LECTURE 23: RAUCH COMPARISON THEOREM

Now we begin to study the so-called comparison theorems. As we have seen last time, a comparison on curvature tensor will induce a comparison on Jacobi fields, which would further give a comparison of geometry (triangles for the Cartan-Hadamard manifolds) or analysis (Hessian of d_p^2 for the Cartan-Hadamard manifolds), or restrict the possible behavior of geodesics (as in the proof of Synge's theorem and Bonnet-Myers theorem). In all these cases we finally arrive at some restrictions on global geometry/topology of the manifold. In the next couple lectures we develop such ideas more systematically.

1. The Index Comparison

¶ Basic index comparison lemma.

In the proof of Synge's Theorem and Bonnet-Myers Theorem, we used parallel vector fields to construct variations of a given geodesic. The advantage of parallel vector fields is that inner products (and thus lengths, angles) are preserved along geodesic. Another class of vector fields that are widely used in constructing variations are Jacobi fields, which are variation fields of geodesic variations. Usually one start with two geodesics on two manifolds whose curvatures are pointwise comparable, then compare two Jacobi fields with same initial value on these geodesics.

So our basic setting for comparison is the following:

- Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds of dimension m.
- Let $\gamma : [0, a] \to M$ and $\widetilde{\gamma} : [0, a] \to \widetilde{M}$ be normal geodesics with $\gamma(0) = p$ and $\widetilde{\gamma}(0) = \tilde{p}$.
- For each $t \in [0, a]$, let

  $K^-(t) = \min\{K(\Pi_{\gamma(t)}) \mid \Pi_{\gamma(t)} \subset T_{\gamma(t)}M, \dim \Pi_{\gamma(t)} = 2 \text{ and } \dot{\gamma}(t) \in \Pi_{\gamma(t)}\},$

  $\widetilde{K}^+(t) = \max\{\widetilde{K}(\widetilde{\Pi}_{\widetilde{\gamma}(t)}) \mid \widetilde{\Pi}_{\gamma(t)} \subset T_{\tilde{\gamma}(t)}\widetilde{M}, \dim \widetilde{\Pi}_{\tilde{\gamma}(t)} = 2 \text{ and } \dot{\tilde{\gamma}}(t) \in \widetilde{\Pi}_{\tilde{\gamma}(t)}\}.$

- We say two vectors $X_c \in T_{\gamma(c)}M$ and $\widetilde{X}_c \in T_{\tilde{\gamma}(c)}\widetilde{M}$ are roughly the same if

$$|X_c| = |\widetilde{X}_c| \quad \text{and} \quad \langle X_c, \dot{\gamma}(c)\rangle = \langle \widetilde{X}_c, \dot{\tilde{\gamma}}(c)\rangle.$$

Now let $X, \widetilde{X}$ be Jacobi fields along γ and $\widetilde{\gamma}$ respectively, with $X(0) = 0$ and $\widetilde{X}(0) = 0$. To compare X and $\widetilde{X}$, we usually assume either

"$X(a)$ and $\widetilde{X}(a)$ are roughly the same",

or

"$\nabla_{\dot{\gamma}(0)}X$ and $\widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)}\widetilde{X}$ are roughly the same".

The following lemma is quite obvious whose proof is left as an exercise:

Lemma 1.1. *Let $X, \widetilde{X}$ be Jacobi fields along γ and $\tilde{\gamma}$ with $X(0) = \widetilde{X}(0) = 0$, and write*

$$X = c\dot{\gamma} + dt\dot{\gamma} + X^{\perp}, \quad \widetilde{X} = \tilde{c}\dot{\tilde{\gamma}} + \tilde{d}t\dot{\tilde{\gamma}} + \widetilde{X}^{\perp}.$$

Suppose either "$X(a)$ and $\widetilde{X}(a)$ are roughly the same", or "$\nabla_{\dot{\gamma}(0)}X$ and $\widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)}\widetilde{X}$ are roughly the same", then $c = \tilde{c}$ and $d = \tilde{d}$.

As a result, to compare two Jacobi fields whose initial or boundary values are roughly the same, it is enough to compare their "normal components".

Now we prove the basic index comparison theorem:

Theorem 1.2. *Let $X, \widetilde{X}$ be Jacobi fields along $\gamma, \widetilde{\gamma}$ such that $X(0) = 0, \widetilde{X}(0) = 0$, and suppose $X(a)$ and $\widetilde{X}(a)$ are roughly the same. Assume further that*

(1) γ has no conjugate points on $[0, a]$,
(2) $\widetilde{K}^+(t) \le K^-(t)$ holds for all $t \in [0, a]$.

then

$$I(X, X) \le I(\widetilde{X}, \widetilde{X}).$$

Moreover, if $K^+(t) < K^-(t)$ for some $t < a$, then $I(X, X) < I(\widetilde{X}, \widetilde{X})$.

Proof. By Lemma 1.1 we may assume $X, \widetilde{X}$ are normal. Let $\{e_1(t), \cdots, e_m(t)\}$ and $\{\tilde{e}_1(t), \cdots, \tilde{e}_m(t)\}$ be orthonormal frames that are parallel along γ and $\tilde{\gamma}$, such that

$$e_1(t) = \dot{\gamma}(t), \quad \tilde{e}_1(t) = \dot{\tilde{\gamma}}(t), \quad \text{and} \quad e_2(a) = X(a)/\alpha, \quad \tilde{e}_2(a) = \widetilde{X}(a)/\alpha,$$

where $\alpha = |X(a)| = |\widetilde{X}(a)| \neq 0$ since γ has no conjugate point. If we denote

$$X(t) = X^i(t)e_i(t), \quad \widetilde{X}(t) = \widetilde{X}^i(t)\tilde{e}_i(t)$$

respectively, then obviously we have

- $X^i(0) = \widetilde{X}^i(0) = 0$ for all i,
- $X^2(a) = \widetilde{X}^2(a) = \alpha$ and $X^i(a) = \widetilde{X}^i(a) = 0$ for all $i \neq 2$,
- $X^1(t) = \tilde{X}^1(t) = 0$ for all $t \in [0, a]$ (since both X and $\widetilde{X}$ are normal).

As in the proof of Bonnet-Myers theorem we transplant $\widetilde{X}$ to γ by defining

$$Y(t) = \widetilde{X}^i(t)e_i(t).$$

Then $Y(0) = 0$, $Y(a) = X(a)$. Since X is a Jacobi field,

$$I(X, X) \le I(Y, Y).$$

On the other hand,

$$\begin{aligned}
I(Y,Y) &= \int_0^a \left(|\nabla_{\dot\gamma} Y|^2 + \langle R(\dot\gamma, Y)\dot\gamma, Y\rangle\right) dt \\
&= \int_0^a \Big(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 K(\dot\gamma, Y)\Big) dt \\
&\le \int_0^a \Big(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 K^-(t)\Big) dt \\
&\le \int_0^a \Big(\sum (\dot{\tilde{X}}^i(t))^2 - \sum (\tilde{X}^i(t))^2 \widetilde{K}^+(t)\Big) dt \\
&\le \int_0^a \Big(|\widetilde{\nabla}_{\dot{\tilde\gamma}} \widetilde{X}|^2 + \langle \widetilde{R}(\dot{\tilde\gamma}, \widetilde{X})\dot{\tilde\gamma}, \widetilde{X}\rangle\Big) dt \\
&= I(\widetilde{X}, \widetilde{X}).
\end{aligned}$$

It follows that $I(X,X) \le I(\widetilde{X}, \widetilde{X})$.

Finally if $K^+(t) < K^-(t)$ for some $t < a$, then the second inequality above is strict, and thus $I(X,X) < I(\widetilde{X}, \widetilde{X})$. □

¶ Local Hessian comparison.

As a consequence of the basic index comparison theorem, we prove

Theorem 1.3 (Local Hessian Comparison)**.** *Let $(M,g), (\widetilde{M}, \tilde g)$ be complete Riemannian manifolds, $\gamma : [0,a] \to M$ and $\tilde\gamma : [0,a] \to \widetilde{M}$ be minimizing normal geodesics in M and $\widetilde{M}$ respectively, so that*

$$\widetilde{K}^+(t) \le K^-(t) \text{ holds for all } t \in [0,a].$$

Fix $0 < b < a$ and write $q = \gamma(b)$, $\tilde q = \tilde\gamma(b)$. Suppose $X_q \in T_qM$ and $\widetilde{X}_{\tilde q} \in T_{\tilde q}\widetilde{M}$ are roughly the same. Then

$$\nabla^2 d_p(X_q, X_q) \le \widetilde{\nabla}^2 \tilde d_{\tilde p}(\widetilde{X}_{\tilde q}, \widetilde{X}_{\tilde q}).$$

Moreover, the equality holds if and only if $\widetilde{K}^+(t) = K^-(t)$ for all $t \in [0,b]$.

Proof. Since γ and $\widetilde\gamma$ are length minimizing, and $b < a$, we see $q \notin \mathrm{Cut}(p)$ and $\widetilde q \notin \mathrm{Cut}(\tilde p)$. Let X be the Jacobi field with $X(0) = 0, X(b) = X_q$, then as we have seen in Lecture 21,

$$(\nabla^2 d_p)_q(X_q, X_q) = \langle \nabla_{\dot\gamma(q)} X, X_q\rangle = I(X,X).$$

Now the conclusion follows from the index comparison theorem above. □

Since $\Delta = \mathrm{Tr}\nabla^2$, by taking trace we get, under the same assumptions,

$$\Delta d_p(q) \le \widetilde{\Delta} \tilde d_{\tilde p}(\tilde q).$$

2. Rauch's Jacobi field comparison theorem

¶ Rauch comparison theorem.

Now we state and prove Rauch's comparison theorem, which relates the sectional curvature of a Riemannian manifold to the length of Jacobi fields (and thus the rates at which geodesics spread apart).

Theorem 2.1 (Rauch comparison theorem)**.** *Let $X, \widetilde{X}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ with $X(0) = \widetilde{X}(0) = 0$, such that $\nabla_{\dot{\gamma}(0)} X$ and $\widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)} \widetilde{X}$ are roughly the same. Assume*

(1) γ has no conjugate points on $[0, a]$,
(2) $\widetilde{K}^+(t) \le K^-(t)$ holds for all $t \in [0, a]$.

Then $\tilde{\gamma}$ has no conjugate points on $[0, a]$, and for all $t \in [0, a]$,

$$|X(t)| \le |\widetilde{X}(t)|.$$

Moreover, if there is $0 < t_0 < t$ such that $K^+(t_0) < K^-(t_0)$, then $|X(t)| < |\widetilde{X}(t)|$.

Proof. Again by Lemma 1.1 we may assume $X, \widetilde{X}$ are normal. We denote

$$u(t) = |X(t)|^2, \quad \tilde{u}(t) = |\widetilde{X}(t)|^2.$$

Then $\tilde{u}(t)/u(t)$ is well-defined. Moreover, since

$$X(t) = t\nabla_{\dot{\gamma}(0)} X + O(t^2),$$

we have

$$\lim_{t \to 0} \frac{\tilde{u}(t)}{u(t)} = \lim_{t \to 0} \frac{t^2 |\nabla_{\dot{\tilde{\gamma}}(0)} \widetilde{X}|^2 + O(t^3)}{t^2 |\nabla_{\dot{\gamma}(0)} X|^2 + O(t^3)} = \frac{|\nabla_{\dot{\tilde{\gamma}}(0)} \widetilde{X}|^2}{|\nabla_{\dot{\gamma}(0)} X|^2} = 1.$$

Therefore, to prove $|X| \le |\widetilde{X}|$, it is enough to prove $\frac{d}{dt} \frac{\tilde{u}(t)}{u(t)} \ge 0$, or equivalently,

$$\dot{\tilde{u}}(t) u(t) - \tilde{u}(t) \dot{u}(t) \ge 0.$$

Since γ has no conjugate point, $u(t) > 0$ for all $t \in (0, a]$. Let $c \le a$ be the greatest number so that $\tilde{u}(t) > 0$ on $(0, c)$. For any $b \in (0, c)$, we define

$$X_b(t) = \frac{X(t)}{|X(b)|}, \quad \widetilde{X}_b(t) = \frac{\widetilde{X}(t)}{|\widetilde{X}(b)|}.$$

Applying Theorem 1.2 to $X_b, \widetilde{X}_b$ on $[0, b]$ we get

$$\langle \nabla_{\dot{\gamma}(b)} X_b, X_b(b) \rangle = I(X_b, X_b) \le I(\widetilde{X}_b, \widetilde{X}_b) = \langle \widetilde{\nabla}_{\dot{\tilde{\gamma}}(b)} \widetilde{X}_b, \widetilde{X}_b(b) \rangle,$$

i.e.

$$\frac{1}{2} \frac{\dot{u}(b)}{u(b)} = \frac{\langle \nabla_{\dot{\gamma}(b)} X, X(b) \rangle}{\langle X(b), X(b) \rangle} \le \frac{\langle \widetilde{\nabla}_{\dot{\tilde{\gamma}}(b)} \widetilde{X}, \widetilde{X}(b) \rangle}{\langle \widetilde{X}(b), \widetilde{X}(b) \rangle} = \frac{1}{2} \frac{\dot{\tilde{u}}(b)}{\tilde{u}(b)}.$$

So for any $t \in (0, c)$, we have $\frac{\dot{u}(t)}{u(t)} \le \frac{\dot{\tilde{u}}(t)}{\tilde{u}(t)}$. This is exactly what we need.

To summary, we proved that $|X(t)| \le |\widetilde{X}(t)|$ for $t \in (0, c)$. If $c < a$, then

$$|\widetilde{X}(c)| \ge |X(c)| > 0,$$

contradicting with the choice of c. So we must have $c = a$. In particular, $\tilde{\gamma}$ has no conjugate points on $[0, a]$.

Finally if there is $0 < t_0 < t$ such that $K^+(t_0) < K^-(t_0)$, then

$$I(X_{t_0}, X_{t_0}) < I(\widetilde{X}_{t_0}, \widetilde{X}_{t_0}),$$

and thus the inequality for t is strict. This completes the proof. □

Although Rauch comparison theorem is stated for two general manifolds whose sectional curvatures are comparable, in most applications one of the two manifolds is a model space that has constant sectional curvature, and the second one is the manifold under study whose curvature is bounded either from below or from above.

For example, according to the explicit formula for Jacobi fields on spheres, the distance between any two consecutive conjugate points of the sphere with sectional curvature κ is $\pi/\sqrt{\kappa}$. As a result we get

Corollary 2.2. *Suppose the sectional curvature of (M, g) satisfies*

$$0 < C_1 \le K \le C_2,$$

where C_1, C_2 are constants. Let γ be any geodesic in M. Then the distance D between any two consecutive conjugate points of γ satisfies

$$\frac{\pi}{\sqrt{C_2}} \le D \le \frac{\pi}{\sqrt{C_1}}.$$

One should compare this with Sturm comparison theorem in ODE.

¶ Rauch comparison theorem: second form.

Since any Jacobi field X along γ with $X(0) = 0$ can be written explicitly as

$$X(t) = t(d\exp_p)_{t\dot{\gamma}(0)}(\nabla_{\dot{\gamma}(0)X}),$$

one can rewrite Rauch comparison theorem above as

Theorem 2.3 (Rauch comparison theorem, Second form)**.** *Let γ, $\tilde{\gamma}$ be geodesics with $p = \gamma(0)$, $\tilde{p} = \tilde{\gamma}(0)$, and suppose $X_p \in T_pM, \widetilde{X}_{\tilde{p}} \in T_{\tilde{p}}\widetilde{M}$ are roughly the same. Assume also*

(1) γ has no conjugate points on $[0, a]$,
(2) $\widetilde{K}^+(t) \le K^-(t)$ holds for all $t \in [0, a]$.

Then

$$|(d\exp_p)_{t\dot{\gamma}(0)}X_p| \le |(d\exp_{\tilde{p}})_{t\dot{\tilde{\gamma}}(0)}\widetilde{X}_p|.$$

Moreover, the equality is strict for t if there exists $0 < t_0 < t$ with $K^+(t_0) < K^-(t_0)$.

Note that for any Riemannian manifold (M, g) and any point $p \in M$, (T_pM, g_p) is a Riemannian manifold that has constant sectional curvature 0. Applying Theorem 2.3 to (M, g) and (T_pM, g_p) we get

Corollary 2.4. *Let (M, g) be a complete Riemannian manifold with sectional curvature $K \le 0$. Then for any $p \in M$, any $X_p \in T_pM$ and $Y_p \in T_pM = T_{X_p}(T_pM)$,*

$$|(d\exp_p)_{X_p}(Y_p)| \ge |Y_p|.$$

In particular, for any curve γ in T_pM, one has

$$L(\gamma) \le L(\exp_p \circ \gamma).$$

Moreover, the equality is strict if $K < 0$.

Note that by using this corollary one can give another proof of Proposition 1.5 (the weak cosine law for Cartan-Hadamard manifolds) in Lecture 22.