LECTURE 22: THEOREMS ON CURVATURE V.S. TOPOLOGY

1. Complete Riemannian manifolds with non-positive curvature

¶ A Hessian comparison for d_p^2.

Let (M,g) be a Cartan-Hadamard manifold, i.e. a complete simply-connected Riemannian manifold with non-positive curvature. We have seen that there is no conjugate point for such manifolds, and by Cartan-Hadamard theorem, $\exp_p : T_pM \to M$ is diffeomorphism. In particular, between any pair of points there is a unique geodesic [which has to be minimizing], and there is no cut point for Cartan-Hadamard manifolds.

We first prove that d_p^2 is strictly convex on Cartan-Hadamard manifolds:

Proposition 1.1. *For any p in a Cartan-Hadamard manifold (M,g), $\nabla^2 d_p^2 \ge 2g$. Moreover, if the sectional curvature is negative, then $\nabla^2 d_p^2 > 2g$ at any $q \ne p$.*

Proof. Let γ be a normal geodesic with $\gamma(0)=p$. For any $v \in T_qM$, where $q=\gamma(l)$,

$$(\nabla^2 d_p^2)_q(\dot\gamma(l), v) = \langle \nabla_{\dot\gamma(l)} \nabla d_p^2, v\rangle = \langle \nabla_{\dot\gamma(l)}(2t\dot\gamma), v\rangle = 2\langle \dot\gamma(l), v\rangle.$$

It remains to prove that for $v \perp \dot\gamma(l)$, one has $(\nabla^2 d_p^2)_q(v,v) \ge 2\langle v, v\rangle$.

Let V be the normal Jacobi field along γ with $V(0)=0, V(l)=v$. Then

$$(\nabla^2 d_p^2)_{\gamma(t)}(V,V) = \langle \nabla_{V(t)} \nabla d_p^2, V\rangle = \langle \nabla_{V(t)}(2t\dot\gamma), V\rangle = 2t\langle \nabla_{V(t)}(\dot\gamma), V\rangle = 2t\langle \nabla_{\dot\gamma(t)} V, V\rangle.$$

Denote $f(t) = \frac{\langle \nabla_{\dot\gamma(t)} V, V(t)\rangle}{\langle V(t), V(t)\rangle}$. Then $\lim_{t\to 0+} f(t) = +\infty$, since $V(t) = t\nabla_{\dot\gamma(0)}V + O(t^2)$. By definition we have

$$f'(t) = \frac{\langle \nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)} V, V\rangle + \langle \nabla_{\dot\gamma(t)} V, \nabla_{\dot\gamma(t)} V\rangle}{\langle V, V\rangle} - 2\frac{\langle \nabla_{\dot\gamma(t)} V, V\rangle^2}{\langle V, V\rangle^2} \ge -f^2(t),$$

where we used the fact $\langle \nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)} V, V\rangle = \langle R(\dot\gamma, V)\dot\gamma, V\rangle = -K(\dot\gamma, V)|\dot\gamma \wedge V|^2 \ge 0$ and Cauchy-Schwartz inequality $\langle \nabla_{\dot\gamma(t)} V, \nabla_{\dot\gamma(t)} V\rangle\langle V, V\rangle \ge \langle \nabla_{\dot\gamma(t)} V, V\rangle^2$. It follows

$$t \ge \int_0^t \frac{-f'(\tau)}{f^2(\tau)} d\tau = \int_0^t \left(\frac{1}{f(\tau)}\right)' d\tau = \frac{1}{f(t)} - \lim_{\tau\to 0+} \frac{1}{f(\tau)} = \frac{1}{f(t)},$$

So we get $tf(t) \ge 1$ for all t, and the first conclusion follows.

For the second conclusion, one just notice $f'(t) > -f^2(t)$ for $t > 0$. $\square$

¶ The fundamental group of Riemannian manifolds with $K \le 0$.

As a first application, we prove

Theorem 1.2 (Cartan). *Let (M, g) be a Cartan-Hadamard manifold, and $\varphi : M \to M$ an isometry with $\varphi^n = \mathrm{Id}$ for some n. Then φ admits a fixed point.*

Proof. Fix $p \in M$ and consider the function

$$g : M \to \mathbb{R}, \quad q \mapsto g(q) = d^2(q, p) + d^2(q, \varphi(p)) + \cdots + d^2(q, \varphi^{n-1}(p)).$$

Then g is strictly convex and $g(q) \to +\infty$ as $d(q, p) \to +\infty$. So g admits a unique minimum at some point $\tilde{p}$. Since $g(\varphi(q)) = g(q)$ we conclude $\varphi(\tilde{p}) = \tilde{p}$. □

As a corollary we get

Corollary 1.3. *Let (M, g) be a complete Riemannian manifold with non-positive sectional curvature. Then $\pi_1(M)$ is torsion free [i.e. no nontrivial finite order element].*

Proof. If $\pi_1(M)$ admits a finite order element τ, then the corresponding Deck transformation $f_\tau : \widetilde{M} \to \widetilde{M}$ is of finite order, and thus by Cartan's theorem above, f_τ admits a fixed point. This implies $f_\tau = \mathrm{Id}$ and $\tau = e$. □

As an immediate consequence, we see

Corollary 1.4. *For any compact manifold M, $\mathbb{RP}^m \times M$ admits no metric of non-positive sectional curvature.*

¶ A weak cosine law for Cartan-Hadamard manifolds.

As a second application of the convexity of d_p^2, we prove the following weak cosine law for Cartan-Hadamard manifolds.

Proposition 1.5. *Let (M, g) be Cartan-Hadamard manifold. Consider the geodesic triangle with vertices $p_1, p_2, p_3 \in M$. Let a, b, c be the lengths of sides and A, B, C be the corresponding opposite angles. Then*

(1) $a^2 + b^2 - 2ab\cos C \le c^2$.
(2) $A + B + C \le \pi$.

Further more, if the sectional curvature is negative, then these inequalities are strict.

Proof. Let γ be a normal geodesic from p_3 to p_1, and let $f(t) = d^2(p_2, \gamma(t))$. Then

$$f(0) = d^2(p_2, p_3) = a^2$$

and

$$f'(0) = 2d(p_2, p_3)\langle \nabla d_{p_2}, \dot\gamma(0)\rangle = -2a\cos C.$$

By Lemma 2.4 in Lecture 21, $f''(\tau) = (\nabla^2 d_p^2)_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))$. By Proposition 1.1, $f''(\tau) \ge 2$ for all τ. Thus we get

$$c^2 = f(b) \ge f(0) + f'(0)b + b^2 = a^2 + b^2 - 2ab\cos C.$$

To prove (2), one may compare the triangle in the plane with sides a, b, c [which satisfies the triangle inequality since they are distances of three points in a Riemannian manifold]. Denote the angles by A', B', C'. Then by the cosine law in $\mathbb{R}^2$ we get

$$A \leq A', B \leq B', C \leq C',$$

which implies $A + B + C \leq \pi$.

Finally if the sectional curvature is negative, then by Proposition 1.1, $f''(\tau) > 2$ for $\tau \neq 0$ and the conclusion follows. □

¶ Preissman's theorem.

What if M is not simply connected? We have just seen that if M admits a non-positive sectional curvature metric, then $\pi_1(M)$ is torsion free. It turns out that if M admits a negative sectional curvature metric, then any nontrivial abelian subgroup of $\pi_1(M)$ is an infinite cyclic group generated by one element:

Theorem 1.6 (Preissman)**.** *Let (M, g) be a compact Riemannian manifold with negative sectional curvature, and let $\{1\} \neq H \subset \pi_1(M)$ be a nontrivial abelian subgroup of the fundamental group. Then $H \cong \mathbb{Z}$.*

Remark. The theorem was strengthened by Byers to: under the same assumption, any nontrivial solvable subgroup of $\pi_1(M)$ is infinite cyclic.

Example. For any closed surface M_g of genus $g \geq 2$, there is Riemannian metric of constant negative sectional curvature. The fundamental group of M_g is

$$\langle a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_1 b_1 a_1^{-1} b_1^{-1} = e \rangle,$$

which is not abelian, but all its abelian subgroups are isomorphic to $\mathbb{Z}$.

As an immediate consequence, we see

Corollary 1.7. *Suppose $m \geq 2$. For any compact manifold M, $\mathbb{T}^m \times M$ admits no metric of negative sectional curvature.*

Remark. It was first proved by Gao and Yau in 1986 that any compact manifold of dimension 3 admits a metric with negative Ricci curvature. Then in 1994, Lohkamp proved that any manifold of dimension at least 3 admits a complete Riemannian metric of negative Ricci curvature. So there is no topological constraint for a manifold of dimension ≥ 3 to admit Riemannian metrics with negative Ricci curvature.

The idea of proof is as follows: Realize Deck transformations associated with all $\alpha \in H$ as "a discrete family of translations along a fixed geodesic". As a result, a nontrivial discrete subgroups of H corresponds to a nontrivial subgroup of $\mathbb{R}$, which is isomorphic to $\mathbb{Z}$.

¶ Translations in Cartan-Hadamard manifolds.

So we need to introduce the concept of translation.

Definition 1.8. Let (M, g) be a complete simply-connected Riemannian manifold, and $\gamma : \mathbb{R} \to M$ a geodesic. An isometry $f : (M, g) \to (M, g)$ is called a *translation* along γ if f has no fixed point, and $f(\mathrm{Im}(\gamma)) = \mathrm{Im}(\gamma)$.

Let (M, g) be any complete Riemannian manifold and $\pi : \widetilde{M} \to M$ be the universal covering. We endow with $\widetilde{M}$ the pull back metric $\tilde{g} = \pi^* g$. Recall that for each element $\alpha \in \pi_1(M)$, one can define a deck transformation $f_\alpha : \widetilde{M} \to \widetilde{M}$:

> for each $\tilde{p} \in \widetilde{M}$, there is a loop γ based at $p = \pi(\tilde{p})$ whose homotopy class is α. Let $\tilde{\gamma}$ be the lift of γ with starting point $\tilde{p}$. Define $f_\alpha(\tilde{p})$ be the endpoint of $\tilde{\gamma}$.

One can prove that

- f_α is well-defined,
- f_α is an isometry,
- $f_\beta \circ f_\alpha = f_{\beta\alpha}$ for all $\alpha, \beta \in \pi_1(M)$,
- f_α has no fixed point if $\alpha \neq e$.

Now suppose $e \neq \alpha \in \pi_1(M)$, and let γ be a minimizing closed geodesic in the homotopy class α. Let $\tilde{\gamma}$ be a lift of γ to $\widetilde{M}$. Then by definition $\tilde{\gamma}$ is a geodesic in $(\widetilde{M}, \tilde{g})$, and $f_\alpha(\mathrm{Im}(\tilde{\gamma})) = \mathrm{Im}(\tilde{\gamma})$. So we get

Lemma 1.9. $f_\alpha : (\widetilde{M}, \tilde{g}) \to (\widetilde{M}, \tilde{g})$ *is a translation along* $\tilde{\gamma}$ *for any* $e \neq \alpha \in \pi_1(M)$.

As a consequence of this lemma, we prove

Corollary 1.10. *Suppose* (M, g) *has negative sectional curvature, then any translation* $f : \widetilde{M} \to \widetilde{M}$ *fixes only one geodesic*[1].

Proof. Suppose there are two geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\widetilde{M}$ such that $f(\tilde{\gamma}_i) = \tilde{\gamma}_i$. First we claim that $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \emptyset$. Otherwise either $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \{\tilde{p}\}$ for some $\tilde{p}$, which implies $f(\tilde{p}) = \tilde{p}$ which is a contradiction (since f has no fixed point), or there are at least two points in $\tilde{\gamma}_1 \cap \tilde{\gamma}_2$, which contradicts with Cartan-Hadamard theorem.

Now choose $\tilde{p}_i \in \tilde{\gamma}_i$, and let $\tilde{\gamma}_3$ and $\tilde{\gamma}_5$ be the minimizing geodesic connecting $\tilde{p}_1$, $\tilde{p}_2$ and connecting $f(\tilde{p}_1), \tilde{p}_2$ respectively. Let $\tilde{\gamma}_4 = f(\tilde{\gamma}_3)$ be the minimizing geodesic connecting $f(\tilde{p}_1)$, $f(\tilde{p}_2)$. Since f is an isometry, the "corresponding angles" at $\tilde{p}_1$ and at $f(\tilde{p}_1)$ are the same, similarly for $\tilde{p}_2$. As a result, the angles of the two geodesic triangle $\tilde{p}_1\tilde{p}_2 f(\tilde{p}_1)$ and $\tilde{p}_2 f(\tilde{p}_1) f(\tilde{p}_2)$ add up to at least 2π, which contradicts with Proposition 1.5. □

[1] Here different parametrizations will be viewed as the same.

¶ Proof of Preissman's theorem.

As above we denote by $\widetilde{M}$ the universal covering of M, and f_α the deck transformation described above associated to $\alpha \in \pi_1(M)$.

First fix $\alpha \in H$ and let $\tilde{\gamma}$ be the geodesic that is invariant under f_α. Then for any $\beta \in H$, one has $f_{\beta\alpha} = f_{\alpha\beta}$ since H is abelian. So

$$f_\beta(\mathrm{Im}(\tilde{\gamma})) = f_\beta(f_\alpha(\mathrm{Im}(\tilde{\gamma}))) = f_\alpha(f_\beta(\mathrm{Im}(\tilde{\gamma}))).$$

By the corollary above, one must have

$$f_\beta(\mathrm{Im}(\tilde{\gamma})) = \mathrm{Im}(\tilde{\gamma}), \quad \forall \beta \in H.$$

As a consequence, $\tilde{\gamma}$ is invariant under all f_α's for $\alpha \in H$.

Now we denote $\tilde{p}_0 = \tilde{\gamma}(0)$. Since $\tilde{\gamma}$ is invariant under f_β, for each $\beta \in H$, there is a unique $t_\beta \in \mathbb{R}$ so that

$$\tilde{\gamma}(t_\beta) = f_\beta(\tilde{p}_0).$$

Note that this implies

$$\tilde{\gamma}(t_\beta + t) = f_\beta(\tilde{\gamma}(t))$$

for any t, since as t varies, both sides are geodesics with the same initial condition.

Now we define a map

$$\varphi : H \to \mathbb{R}, \qquad \varphi(\beta) = t_\beta.$$

Claim 1: φ is a group homomorphism:

For any $\beta_1, \beta_2 \in H$,

$$\tilde{\gamma}(t_{\beta_1} + t_{\beta_2}) = f_{\beta_1} \circ f_{\beta_2}(\tilde{p}_0) = f_{\beta_1\beta_2}(\tilde{p}_0) = \tilde{\gamma}(t_{\beta_1\beta_2}).$$

So we have $\varphi(\beta_1\beta_2) = t_{\beta_1\beta_2} = t_{\beta_1} + t_{\beta_2}$.

Claim 2: φ is injective:

Suppose $\varphi(\beta) = 0$, then $\tilde{p}_0 = \tilde{\gamma}(0) = f_\beta(\tilde{p}_0)$. So $\beta = e \in \pi_1(M)$.

Claim 3: The image of φ is not dense in $\mathbb{R}$.

Pick a strongly convex geodesic ball $U = B_r(p)$ centered at $p = \pi(\tilde{p}_0)$ so that $\pi^{-1}(U) = \cup_\delta U_\delta$, where each U_δ is diffeomorphic to U under the covering map $\pi : \widetilde{M} \to M$ and are disjoint. Denote U_0 be the one so that $\tilde{p}_0 \in U_0$. Then for each $\beta \neq e$, $f_\beta(\tilde{p}_0) \notin U_0$. So

$$|t_\beta| = d(\tilde{p}_0, f_\beta(\tilde{p}_0)) \geq r$$

for any $\beta \neq e$.

As a consequence of the first two claims, H is an additive subgroup of $\mathbb{R}$. But we know that any additive subgroup of $\mathbb{R}$ is either dense or infinite cyclic. So the theorem is proved.

2. Complete Riemannian manifolds with positive curvature

Now let's turn to Riemannian manifolds with positive curvature.

¶ Synge's Theorem.

Another application of the second variation formula to Riemannian manifolds with positive curvature is

Theorem 2.1 (Synge)**.** *Let (M, g) be a compact Riemannian manifold with positive sectional curvature.*

(1) If M is even dimensional and orientable, then M is simply connected.
(2) If M is odd dimensional, then M is orientable.

Since $\mathbb{R}P^m$ admits a positive sectional curvature metric, given the fact "$\pi_1(\mathbb{R}P^m) \cong \mathbb{Z}_2$ for $m \geq 2$", we conclude that "$\mathbb{R}P^m$ is orientable if and only if m is odd".

Corollary 2.2. *If (M, g) is a compact even dimensional Riemannian manifold of positive sectional curvature, and M is not orientable, then $\pi_1(M) \cong \mathbb{Z}_2$.*

Proof. Let $\overline{M}$ be the orientable double covering of M, endowed with the induced pull-back metric. Then $\overline{M}$ is orientable and satisfies all the conditions in Synge Theorem. It follows that $\overline{M}$ is simply connected and thus $\pi_1(M) \cong \mathbb{Z}_2$. □

As a consequence, $\mathbb{RP}^2 \times \mathbb{RP}^2$ admits no metric of positive sectional curvature since $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong (\mathbb{Z}_2)^2$. We remark that it is still unknown whether $S^2 \times S^2$ admits a positive sectional curvature metric: this is the well-known Hopf conjecture.

Remark. In the odd dimensional case we cannot say too much of its fundamental group. For example, for each k there is a lens space $S^3/\mathbb{Z}_k$ that has constant sectional curvature 1 and fundamental group $\mathbb{Z}_k$.

¶ Proof of Synge's Theorem.

We first prove

Lemma 2.3. *Let (M, g) be an orientable Riemannian manifold, and $\gamma : [a, b] \to M$ be a smooth loop, i.e. $\gamma(a) = \gamma(b) := p$. Then the parallel transport $P^\gamma_{a,b} : T_pM \to T_pM$ has determinant 1.*

Proof. In Lecture 6 we have seen $P^\gamma_{a,b} \in O(T_pM)$. It remains to show $\det P^\gamma_{a,b} > 0$. For this purpose we take a *positive* m-form ω on M, and let $\{e_i\}$ be a *positive* basis of T_pM, i.e. $\omega(e_1, \cdots, e_m) > 0$. Let $e_j(t) = P^\gamma_{a,t}(e_j)$ be the parallel transport of $\{e_i\}$ along γ. Then

$$\omega(e_1(t), \cdots, e_m(t)) \neq 0$$

for all t. It follows that $\omega(e_1(b), \cdots, e_m(b)) > 0$. But

$$\omega(e_1(b), \cdots, e_m(b)) = (\det P^\gamma_{a,b})\omega(e_1, \cdots, e_m),$$

so we must have $\det P^\gamma_{a,b} > 0$. □

Proof of Synge's Theorem. (1) Suppose M is not simply connected. Then there exists a nontrivial closed geodesic $\gamma : [0,1] \to M$ which is length minimizing in its free homotopy class. Since the parallel transport $P^\gamma_{0,1} \in SO(T_pM)$ and satisfies

$$P^\gamma_{0,1}(\dot\gamma(0)) = \dot\gamma(0),$$

we can find $X_p \in E_p = (\dot\gamma(0))^\perp$ such that

$$P^\gamma_{0,1}(X_p) = X_p.$$

(Here, we used the condition that $\dim M$ is even, so that $\dim E$ is odd!)

Now let $X(t)$ be the parallel vector field along γ with $X(0) = X_p$. Then

$$X(1) = P^\gamma_{0,1}(X_p) = X_p.$$

Thus for the variation γ_s of γ whose variation field is X, we have

$$\frac{d^2}{ds^2}\bigg|_{s=0} E(\gamma_s) = \int_0^1 \langle R(\dot\gamma, X)\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}X, X\rangle dt = \int_0^1 R(\dot\gamma, X, \dot\gamma, X\rangle dt < 0.$$

This contradicts with the fact that γ is minimizing in its homotopy class.

(2) Suppose M is not orientable, then there is a smooth loop $\gamma : [0,1] \to M$ and a frame $\{e_1(t), \cdots, e_m(t)\}$ so that $e_1(1) \wedge \cdots \wedge e_m(1) = -e_1(0) \wedge \cdots \wedge e_m(0)$. On the other hand, we write $\tilde e_i(t) = P^\gamma_{0,t}(e_i(0))$, then there is a nonzero function $f(t)$ so that $\tilde e_1(t) \wedge \cdots \wedge \tilde e_m(t) = f(t) e_1(t) \wedge \cdots \wedge e_m(t)$. So we must have $\det(P^\gamma_{0,1}) = -1$ since m is odd. By continuity, any smooth loop in the same homotopy class satisfies $\det(P^\gamma_{0,1}) = -1$. Let γ be the minimizing geodesic in this class. Since $P^\gamma_{0,1}(\dot\gamma(0)) = \dot\gamma(0)$, we see

$$\det P^\gamma_{0,1}|_E = -1,$$

where $E = (\dot\gamma(0))^\perp$ is the orthogonal complement of $\dot\gamma(0)$ in T_pM. Since E is even dimensional, complex eigenvalues appear in conjugate pairs and -1 appear by odd times. So again we conclude that there exists $X_p \in E$ so that $P^\gamma_{0,1}(X_p) = X_p$. Now repeat the variation argument above we conclude that γ is not minimizing in its homotopy class, a contradiction. □

¶ Bonnet-Myers Theorem.

What about Ricci curvature?

Theorem 2.4 (Bonnet-Myers). *Let (M, g) be a complete connected Riemannian manifold. Suppose there is a constant $\kappa > 0$ such that*

$$\mathrm{Ric}(X_p) \ge (m-1)\kappa, \qquad \forall X_p \in SM.$$

Then M is compact, and its diameter is bounded by

$$\mathrm{diam}(M) := \sup_{p,q\in M} \mathrm{dist}(p,q) \le \frac{\pi}{\sqrt{\kappa}}.$$

Remarks. (1) One cannot weaken the condition on Ricci curvature to $Ric > 0$ or even $K > 0$. For example, consider the paraboloid

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}.$$

It is a surface of revolution with $K > 0$, which is not compact.

(2) The estimate is optimal in the following sense: Let M be the standard sphere of radius $\frac{1}{\sqrt{\kappa}}$, then it has Ricci curvature $(m-1)\kappa$ and diameter $\frac{\pi}{\sqrt{\kappa}}$. (Note: the diameter here is not the standard diameter as a subset in $\mathbb{R}^m$.)

(3) We will prove the following result of S. Y. Cheng later: If (M, g) satisfies the conditions of the Bonnet-Myers theorem and $\mathrm{diam}(M) = \frac{\pi}{\sqrt{\kappa}}$, then (M, g) is isometric to the standard sphere of radius $\frac{1}{\sqrt{\kappa}}$.

Corollary 2.5. *Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded below by a positive number. Then $\pi_1(M)$ is finite.*

Proof. Let $\widetilde{M}$ be the universal covering of M, endowed with the pull-back metric $\tilde{g} = \pi^* g$. Then $(\widetilde{M}, \tilde{g})$ is also a complete Riemannian manifold whose Ricci curvature is bounded below by a positive number. By Bonnet-Myers theorem, $\widetilde{M}$ is compact. As a consequence, $\pi : \widetilde{M} \to M$ has to be a finite covering. So $\pi_1(M)$ is finite. □

It particular, we see that if M, N are compact, $\pi_1(M)$ is infinite, then $M \times N$ admits no Riemannian metric of positive Ricci curvature.

¶ Proof of Bonnet-Myers Theorem.

Proof. For any $p, q \in M$, let $\gamma : [0, 1] \to M$ be a minimizing geodesic joining p to q. It's enough to show $L(\gamma) \le \frac{\pi}{\sqrt{\kappa}}$ (which implies compactness of M by Hopf-Rinow theorem).

By contradiction, suppose that $L(\gamma) = l > \frac{\pi}{\sqrt{\kappa}}$. Let $\{e_i(t)\}$ be parallel vector fields along γ which form an orthonormal basis at each point $\gamma(t)$ and so that $e_1(t) = \frac{\dot{\gamma}(t)}{l}$. For $j = 2, \cdots, m$, we define

$$V_j(t) = \sin(\pi t) e_j(t).$$

Then $V_j(0) = V_j(1) = 0$, and $\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V_j = -\pi^2 \sin(\pi t) e_j(t)$. Thus

$$I(V_j, V_j) = \int_0^1 \langle R(\dot{\gamma}, V_j)\dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V_j, V_j \rangle dt = \int_0^1 \sin^2(\pi t)(\pi^2 + l^2 R(e_1, e_j, e_1, e_j)) dt.$$

Summing over $2 \le j \le m-1$, we get

$$\sum_{j=2}^m I(V_j, V_j) = \int_0^1 \sin^2(\pi t)((m-1)\pi^2 - l^2 \mathrm{Ric}(e_1)) dt < 0.$$

So there exists some $j \ge 2$ so that $I(V_j, V_j) < 0$. If follows that there exists $\bar{q} = \gamma(t_0)$ with $0 < t_0 < 1$ which is conjugate to p along γ. In particular, γ is not length minimizing. A contradiction. □