LECTURE 21: CUT LOCUS

1. The conjugate and cut Locus

¶ The conjugate locus.

Let (M, g) be a complete Riemannian manifold.

Definition 1.1. For any $p \in M$, we call

$$\widetilde{\mathrm{Con}}(p) = \{v \in T_pM \mid (d\exp_p)_v \text{ is singular}\}$$

the *conjugate locus* of p of p in T_pM, and call

$$\mathrm{Con}(p) = \{q \in M \mid q \text{ is a conjugate point of } p \text{along some geodesic}\}$$

the *conjugate locus* of p in M.

Obviously $\widetilde{\mathrm{Con}}(p)$ is a closed subset in T_xM. We first prove

Lemma 1.2. *The conjugate locus $\widetilde{\mathrm{Con}}(p)$ and $\mathrm{Con}(p)$ are measure zero sets in T_pM and M respectively.*

Proof. By Morse index theorem, for each $v \in S_pM$, the set $\{tv \mid t > 0\} \cap \widetilde{\mathrm{Con}}(p)$ is discrete. So the conclusion follows. □

As we have seen last time, the first conjugate point in each direction is important: a geodesic γ starting from p is locally length minimizing if the endpoint is before the first conjugate point, and is not locally length minimizing if the endpoint is after the first conjugate point. We define $k : S_pM \to \mathbb{R} \cup \{+\infty\}$ to be the function

$$k(v) = \inf\{t > 0 \mid tv \in \widetilde{\mathrm{Con}}(p)\},$$

and we set $k(v) = +\infty$ if there is no conjugate point in the direction v.

Proposition 1.3. *The function k is continuous.*

Proof. Suppose $v_i \in S_pM$ and $v_i \to v$. We want to prove $k(v_i) \to k(v)$.

We need an observation:

> **Continuity Observation:** Suppose $v_i \in S_pM$ and $v_i \to v$. Fix l and denote
>
> $$\gamma_i = \exp(tv_i) \quad \text{and} \quad \gamma(t) = \exp(tv) \quad (0 \le t \le l),$$
>
> By the smooth dependence of geodesics on initial values, γ_i converges uniformly to γ. By using parallel transport [first from $\gamma_i(t)$

to $\gamma_i(0) = p$ along γ_i, then from $p = \gamma(0)$ to $\gamma(t)$ along γ] one can identify each $T_{\gamma_i(t)}M$ with $T_{\gamma(t)}M$. As a result, we get an identification $\Psi_i : \mathcal{V}^0_{\gamma_i} \to \mathcal{V}^0_\gamma$. Furthermore, for any $X \in \mathcal{V}^0_\gamma$, again by smooth dependence, $I^{\gamma_i}(\Psi_i^{-1}(X), \Psi_i^{-1}(X)) \to I^\gamma(X, X)$.

Case 1: $k(v) = c < +\infty$ We want to prove $k(v_i) \to k(v) = c$.

- Take $l = c + \varepsilon$. Then $\mathrm{ind}(\gamma) \geq 1$, i.e. $I^\gamma(X, X) < 0$ for some $X \in \mathcal{V}^0_\gamma$. By continuity observation above, $I^{\gamma_i}(\Psi_i^{-1}(X), \Psi_i^{-1}(X)) < 0$ for i large enough. So $\mathrm{ind}(\gamma_i) \geq 1$ and thus $k(v_i) \leq l + \varepsilon$ for i large enough.
- Take $l = c - \varepsilon$. Since $\mathcal{V}^0_\gamma$ is infinite dimensional, we can't conclude that I^{γ_i} is positive definite directly from the fact I^γ is positive definite. However, from the proof of Morse index theorem, the maximal negative definite space of I^γ can be taken to be a subspace in
$$\mathcal{T}^\gamma_1 = \{X \in \mathcal{V}^0_\gamma \mid X \text{ is Jacobi on each } [t_i, t_{i+1}]\},$$
or any other direct sum complement of
$$\mathcal{T}^\gamma_2 = \{V \in \mathcal{V}^0_\gamma \mid X(t_1) = \cdots = X(t_k) = 0\},$$
where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$ are chosen so that $\gamma([t_j, t_{j+1}]) \subset U_j$, and $U_0, U_1, \cdots, U_k$ are strongly convex open subsets that cover γ. Since γ_i converges to γ uniformly, we have $\gamma_i([t_j, t_{j+1}]) \subset U_j$ for i large enough. We may use the same partition for all γ_i. Now we have $\Psi_i(\mathcal{T}^{\gamma_i}_2) = \mathcal{T}^\gamma_2$, and thus
$$\mathcal{V}^0_{\gamma_i} = \Psi_i^{-1}(\mathcal{V}^0_\gamma) = \Psi^{-1}(\mathcal{T}^\gamma_1 \oplus \mathcal{T}^\gamma_2) = (\Psi_i)^{-1}(\mathcal{T}^\gamma_1) \oplus \mathcal{T}^{\gamma_i}_2,$$
so $\mathrm{ind}(\gamma_i)$ also equals the maximal dimension of subspace in $(\Psi_i)^{-1}(\mathcal{T}^\gamma_1)$ on which I^{γ_i} is negative definite. Now suppose $I^{\gamma_i}(X_i, X_i) < 0$ for $\Psi_i(X_i) \in \mathcal{T}^\gamma_1$ which can be taken so that $\sup |X_i| = 1$, then we may take a convergent subsequence of $\Psi_i(X_i) \in \mathcal{T}^\gamma_1$ and conclude the existence of $X \neq 0$ with $I^\gamma(X, X) \leq 0$, which contradicts with the fact I^γ is positive definite. So we get $k(v_i) \geq l - \varepsilon$ for i large enough.

Case 2: $k(v) = +\infty$ Suppose to the contrary that $k(v_i)$ has a bounded subsequence. Without less of generality, suppose $k(v_i) \leq c$ for all i. Take $l = c$. By the same argument above we get a contradiction. □

As a consequence we get

Corollary 1.4. *The set of first conjugate points of p in T_pM is closed.*

Proof. If t_iv_i are first conjugate points of p and $t_iv_i \to v$, then
$$k(v/|v|) = \lim_{i\to\infty} k(v_i) = \lim_{i\to\infty} t_i = |v|.$$
So v is the first conjugate point of p in the direction $v/|v|$. □

¶ The cut locus.

Let γ be the normal geodesic in (M, g) with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Suppose (M, g) is complete so that γ can be defined on $\mathbb{R}$. Let's concentrate for $t > 0$, which corresponds to the part of the geodesic in the direction v. For t small $\gamma|_{[0,t]}$ is length minimizing between $\gamma(0)$ and $\gamma(t)$. For general t, it may happen that either $\gamma|_{[0,t]}$ is length minimizing between $\gamma(0)$ and $\gamma(t)$ for all $t > 0$, or there exists t_0 such that $\gamma|_{[0,t]}$ is no longer length minimizing between $\gamma(0)$ and $\gamma(t)$ for all $t > t_0$.

Definition 1.5. Let (M, g) be a complete Riemannian manifold, $p \in M$ a point, and $\gamma : [0, \infty) \to M$ a normal geodesic with $\gamma(0) = p$. If

$$t_0 := \sup\{t \mid \gamma([0, t]) \text{ is a minimizing geodesic}\} < +\infty,$$

then we will call $\gamma(t_0)$ the *cut point* of p along γ.

- The *cut locus* of p in M is defined to be the set $\mathrm{Cut}(p)$ of all cut points of p along all geodesics that start from p
- The *cut locus* of p in T_pM is defined to be the set $\widetilde{\mathrm{Cut}(p)}$ of all vectors $v \in T_pM$ so that $\exp_p(v)$ is a cut point.

Remark. If M is compact, then $\mathrm{Cut}(p) \neq \varnothing$ for all p.

Example. On $\mathbb{R}^m$ and $\mathbb{H}^m$ (endowed with the canonical metrics), there exists only one normal minimizing geodesic joining any two given points. So $\mathrm{Cut}(p) = \varnothing$ for all p.

Example. For S^m with the round metric, $\mathrm{Cut}(p) = \{\bar p\}$ for any $p \in M$, where $\bar p = -p$ is the antipodal point of p. Note that $\bar p$ is also the first conjugate point of p.

Example. For the cylinder $S^1 \times \mathbb{R}$ endowed with the canonical metric, if $p = (e^{i\theta_0}, z_0)$, then $\mathrm{Cut}(p) = \{(e^{i(\theta_0+\pi)}, z) \mid z \in \mathbb{R}\}$ is the vertical line "opposite to p". Note that p has no conjugate points at all.

By definition we have

Lemma 1.6. *For any $q \notin \mathrm{Cut}(p)$, there exists a unique minimizing geodesic joining p to q.*

Proof. If there exist two minimizing geodesics γ, σ of length l joining p to q, then γ is minimizing between p and q, and is no longer minimizing after q:

> The curve $\bar\gamma$ defined by connecting σ with $\gamma|_{[l,l+\varepsilon]}$ is a piecewise smooth but not smooth curve connecting p to $\gamma(l + \varepsilon)$ whose length is $l + \varepsilon$. But according to the first variation formula, any piecewise smooth but not smooth curve is not a minimizing curve[c.f. Corollary 2.7 in Lecture 16]. We conclude that $\gamma|_{[0,l+\varepsilon]}$ is also not a minimizing curve, since it has the same length as $\bar\gamma$.

So $q \in \mathrm{Cut}(p)$. □

¶ Cut points v.s. first conjugate points.

The following theorem relates cut points with first conjugate points:

Theorem 1.7. *Suppose $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along a normal geodesic γ, then at least one of the following assertion holds:*

(1) *$\gamma(t_0)$ is the first conjugate point of p along γ.*

(2) *$\gamma(t_0)$ is the first point along γ so that there exists another normal geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ with length $L(\sigma) = t_0 = L(\gamma|_{[0,t_0]})$.*

Proof. Take a decreasing sequence $t_i \to t_0^+$. Let σ_i be a normal minimizing geodesic connecting p to $\gamma(t_i)$. Then by definition of cut point, $s_i := L(\sigma_i) < t_i$. Note that $\{\dot\sigma_i(0)\}$ is a sequence in the unit sphere S_pM. By passing to a subsequence, we may assume $\dot\sigma_i(0) \to X_p \in S_pM$. Let σ be the normal geodesic with $\sigma(0) = p$, $\dot\sigma(0) = X_p$. Then by continuity, σ is a minimizing geodesic connecting p to $\gamma(t_0)$, thus $L(\sigma) = t_0$.

Case 1: $X_p = \dot\gamma(0)$. Since $s_i < t_i$, we have $s_i\dot\sigma_i(0) \neq t_i\dot\gamma(0)$. But

$$\exp_p(s_i\dot\sigma_i(0)) = \sigma_i(s_i) = \gamma(t_i) = \exp_p(t_i\dot\gamma(0)),$$

so $\exp_p$ is not a local diffeomorphism near $t_0\dot\gamma(0)$. So $\gamma(t_0)$ is a conjugate point of p. Obviously it is the first conjugate point, otherwise $\gamma([0, t_0])$ is not minimizing.

Case 2: $X \neq \dot\gamma(0)$. Then σ is a geodesic that is different from γ. We have

$$t_0 = L(\gamma|_{[0,t_0]}) \leq L(\sigma) = \lim_i L(\sigma_i) \leq \lim_i t_i = t_0.$$

So $L(\sigma) = t_0$. To show that $\gamma(t_0)$ is the first point along γ with this property, we argue by contradiction. If there exists a $\bar t < t_0$ and a normal geodesic $\bar\sigma$ connecting p to $\gamma(\bar t)$ so that $L(\bar\sigma) = \bar t$, then by the argument in the proof of Lemma 1.6, $\gamma|_{[0,t_0]}$ is not a minimizing curve. This contradicts with the definition of cut point. □

Corollary 1.8. *If $q \in \mathrm{Cut}(p)$, then $p \in \mathrm{Cut}(q)$.*

Proof. If q is the cut point of p along γ, then γ is minimizing between p and q. It follows that the "opposite geodesic" $-\gamma$ is also minimizing between q and p. Moreover, by the theorem above, either q is the first conjugate point of p along γ, or there exists a different normal geodesic σ joint p to q which has length $L(\sigma) = \mathrm{dist}(p, q)$. In both cases $-\gamma$ is no longer minimizing after p. So $p \in \mathrm{Cut}(q)$. □

Remark. One can show that the function $f : SM \to \mathbb{R} \cup \{\infty\}$ defined by

$$f(p, X_p) = \begin{cases} t_0, & \text{if } \gamma_{p,X_p}(t_0) \text{ is the cut point of } p \text{ along } \gamma, \\ +\infty, & \text{if } p \text{ has no cut point along } \gamma_{p,X_p}. \end{cases}$$

is a continuous function. It follows that $\mathrm{Cut}(p)$ is a closed subset in M. It has measure zero since there is at most one cut point in each directionin M. Note that by definition $f \leq k$.

2. The distance function

¶ Smoothness of Distance Function.

Now let's fix $p \in M$ and consider the distance function

$$d_p : M \to \mathbb{R}, \quad d_p(q) = \text{dist}(p, q).$$

As we have already seen, d_p is a continuous function. However, it is not hard to see that $d_p \notin C^\infty(M)$. In fact, d_p is never smooth at p.

Example. Consider (S^2, g_{S^2}). Let $\bar{p} = -p$ be the antipodal point of p. Then for q near $\bar{p}$, $d_p(q) = \pi - d_{\bar{p}}(q)$. It follows that d_p is also not smooth at $\bar{p}$.

Theorem 2.1. *The function d_p is smooth on $M \setminus \text{Cut}(p) \cup \{p\}$. Moreover, for each $q \in M \setminus \text{Cut}(p) \cup \{p\}$, if we let γ^q be the unique normal minimizing geodesic from p to q, then the gradient of d_p at q is*

$$(\nabla d_p)(q) = \dot{\gamma}^q(d_p(q)).$$

Proof. For each $q \in M \setminus \text{Cut}(p) \cup \{p\}$, let γ^q be the unique normal minimizing geodesic from p to q and denote $X^q = \dot{\gamma}^q(0) \in S_pM$. Let

$$A = \{L(\gamma^q)X^q \mid q \in M \setminus \text{Cut}(p) \cup \{p\}\}.$$

Then $A \subset T_pM \setminus \{0\}$ is an open set and

$$\exp_p : A \to M \setminus \text{Cut}(p) \cup \{p\}$$

is smooth. Moreover, at each vector in A, $\exp_p$ is non-singular and thus a local diffeomorphism. Since $\exp_p$ is globally one-to-one on A, it is a diffeomorphism from A to $M \setminus \text{Cut}(p) \cup \{p\}$. It follows that

$$\exp_p^{-1} : M \setminus \text{Cut}(p) \cup \{p\} \to A \subset T_pM \setminus \{0\}]$$

is smooth. Thus $d_p(q) = |\exp_p^{-1}(q)|$ is smooth on $M \setminus \text{Cut}(p) \cup \{p\}$.

To calculate its gradient at q, we choose any $X_q \in T_qM$ and let $\sigma(s)$ be a smooth curve in $M \setminus \text{Cut}(p) \cup \{p\}$ tangent to X_q at $q = \sigma(0)$. Now we consider the variation of γ^q so that γ^q_s be the unique minimizing geodesic from p to $\sigma(s)$. Observe that the variation field vector [which is a Jacobi field] of this variation at the point q is exactly X_q. So according to the first variation formula,

$$X_q(d_p) = \left.\frac{d}{ds}\right|_{s=0} d_p(\sigma(s)) = \left.\frac{d}{ds}\right|_{s=0} L(\gamma^q_s) = \langle X_q, \dot{\gamma}^q(d_p(q))\rangle.$$

It follows that $(\nabla d_p)(q) = \dot{\gamma}^q(d_p(q))$. □

Remarks. One can show that if there exists two minimizing geodesic from p to q, then d_p is not differentiable at q.

¶ Hessian of the distance function.

By using the second variation formula one can calculate the Hessian of d_p on $M \setminus \mathrm{Cut}(p) \cup \{p\}$. Recall that the Hessian of a smooth function f is

$$\begin{aligned}(\nabla^2 f)_q(X_q, Y_q) = (X_q Y f - \nabla_{X_q} Y) f &= \nabla_{X_q}(\langle \nabla f, Y\rangle) - \langle \nabla f, \nabla_{X_q} Y\rangle \\ &= \langle \nabla_{X_q} \nabla f, Y\rangle.\end{aligned}$$

Now Let $\gamma_s : [0, l] \to M$ be geodesic variation of γ by minimizing geodesics with $\gamma_s(0) = p$ [so its variation field X is a normal Jacobi field along γ with $X(0) = 0$]. Then

$$(\nabla^2 d_p)_q(X_q, Y_q) = \langle (\nabla_X \nabla d_p)_q, Y_q\rangle = \langle \nabla_{X(q)} \dot\gamma^q, Y_q\rangle = \langle \nabla_{\dot\gamma^q(l)} X, Y_q\rangle, \quad \forall Y_q \in T_q M.$$

So we proved

Proposition 2.2. *Suppose $q \notin \mathrm{Cut}(p) \cup \{p\}$. Let $\gamma : [0, l] \to M$ be the unique length minimizing normal geodesic connecting p to q, and let X be a normal Jacobi field along γ with $X(0) = 0$. Denote $X_q = X(l)$. Then for any $Y_q \in T_q M$,*

$$(\nabla^2 d_p)_q(X_q, Y_q) = \langle \nabla_{\dot\gamma(l)} X, Y_q\rangle.$$

Here are two special cases that will be quite useful later:

Corollary 2.3. *Suppose $q \notin \mathrm{Cut}(p) \cup \{p\}$, and $\gamma : [0, l] \to M$ the unique length minimizing normal geodesic connecting p to q.*

(1) For any $Y_q \in T_q M$, $(\nabla^2 d_p)_q(\dot\gamma(l), Y_q) = \langle \nabla_{\dot\gamma(l)} \dot\gamma, Y_q\rangle = 0$.
(2) For any normal Jacobi field X along γ with $X(0) = 0$,

$$(\nabla^2 d_p)_q(X_q, X_q) = \langle \nabla_{\dot\gamma(q)} X, X_q\rangle = I(X, X).$$

Note that the "singularity" of d_p at the point p is not too bad: one can always remove the singularity at p by considering the function d_p^2 instead. So it is reasonable to study $(\nabla^2 d_p^2)_p$. In general, for any smooth function $f \in C^\infty(M)$ and $X_p \in T_p M$, if we let γ be the geodesic $\gamma(t) = \exp_p(t X_p)$, then "the second order derivative of f along γ" is

$$\frac{d^2}{dt^2} f \circ \gamma(t) = \frac{d}{dt}\frac{d}{dt}(f \circ \gamma) = \frac{d}{dt}\langle \nabla f, \dot\gamma\rangle = \langle \nabla_{\dot\gamma} \nabla f, \dot\gamma\rangle = (\nabla^2 f)_{\gamma(t)}(\dot\gamma, \dot\gamma).$$

So we get

Lemma 2.4. $(f \circ \gamma)''(t) = (\nabla^2 f)_{\gamma(t)}(\dot\gamma, \dot\gamma)$.

On the other hand, if X_p is a normal vector, i.e. γ is a normal geodesic, then for t small enough we have $d_p^2(\gamma(t)) = t^2$. So we get, for $X_p \in S_p M$,

$$(\nabla^2 d_p^2)_p(X_p, X_p) = 2g(X_p, X_p)$$

and thus by polarization, we get

Proposition 2.5. *The Hessian of d_p^2 at p is*

$$(\nabla^2 d_p^2)_p = 2g.$$