}^0$ is the variation field of γ_s . So what about $I(X, Y)$? It is also second variation of the energy functional E or the length functional L , but with respect to “mixed directions”: Let $\gamma_{r,s}(t)$ be a two-parameter variation of $\gamma = \gamma_{0,0}$ with fixed endpoints. Denote the variation fields corresponding to the two parameter directions by X and Y . Obviously $X, Y \in \mathcal{V}^0$. Then as in PSet 3 one can prove

$$\left. \frac{\partial^2}{\partial r \partial s} \right|_{r=s=0} E(\gamma_{r,s}) = I(X, Y).$$

So the index form I of γ , restricted the subspace \mathcal{V}^0 , can be regarded as the Hessian of the energy/length functional defined on $\mathcal{C}_{p,q}^{0,l}$ at the critical point γ .

Recall from calculus: at a critical point p of a multi-variable smooth function f ,

- if $\text{Hess}_p(f)$ is positive definite, then p is an isolated local minimum,
- if $\text{Hess}_p(f)(v, v) < 0$ for some v , then p cannot be a local minimum,
- if $\text{Hess}_p(f)$ is positive semi-definite but not positive definite, we can't draw a conclusion on the behavior of f near p .

It turns out that the same phenomena happens for the energy/length functional:

Theorem 1.1. *Let $\gamma : [0, l] \rightarrow M$ be a geodesic from $p = \gamma(0)$ to $q = \gamma(l)$. Then*

- (1) *The index form I is positive definite on $\mathcal{V}^0 \iff p$ has no conjugate point along γ .*
 - Moreover *in this case γ is an “isolated” length minimizing among nearby curves: there exists $\varepsilon > 0$ so that for any piecewise smooth curve $\bar{\gamma} : [0, l] \rightarrow M$ from p to q satisfying $\text{dist}(\gamma(t), \bar{\gamma}(t)) < \varepsilon$, we have $L(\bar{\gamma}) \geq L(\gamma)$, with equality hold if and only if $\bar{\gamma}$ is a re-parametrization of γ .*
- (2) *There exists $X \in \mathcal{V}^0$ with $I(X, X) < 0 \iff$ there exists $\bar{t} < l$ such that $\bar{q} = \gamma(\bar{t})$ is conjugate to p along γ .*
 - Moreover *in this case γ is not length minimizing among nearby curves: there is a proper variation of γ so that $L(\gamma_s) < L(\gamma)$ for $0 < |s| < \varepsilon$.*
- (3) *The index form I is positive semi-definite but not positive definite on $\mathcal{V}^0 \iff q$ is the first conjugate point of p along γ .*

Remark. Note that we only claim that γ is minimizing among nearby curves. It is possible that there exists other shorter geodesics from p to q . For example, for the cylinders there is no conjugate point (since the sectional curvature is 0), but there are infinitely many geodesics between any given two points: each is minimizing among “nearby curves”, but only one of them is minimizing among all curves.

As a corollary of part (1), we get the following important property:

Corollary 1.2. *Suppose $p = \gamma(0)$ has no conjugate point along $\gamma : [0, l] \rightarrow M$. If V is a Jacobi field along γ , and $X \in \mathcal{V}$ satisfies $X(0) = V(0), X(l) = V(l)$, then*

$$I(V, V) \leq I(X, X),$$

with equality holds if and only if $X = V$.

Proof. Since X is a Jacobi field and $X(0) = V(0), X(l) = V(l)$, by the equation(1),

$$I(V, V) = I(V, X).$$

It follows from part (1) of Theorem 1.1 that

$$0 \leq I(V - X, V - X) = I(V, V) - 2I(V, X) + I(X, X) = -I(V, V) + I(X, X),$$

with equality holds if and only if $V - X = 0$. □

¶ Proof of Theorem 1.1, part (1).

We need the following lemma which is essentially Theorem 2.6 in Lecture 13:

Lemma 1.3. *Suppose $\mathcal{E}_p = \mathcal{E} \cap T_p M$ contains a line segment $[0, l]X_p$. Let $\varphi : [0, l] \rightarrow \mathcal{E}_p$ be a piecewise smooth curve with $\varphi(0) = 0, \varphi(l) = lX_p$. Then for*

$$\gamma(t) := \exp_p(tX_p) \quad (0 \leq t \leq l) \quad \text{and} \quad \bar{\gamma}(t) := \exp_p(\varphi(t)) \quad (0 \leq t \leq l),$$

we have $L(\bar{\gamma}) \geq L(\gamma)$. Moreover, if \exp_p is non-singular along $[0, l]X_p$, then the equality $L(\bar{\gamma}) = L(\gamma)$ holds if and only if $\bar{\gamma}$ is a re-parametrization of γ .

Proof. WLOG, we may assume $\varphi(t) \neq 0$ for all $t \in (0, l)$. Write $\varphi(t) = r(t)e(t)$, where $r(t) = |\varphi(t)|$ and $e(t) \in S_p M$ has unit length. Then

$$\dot{\varphi}(t) = \dot{r}(t)e(t) + r(t)\dot{e}(t),$$

and $e(t) \perp \dot{e}(t)$. According to Gauss lemma,

$$|\dot{\gamma}(t)| = |(d \exp_p)_{\varphi(t)} \dot{\varphi}(t)| \geq |(d \exp_p)_{\varphi(t)}(\dot{r}(t)e(t))| = |\dot{r}(t)|.$$

Therefore,

$$L(\bar{\gamma}) \geq \int_0^l |\dot{r}(t)| dt \geq |r(l) - r(0)| = ||\varphi(l)| - |\varphi(0)|| = l|X_p| = L(\gamma).$$

If \exp_p is non-singular along $[0, l]X_p$, then for ε small enough \exp_p is non-singular in the ε -neighborhood of $[0, l]X_p$. Now suppose the equality holds, then we have

$$(d \exp_p)_{\varphi(t)} \dot{e}(t) = 0,$$

thus $\dot{e}(t) = 0$ for all t . It follows that $e(t)$ is a constant unit vector, i.e. $\varphi(t) = r(t)e$ for some $e \in S_p M$. Obviously $e = X_p/|X_p|$ is the direction vector of X_p . Moreover, \dot{r} cannot change sign. So $\bar{\gamma}(t) = \exp_p(\frac{r(t)}{|X_p|}X_p)$ is a re-parametrization of γ . \square

Remark. There is no conflict with Theorem 1.1(2). Suppose γ contains a conjugate point of p . Since the exponential map \exp_p is not even a local diffeomorphism at a conjugate point of p , it is possible that a curve that is close to γ cannot be realized as the image of a curve in $T_p M$ with the same endpoints under the exponential map.

Now we prove the “moreover” and \Leftarrow part in (1) of Theorem 1.1.

(1)Moreover: Suppose p has no conjugate point along γ . Find a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$$

and open neighborhoods V_i , $1 \leq i \leq k$, of the line segment $[t_i, t_{i+1}] \dot{\gamma}(0)$ in $T_p M$ so that \exp_p is a diffeomorphism on each V_i . Denote $U_i = \exp_p(V_i)$. According to our assumption on $\bar{\gamma}$, for ε small enough, $\bar{\gamma}([t_i, t_{i+1}]) \subset U_i$. Now define

$$\varphi(t) = (\exp_p|_{V_i})^{-1}(\bar{\gamma}(t)), \quad t_{i-1} \leq t \leq t_i.$$

Then $\varphi(t)$ is a piecewise smooth curve in $T_p M$ connecting 0 to $l\dot{\gamma}(0)$ with $\exp_p(\varphi(t)) = \bar{\gamma}(t)$. So the conclusion follows from Lemma 1.3.

(1) \Leftarrow : According to the part **(1)Moreover** that we just proved, if $p = \gamma(0)$ has no conjugate point along γ , then for any $X \in \mathcal{V}^0$, $I(X, X) \geq 0$. (Otherwise one can construct a variation with $L(\gamma_s) < L(\gamma)$.) If I is not positive definite on \mathcal{V}^0 , then $I(Y, Y) = 0$ for some $Y \in \mathcal{V}^0$. It follows that for any $Z \in \mathcal{V}^0$ and any $\lambda \in \mathbb{R}$,

$$0 \leq I(Y - \lambda Z, Y - \lambda Z) = -2\lambda I(Y, Z) + \lambda^2 I(Z, Z).$$

As a consequence, $I(Y, Z) = 0$ for any $Z \in \mathcal{V}^0$. In other words, Y is a Jacobi field. Since $q = \gamma(l)$ is not a conjugate point of p , and $Y(0) = 0, Y(l) = 0$, we must have $Y \equiv 0$. So I is positive definition on \mathcal{V}^0 .

¶ **Proof of Theorem 1.1, part (2) and (3).**

Lemma 1.4. *Suppose $q = \gamma(t_0)$ is NOT conjugate to $p = \gamma(0)$ along a geodesic $\gamma : [0, l] \rightarrow M$. Then for any $X_p \in T_pM$ and $X_q \in T_qM$, there exists a unique Jacobi field V along γ so that $V(0) = X_p$ and $V(t_0) = X_q$.*

Proof. Let \mathcal{J}_γ be the space of all Jacobi fields along γ . Define a mapping

$$\Theta : \mathcal{J}_\gamma \rightarrow T_pM \times T_qM, \quad V \mapsto \Theta(V) = (V(0), V(t_0)).$$

Since q is not a conjugate point of p , Θ is injective. But Θ is linear, and $\dim \mathcal{J}_\gamma = \dim(T_pM \times T_qM) = 2m$ are of same dimension, so Θ is a linear isomorphism. \square

Now we prove part (2) and (3) of Theorem 1.1.

(2) Moreover and **(2) \Leftarrow** : Let X be a nonzero Jacobi field along γ with $X(0) = 0, X(\bar{t}) = 0$. Note that $\nabla_{\dot{\gamma}(\bar{t})}X \neq 0$, otherwise X will be identically zero. Let Z be a smooth vector field along γ with

$$Z(0) = 0, \quad Z(l) = 0, \quad Z(\bar{t}) = -\nabla_{\dot{\gamma}(\bar{t})}X.$$

Denote $\gamma_1 = \gamma|_{[0, \bar{t}]}$, $\gamma_2 = \gamma|_{[\bar{t}, l]}$, $Z_1 = Z|_{[0, \bar{t}]}$ and $Z_2 = Z|_{[\bar{t}, l]}$. For any $\eta > 0$ we put

$$Y_\eta(t) := \begin{cases} Y_\eta^1 = X(t) + \eta Z_1(t), & \text{for } 0 \leq t \leq \bar{t}, \\ Y_\eta^2 = \eta Z_2(t), & \text{for } \bar{t} \leq t \leq l. \end{cases}$$

Then

$$I^{\gamma_1}(Y_\eta^1, Y_\eta^1) = -\eta |\nabla_{\dot{\gamma}(\bar{t})}X|^2 + \eta^2 I^{\gamma_1}(Z_1, Z_1)$$

and thus

$$I^\gamma(Y_\eta, Y_\eta) = -\eta |\nabla_{\dot{\gamma}(\bar{t})}X|^2 + \eta^2 I^{\gamma_1}(Z_1, Z_1) + \eta^2 I^{\gamma_2}(Z_2, Z_2) < 0$$

for η small enough. This proves the theorem.

(3) \Leftarrow : Fix any $c \in (0, l)$. For any $X \in \mathcal{V}^0$, we may write $X = X^i(t)e_i(t)$, where $\{e_i\}$ are orthonormal and parallel along γ with $e_1 = \dot{\gamma}$. Let

$$X^c(t) = X^i\left(\frac{lt}{c}\right)e_i(t), \quad 0 \leq t \leq c.$$

Then X^c is a vector field along $\gamma^c := \gamma|_{[0, c]}$ with $X^c(0) = 0, X^c(c) = 0$. Since p has no conjugate point along $\gamma|_{[0, c]}$, we get from (1) that $I(X^c, X^c) \geq 0$. It follows that $I(X, X) = \lim_{c \rightarrow a} I(X^c, X^c) \geq 0$. So I is positively semi-definite on \mathcal{V}^0 .

Obviously I is not positively definite on \mathcal{V}^0 , since if X is a nonzero Jacobi field along γ with $X(a) = 0, X(b) = 0$, then we have $I(X, X) = 0$.

(1) \Rightarrow : This follows from (2) (\Leftarrow) and (3) (\Leftarrow).

(2) \Rightarrow : This follows from (1) (\Leftarrow) and (3) (\Leftarrow).

(3) \Rightarrow : This follows from (1) (\Leftarrow) and (2) (\Leftarrow).

2. MORSE INDEX THEOREM

¶ Morse index of a geodesic.

Let's continue our "finite dimension manifold" v.s. "infinite dimension space $\mathcal{C}_{p,q}^{0,l}$ " analogue a bit further. For a finite dimensional manifold M , there is a remarkable theory, known as Morse theory, that relates the topology of M to the behavior of critical points of a Morse function [the existence is guaranteed by Sard's theorem]:

Suppose $f \in C^\infty(M)$ is a Morse function, i.e. all critical points of f are non-degenerate [i.e. the Hessian $\text{Hess}_p(f)$ at any critical point p of f is non-singular], then M has the homotopy type of a CW-complex whose λ -cells are in one-to-one correspondence with critical points of index λ of f [the index of a critical point p is the number of negative eigenvalues of $\text{Hess}_p(f)$]. For example, if a compact manifold M admits a Morse function with only two critical points, then they have to be maximum and minimum, so their indexes have to be m and 0. In this case one can prove that M is homeomorphic to S^m .

What is the analogue in our setting? We already have

- critical points \leftrightarrow geodesics $\gamma : [0, l] \rightarrow M$,
- Hessian at the critical point \leftrightarrow the index form I of γ defined on \mathcal{V}^0 .

So "non-degenerate critical point" should be translated to "the index form I of γ defined on \mathcal{V}^0 has trivial null space". On the other hand, we have seen last time that $I(V, W) = 0$ for all $W \in \mathcal{V}^0$ if and only if V is Jacobi field. So

$$\begin{aligned} & \text{the index form } I \text{ of } \gamma \text{ defined on } \mathcal{V}^0 \text{ has trivial null space} \\ \iff & \text{there is no Jacobi field } V \text{ along } \gamma \text{ with } V(0) = V(l) = 0 \\ \iff & q = \gamma(l) \text{ is not a conjugate point of } p = \gamma(0) \text{ along } \gamma. \end{aligned}$$

So we get

- non-degenerate critical point \leftrightarrow geodesics $\gamma : [0, l] \rightarrow M$ so that $q = \gamma(l)$ is not a conjugate point of $p = \gamma(0)$ along γ ,
- the index of a non-degenerate critical point \leftrightarrow the dimension of the maximal subspace of \mathcal{V}^0 on which I is negative definite.

By the explanations above, we are naturally led to study

Definition 2.1. Let (M, g) be a Riemannian manifold, and $\gamma : [0, l] \rightarrow M$ a geodesic. We will call

$$\text{ind}(\gamma) = \max \dim \{ \mathcal{A} \subset \mathcal{V}^0 \mid I|_{\mathcal{A}} \text{ is negatively definite} \}$$

the *index* of γ , and call

$$N(\gamma) = \dim \{ X \in \mathcal{V}^0 \mid I(X, Y) = 0 \text{ for all } Y \in \mathcal{V}^0 \}$$

the *nullity* of γ .

Remark. The geometric meanings of $N(\gamma)$ and $\text{ind}(\gamma)$ are clear:

- (1) The nullity $N(\gamma)$ equals the multiplicity of $q = \gamma(l)$ as a conjugate point of $p = \gamma(0)$. In particular, $N(\gamma) = 0$ if $q = \gamma(l)$ is not conjugate to p along γ .
- (2) The index $\text{ind}(\gamma)$ is “the number of independent directions” towards which γ can be deformed to shorter curves with the same endpoints.

¶ Morse index theorem.

It turns out that there is a well-developed infinite dimensional Morse theory:

As in the finite dimensional case one can show that there exists p, q so that all geodesics connecting p and q are non-degenerate in the above sense. Again $\mathcal{C}_{pq} = \mathcal{C}_{p,q}^{0,l}$ has the homotopy type of a CW-complex whose λ -cells are in one-to-one correspondence with geodesics of index λ . This also gives the topological information of M , since \mathcal{C}_{pq} is homotopy equivalent to \mathcal{C}_{pp} , and $\pi_{k+1}(M) \cong \pi_k(\mathcal{C}_{pp})$. As an application we can explain Serre’s result we mentioned in the remark on page 3 in Lecture 14: by using algebraic topology Serre proved $H_i(\mathcal{C}_{pq}, \mathbb{R}) \neq 0$ for infinitely many i , which implies that for any k , there exists a geodesic whose index is great than k . By Morse index theorem below, there must be infinitely many geodesics connecting p and q .

So it is important to study the index of a geodesic. In what follows we will denote by γ^t the geodesic $\gamma|_{[0,t]}$, so that the corresponding index and nullity are $\text{ind}(\gamma^t)$ and $N(\gamma^t)$. The following theorem is fundamental in this direction:

Theorem 2.2 (Morse Index Theorem). *For any geodesic $\gamma : [0, l] \rightarrow M$, we have*

$$\text{ind}(\gamma) = \sum_{0 < t < l} N(\gamma^t) < \infty.$$

As a consequence, we get a quantitative version of Theorem 1.1(2) “moreover”,

Corollary 2.3. *For any geodesic $\gamma : [0, l] \rightarrow M$, $\gamma(0)$ has only finitely many conjugate points along γ . If we denote these conjugate points (except possibly $\gamma(b)$) by $\gamma(t_1), \dots, \gamma(t_k)$ ($a < t_1 < \dots < t_k < b$), then*

$$\text{ind}(\gamma) = \sum_{j=1}^k n_{\gamma,0}(t_j).$$

¶ Proof of Morse index theorem.

To prove the theorem, the first step is to reduce the infinite dimension space \mathcal{V}^0 to a finite dimensional subspace \mathcal{T}_1 , so that the index and nullity of $I|_{\mathcal{V}^0}$ is the same as $I|_{\mathcal{T}_1}$. As a consequence, $\text{ind}(\gamma)$ is finite.

To construct \mathcal{T}_1 , let’s recall that near any point one can find a strongly convex neighborhood U , so that any two points in U can be connected by a unique minimal geodesic which is contained in U . Now we use finitely many such strongly convex

neighborhoods U_0, U_1, \dots, U_k to cover γ , and take $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = l$ so that $\gamma([t_j, t_{j+1}]) \subset U_j$. Define

$$\mathcal{T}_1 = \{X \in \mathcal{V}^0 \mid X \text{ is a Jacobi field along each } \gamma|_{[t_j, t_{j+1}]}\}.$$

Note that $\gamma|_{[t_j, t_{j+1}]}$ contains no pair of conjugate points since \exp_q is non-singular on U_j for all $q \in U_j$. By Lemma 1.4, if $X \in \mathcal{T}_1$, then $X(t_j)$ and $X(t_{j+1})$ determine $X|_{[t_j, t_{j+1}]}$. As a consequence, the map

$$\Phi : \mathcal{T}_1 \rightarrow T_{\gamma(t_1)}M \oplus \dots \oplus T_{\gamma(t_k)}M, \quad \Phi(X) = (X(t_1), \dots, X(t_k))$$

is a linear isomorphism. In particular,

$$\dim \mathcal{T}_1 = mk < \infty.$$

Next define

$$\mathcal{T}_2 = \{X \in \mathcal{V}^0 \mid X(t_1) = 0, \dots, X(t_k) = 0\}.$$

Then

- Obviously $\mathcal{T}_1 \cap \mathcal{T}_2 = \{0\}$. On the other hand, for any $X \in \mathcal{V}^0$, if we let $V = \Phi^{-1}(X(t_1), \dots, X(t_k)) \in \mathcal{T}_1$, then $X - V \in \mathcal{T}_2$. So $\mathcal{V}^0 = \mathcal{T}_1 \oplus \mathcal{T}_2$.
- By formula (1), for any $X \in \mathcal{T}_1, Y \in \mathcal{T}_2$ we have $I(X, Y) = 0$.
- By Corollary 1.2, for any $Y \in \mathcal{T}_2$ we have $I(Y, Y) = \sum I^{\gamma_i}(Y|_{\gamma_i}, Y|_{\gamma_i}) > 0$. Thus $I|_{\mathcal{T}_2}$ is positive definite.

As a result, we get

$$(2) \quad \text{ind}(\gamma) = \max \dim \{\mathcal{A} \subset \mathcal{T}_1 \mid I|_{\mathcal{A}} \text{ is negatively definite}\}$$

and thus $\text{ind}(\gamma) \leq \dim \mathcal{T}_1 < +\infty$.

To get the precise formula of $\text{ind}(\gamma)$ in the theorem, we need the following lemmas which reveals the continuity property of the index $\text{ind}(\gamma^t)$:

Lemma 2.4. $\text{ind}(\gamma^t)$ is non-decreasing in t .

Lemma 2.5. $\text{ind}(\gamma^t) = \text{ind}(\gamma^{t-\varepsilon})$ for ε small enough.

Lemma 2.6. $\text{ind}(\gamma^{t+\varepsilon}) = \text{ind}(\gamma^t) + N(\gamma^t)$ for ε small enough.

Proof of Moser Index Theorem. According to the previous lemmas, we have

- $\text{ind}(\gamma^t)$ is a non-decreasing and left-continuous *step* function.
- For t small, $\text{ind}(\gamma^t) = 0$ (because γ^t is a minimal geodesic for t small).
- If $\gamma(t)$ is not a conjugate point of $\gamma(0)$, then $\text{ind}(\gamma^\tau)$ is constant for $\tau \in (t - \varepsilon, t + \varepsilon)$.
- If $\gamma(t)$ is a conjugate point of $\gamma(0)$, then $\text{ind}(\gamma^\tau)$ is a step function in $(t - \varepsilon, t + \varepsilon)$ with a “jump from right” of size $N(\gamma^t)$ at $\tau = t$

The theorem follows from these facts. □

¶ Proof of Lemmas.

Proof of Lemma 2.4. Let $0 < t < s$. For any $X \in \mathcal{V}_{\gamma^t}^0$, we can extend X to $X' \in \mathcal{V}_{\gamma^s}^0$ by zero extension. Since $I^{\gamma^t}(X, X) = I^{\gamma^s}(X', X')$, the conclusion follows. \square

Proof of Lemma 2.5. Suppose $t \in (t_j, t_{j+1})$, where t_j is a subdivision of $(0, l)$ as above. Then by (2),

$$\text{ind}(\gamma^t) = \{\mathcal{A} \subset \mathcal{T}_1^{\gamma^t} \mid I|_{\mathcal{A}} \text{ is negatively definite}\},$$

where $\mathcal{T}_1^{\gamma^t}$ is the set of piecewise smooth vector fields in $\mathcal{V}_{\gamma^t}^0$ which are Jacobi fields on each (t_i, t_{i+1}) . Under the isomorphism $\Phi^t : \mathcal{T}_1^{\gamma^t} \rightarrow T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M$ the restriction of I on $\mathcal{T}_1^{\gamma^t}$ becomes a bilinear form I^t defined on $T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M$. It depends smoothly on t since for $\vec{X}, \vec{Y} \in T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M$, by (1),

$$I^t(\vec{X}, \vec{Y}) = \sum_{i < j} I^{\gamma^i}(X^i, Y^i) + I^{\tilde{\gamma}^t}(X^t, Y^t) = \langle \nabla_{\dot{\gamma}(t_j)} X^t, Y^t \rangle + \text{terms independent of } t,$$

where $\gamma^i = \gamma|_{[t_i, t_{i+1}]}$ and $X^i = (\Phi^t)^{-1}(\vec{X})|_{[t_i, t_{i+1}]}$ for $i < j$, while $\tilde{\gamma}^t = \gamma|_{[t_j, t]}$ and $X^t = (\Phi^t)^{-1}(\vec{X})|_{[t_j, t]}$. So if I^t is negative definite on a subspace, so is $I^{t-\varepsilon}$ for ε small, i.e. $\text{ind}(\gamma^{t-\varepsilon}) \geq \text{ind}(\gamma^t)$. The conclusion follows from Lemma 2.4. \square

Proof of Lemma 2.6. As in the proof of Lemma 2.5, we identify the restriction of I on $\mathcal{T}_1^{\gamma^t}$ with the bilinear form I^t defined on $T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_k)}M$ which depends smoothly on t . Then I^t is positive definite on a subspace of dimension $mk - \text{ind}(\gamma^t) - N(\gamma^t)$. It follows that $I^{t+\varepsilon}$ is positive definite on this subspace for ε small enough. This implies

$$\text{ind}(\gamma^{t+\varepsilon}) \leq mk - (mk - \text{ind}(\gamma^t) - N(\gamma^t)) = \text{ind}(\gamma^t) + N(\gamma^t).$$

On the other hand, we have

- By the same continuity argument, $I^{t+\varepsilon}$ is negative definite on the space where I^t is negative definite.
- $I^{t+\varepsilon}$ is negative definite on the null space of I^t : If $0 \neq \vec{X}$ is in the null space of I^t , then $(\Phi^t)^{-1}(\vec{X})$ is a Jacobi field along γ^t and $X_{t_j} \neq 0$ (since there is no Jacobi field that vanishes at both t_j and t). So if we denote by \tilde{X}^t the zero extension of X^t to $[t_j, t + \varepsilon]$, then by Corollary 1.2,

$$I^{\tilde{\gamma}^{t+\varepsilon}}(X^{t+\varepsilon}, X^{t+\varepsilon}) < I^{\tilde{\gamma}^{t+\varepsilon}}(\tilde{X}^t, \tilde{X}^t) = I^{\tilde{\gamma}^t}(X^t, X^t)$$

and thus $I^{t+\varepsilon}(\vec{X}, \vec{X}) < I^t(\vec{X}, \vec{X}) = 0$.

So we get $\text{ind}(\gamma^{t+\varepsilon}) \geq \text{ind}(\gamma^t) + N(\gamma^t)$ which finishes the proof. \square