LECTURE 25: THE LAPLACIAN AND VOLUME COMPARISON

Today we discuss comparison under Ricci curvature condition. We first prove the Laplacian comparison theorem, which can be viewed as an averaged version of the Hessian comparison (under slightly stronger condition). Then we prove the very useful Bishop-Gromov volume comparison theorem, which can be viewed as an "integrated" version of the Laplacian comparison theorem.

1. The Laplacian Comparison Theorem

Recall that if $(M, g), (\widetilde{M}, \tilde{g})$ are complete Riemannian manifolds, $\gamma : [0, a] \to M$ and $\tilde{\gamma} : [0, a] \to \widetilde{M}$ are minimizing normal geodesics with

$$\widetilde{K}^+(t) \le K^-(t) \text{ holds for all } \ t \in [0, a].$$

and if $X_q \in T_qM$ and $\widetilde{X}_{\tilde{q}} \in T_{\tilde{q}}\widetilde{M}$ are roughly the same, where $q = \gamma(b)$, $\tilde{q} = \tilde{\gamma}(b)$ and $0 < b < a$, then we have

$$\nabla^2 d_p(X_q, X_q) \le \widetilde{\nabla}^2 \tilde{d}_{\tilde{p}}(\widetilde{X}_{\tilde{q}}, \widetilde{X}_{\tilde{q}}).$$

Moreover, the equality holds if and only if $\widetilde{K}^+(t) = K^-(t)$ for all $t \in [0, b]$.

Since $\Delta = \mathrm{Tr}\nabla^2$, the Hessian comparison theorem will imply a Laplacian comparison $\Delta d_p(q) \le \widetilde{\Delta}\tilde{d}_{\tilde{p}}(\tilde{q})$. Note that by taking the trace of the Hessian, what we get is (up to a constant) "the average of the Hessian". As a result, one can anticipate to weaken the comparison condition from sectional curvature to a weaker "averaged version", namely, a comparison condition on Ricci curvature.

Theorem 1.1 (The Laplacian Comparison Theorem)**.** *Let (M, g) be a Riemannian manifold, and $\gamma : [0, l] \to M$ a minimizing normal geodesic with $\gamma(0) = p$. Suppose*

$$Ric(\dot{\gamma}(t)) \ge (m - 1)k.$$

We denote by $\widetilde{\Delta}_k$, $\tilde{d}$, $\tilde{\gamma}$ etc the corresponding objects in M^m_k. Then

$$\Delta d_p(\gamma(t)) \le \widetilde{\Delta}\tilde{d}_{\tilde{p}}(\tilde{\gamma}(t)), \quad \forall 0 < t < l.$$

Moreover, $\Delta d_p(\gamma(b)) \le \widetilde{\Delta}\tilde{d}_{\tilde{p}}(\tilde{\gamma}(b))$ for some $b < l$ if and only if

> *for any $0 \le t \le b$, $Ric(\dot{\gamma}(t)) = (m-1)k$, and any normal Jacobi field X along $\gamma|_{[0,b]}$ with $X(0) = 0$ is almost parallel, i.e. is of the form $X = \frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)}e(t)$, where e is a parallel vector field along γ.*

Proof. Fix $b < l$. As usual we let $\{e_1(t), \cdots, e_m(t)\}$ be a parallel orthonormal frame along γ with $e_1(t) = \dot{\gamma}(t)$, and let $\{\tilde{e}_1(t), \cdots, \tilde{e}_m(t)\}$ be a parallel orthonormal frame along $\tilde{\gamma}$ with $\tilde{e}_1(t) = \dot{\tilde{\gamma}}(t)$. For any $i \ge 2$, let $X_i(\tau)$ be the normal Jacobi field along

$\gamma|_{[0,b]}$ with $X_i(0)=0$ and $X_i(b)=e_i(b)$, and let $\widetilde{X}_i(\tau)$ be the normal Jacobi field along $\tilde{\gamma}|_{[0,b]}$ with $\widetilde{X}_i(0)=0$ and $\widetilde{X}_i(b)=\tilde{e}_i(b)$. Then for $q=\gamma(b)$ we have

$$\Delta d_p(q)=\sum_{i=2}^m(\nabla^2 d_p)_q(e_i(b),e_i(b))=\sum_{i=2}^m I(X_i,X_i)$$

and similarly for $\tilde{q}=\tilde{\gamma}(b)$, $\widetilde{\Delta}\tilde{d}_{\tilde{p}}(\tilde{q})=\sum_{i=2}^m I(\widetilde{X}_i,\widetilde{X}_i)$. It remains to prove

$$\sum_{i=2}^m I(X_i,X_i)\le\sum_{i=2}^m I(\widetilde{X}_i,\widetilde{X}_i).$$

We shall apply the same trick that we played in the proof of the basic index comparison lemma, namely we transplant $\widetilde{X}_i$ to γ. For this purpose we first recall that under the condition " $\widetilde{M}$ has constant sectional curvature k along $\tilde{\gamma}$" (c.f. PSet 4), the normal Jacobi field $\widetilde{X}_i$ is given by $\widetilde{X}^i(t)=\frac{sn_k(t)}{sn_k(b)}\tilde{e}_i(t)$. So for each $2\le i\le m$ we define on $\gamma|_{[0,b]}$ a vector field

$$X_i'(t)=\frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)}e_i(t).$$

Obviously X_i' has the same boundary condition as the Jacobi field X_i. So we get $I(X_i,X_i)\le I(X_i',X_i')$. Now the conclusion follows from

$$\begin{aligned}\sum I(X_i',X_i')&=\sum\int_0^b\left(|\nabla_{\dot\gamma}X_i'|^2+Rm(\dot\gamma,X_i',\dot\gamma,X_i')\right)dt\\&=\sum_i\int_0^b\left((\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(b)})^2-(\frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)})^2K(\dot\gamma,e_i)\right)dt\\&=\int_0^b\left((m-1)(\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(b)})^2-(\frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)})^2Ric(\dot\gamma)\right)dt\\&\le\int_0^b\left((m-1)(\frac{\mathrm{sn}_k'(t)}{\mathrm{sn}_k(b)})^2-(\frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(b)})^2(m-1)k\right)dt\\&=\sum\int_0^b\left(|\nabla_{\dot{\tilde\gamma}}\widetilde{X}_i|^2+\widetilde{Rm}(\dot{\tilde\gamma},\widetilde{X}_i,\dot{\tilde\gamma},\widetilde{X}_i)\right)dt=\sum I(\widetilde{X}_i,\widetilde{X}_i).\end{aligned}$$

From the proof we see that the equality holds if and only if "$\mathrm{Ric}(\dot\gamma)=(m-1)k$, and $X_i=X_i'$ for all i". Since any normal Jacobi field X along γ with $X(0)=0$ is a linear combination of these X_i, the conclusion follows. $\square$

Remark. As we have seen, $(\nabla^2 d_p)_q(X_q,Y_q)=\langle\nabla_{X_1}N,Y_q\rangle$, where $N=\partial_r$ is the outward unit normal vector of the geodesic sphere $S(p,d(p,q))$ at q. In other words, we may replace $\nabla_{X_1}N$ by the shape vector $(\nabla_{X_1}N)^\perp$ (c.f. PSet 2 Problem 9) and conclude that $\nabla^2 d_p$ is the second fundamental form of the geodesic sphere $S(p,d(p,q))$. It follows that $\Delta d_p(q)=\mathrm{Tr}(\nabla^2 d_p)$ is the trace of the second fundamental form, i.e. the mean curvature of the geodesic sphere $S(p,d(p,q))$ at q.

2. THE BISHOP-GROMOV VOLUME COMPARISON THEOREM

¶ The volume measure in coordinates.

Recall that the Riemannian volume density is defined in a chart (φ, U, V) as

$$dV_g = \sqrt{G}\circ\varphi^{-1}dx^1\cdots dx^m,$$

where $G=\det(g_{ij})$ and $dx^1\cdots dx^m$ is the Lebesgue measure on $\mathbb{R}^m$.

Now let (M,g) be a complete Riemannian manifold. Although in general there is no global chart, there is a large chart that covers almost the whole of M, namely $\{\exp_p^{-1}, M\backslash\mathrm{Cut}(p), \Sigma(p)\}$, which is defined except for the measure zero closed subset $\mathrm{Cut}(p)$, where

$$\Sigma(p)=\exp^{-1}(M\setminus\mathrm{Cut}(p))$$

is an open star-shaped domain in T_pM. In particular, on T_pM we may use polar coordinates and write

$$dx^1\cdots dx^m = r^{m-1}drd\Theta,$$

where $d\Theta$ is the usual surface measure on S^{m-1}. Combine this with the chart $\{\exp_p^{-1}, M\backslash\mathrm{Cut}(p), \Sigma(p)\}$ we get

$$dV_g=\sqrt{G}(\exp_p(r\Theta))r^{m-1}drd\Theta, \quad r\Theta\in\Sigma(p).$$

We denote

$$\mu_p(r,\Theta)=\begin{cases}\sqrt{G}(\exp_p(r\Theta))r^{m-1}, & r\Theta\in\Sigma(p),\\ 0, & r\Theta\notin\Sigma(p).\end{cases}$$

Note that by definition

$$\overline{B_r(p)}=\exp_p(\overline{B_r(0)})=\exp_p(\overline{B_r(0)}\cap\overline{\Sigma(p)}\,).$$

Since $\mathrm{Cut}(p)$ is of measure zero in M, we get

$$\mathrm{Vol}(B_r(p))=\int_{B_r(0)\cap\Sigma(p)}\mu(t,\Theta)dtd\Theta=\int_{B_r(0)}\mu(t,\Theta)dtd\Theta.$$

Example. We may calculate the function $\mu_p(r,\Theta)$ for the three model spaces,

- $\mathbb{R}^m$: $\mu(r,\Theta)=r^{m-1}$.
- S^m: $\mu(r,\Theta)=\sin^{m-1}(r)$.
- $\mathbb{H}^m$: $\mu(r,\Theta)=\sinh^{m-1}(r)$.

¶ The volume measure via Jacobi fields.

Observe that the three functions $\mu(r,\Theta)$ in the previous example are closely related to the Jacobi fields on the three model spaces. This is not a coincidence:

Proposition 2.1. *Given any $\Theta \in S_pM$ and write $\gamma(t) = \exp_p(t\Theta)$. Then for any basis $v_2, \cdots, v_m$ of $\dot\gamma(0)^\perp$, if we let $V_j(t)$ $(i \ge 2)$ be the normal Jacobi fields along γ with $V_j(0) = 0$ and $\nabla_{\dot\gamma(0)} V_j = v_j$, then for any point $r\Theta \in \Sigma(p)$,*

$$\mu_p(r,\Theta) = \left| \frac{\det(V_2(r), \cdots, V_m(r))}{\det(\nabla_{\dot\gamma(0)}V_2, \cdots, \nabla_{\dot\gamma(0)}V_m)} \right|.$$

Proof. Using $v_1 = \Theta, v_2, \cdots, v_m$ as a basis one can define a set of global linear coordinates $u_1, \cdots, u_m$ on T_pM,

$$u \in T_pM \quad \rightsquigarrow u = u_1 v_1 + \cdots + u_m v_m.$$

Then we have

$$du_1 \cdots du_m = \frac{r^{m-1}}{|\det(v_2, \cdots, v_m)|} dr d\Theta.$$

Since V_j is a Jacobi field along γ with $V_j(0) = 0$, we have

$$V_j(t) = t(d\exp_p)_{t\Theta}(v_j).$$

Note that under $\exp_p^{-1}$, $(u_1, \cdots, u_m)$ also gives a coordinate system near $\exp_p(r\Theta)$ for $r\Theta \in \Sigma(p)$. With this coordinate system, we have (at $\exp_p(r\Theta)$)

$$\partial_1 = \dot\gamma(r), \quad \partial_j = \frac{1}{r} V_j(r) \ \ (j \ge 2).$$

It follows

$$g_{ij}(\exp_p(r\Theta)) = \langle \partial_i, \partial_j \rangle|_{\exp_p(r\Theta)} = \frac{1}{r^2} \langle V_i(r), V_j(r) \rangle$$

for $i, j \ge 2$. Since γ is a normal geodesic and since each V_j is a normal Jacobi field, we have $\langle \partial_1, \partial_1 \rangle = 1$ and $\langle \partial_1, \partial_i \rangle = 0$ for $i \ge 2$. So we get, at $\exp_p(r\Theta)$,

$$G = \det(g_{ij}) = r^{-2m+2} \det(\langle V_i, V_j \rangle)_{i,j \ge 2} = r^{-2m+2} \det(V_2(r), \cdots, V_m(r))^2.$$

It follows

$$dV_g = \sqrt{G}(\exp_p(r\Theta)) du_1 \cdots du_m = \left| \frac{\det(V_2(r), \cdots, V_m(r))}{\det(\nabla_{\dot\gamma(0)}V_2, \cdots, \nabla_{\dot\gamma(0)}V_m)} \right| dr d\Theta.$$

This completes the proof. □

¶ The volume measure v.s. the Laplacian of distance.

The crucial observation is

Lemma 2.2. *Suppose $\Theta \in S_pM$ and $r\Theta \in \Sigma(p)$, then*

$$\frac{\mu_p'(r,\Theta)}{\mu_p(r,\Theta)} = (\Delta d_p)(\exp_p(r\Theta)),$$

where the derivative is taken with respect to r.

Proof. Let $\gamma(t) = \exp_p(t\Theta)$ $(0 \le t \le r)$ be the normal geodesic starting at p in the direction Θ. Consider a parallel orthonormal frame $\{e_i(t)\}$ along γ with $e_1(t) = \dot\gamma(t)$. Let $V_j(t)$ be a Jacobi field along γ such that

$$V_j(0) = 0 \quad \text{and} \quad V_j(r) = e_j(r).$$

Then at $q = \gamma(r)$ we have

$$\Delta d_p(\exp_p(r\Theta)) = \sum_{i=2}^m (\nabla^2 d_p)_q(e_i(t), e_i(t)) = \sum_{i=2}^m I(V_i, V_i).$$

On the other hand, if we denote $A(t) = (\langle V_i(t), V_j(t)\rangle)_{i,j\ge 2}$, then $A(r) = \mathrm{Id}$, and the derivative of $d(t) = \det A(t) = (\det(V_2(t), \cdots, V_m(t)))^2$ is

$$d'(t) = d(t)\mathrm{Tr}(A^{-1}(t)A'(t)).$$

Thus we get

$$\frac{\mu_p'(r,\Theta)}{\mu_p(r,\Theta)} = \frac{1}{2}\frac{d'(r)}{d(r)} = \frac{1}{2}\mathrm{Tr}(A'(r)) = \sum_{j=2}^m \langle V_j(r), \nabla_{\dot\gamma(r)} V_j(r)\rangle = \sum_{j=2}^m I(V_j, V_j),$$

so the conclusion follows. □

¶ Comparison of volume elements.

In particular if we denote by $\mu_k(r)$ the function $\mu(r,\Theta)$ for the space M_k^m, i.e.

$$\mu_k(r) = \begin{cases} \sin^{m-1}(\sqrt{k}r), & k > 0, \\ r^{m-1}, & k = 0, \\ \sinh^{m-1}(\sqrt{-k}r), & k < 0, \end{cases}$$

then by Laplace comparison theorem,

$$\frac{\mu_p'(r,\Theta)}{\mu_p(r,\Theta)} \le \frac{\mu_k'(r)}{\mu_k(r)}$$

for any complete Riemannian manifold with $\mathrm{Ric} \ge (m-1)k$, as long as $r\Theta \in \Sigma(p)$.

Proposition 2.3. *If (M,g) is a complete Riemannian manifold with* $\mathrm{Ric} \ge (m-1)k$, *then for any fixed $\Theta \in S_pM$,*

(1) the function $\frac{\mu_p(r,\Theta)}{\mu_k(r)}$ is non-increasing in r,

(2) $\lim_{r\to 0+} \frac{\mu_p(r,\Theta)}{\mu_k(r)} = 1$, and thus $\mu_p(r,\Theta) \le \mu_k(r)$ for all $r > 0$,

(3) if $\mu_p(t,\Theta) = \mu_k(t)$ for $t \in [a,r]$ and any Θ, then $B(p,r)$ is isometric to $B_k(r)$.

Proof. (1) The monotonicity of $\frac{\mu_p(r,\Theta)}{\mu_k(r)}$ follows from

$$\frac{d}{dt}\left(\log \frac{\mu_p(t,\Theta)}{\mu_k(t)}\right) = \frac{\mu_p'(t,\Theta)}{\mu_p(t,\Theta)} - \frac{\mu_k'(t)}{\mu_k(t)} \le 0.$$

(2) By $V_j(r) = r\nabla_{\dot\gamma(0)} V_j + O(r^2)$ we get $\mu_p(r,\Theta) = r^{m-1} + O(r^m)$. The result follows.

(3) If $\mu_p(t,\Theta)=\mu_k(t)$ for $t \le r$ and any Θ, then

$$(\Delta d_p)(\exp_p(t\Theta)) = \frac{\mu_p'(t,\Theta)}{\mu_p(t,\Theta)} = \frac{\mu_k'(t)}{\mu_k(t)} = (\Delta_k d_k)(t).$$

It follows that $Ric(\dot\gamma(t)) = (m-1)k$, and (since Θ and thus γ are arbitrary) any normal Jacobi field along any geodesic starting at p is almost parallel. By PSet 4, (M,g) has constant sectional curvature, and thus the constant has to be k. Finally by Cartan's local isometry theorem, $B(p,r)$ is isometric to $B_k(r)$ in M_k^m. □

¶ The Bishop-Gromov volume comparison theorem.

Note that for t large, it may happen that $t\Theta \not\in \Sigma(p)$. However, in this case $\mu_p(t,\Theta)=0$. So the monotonicity holds for all t. It is this simple observation that leads to a global comparison instead of a local comparison inside the interior radius. In fact, by integrating the volume density we get

Theorem 2.4 (Bishop-Gromov). *If (M,g) is a complete Riemannian manifold with* $\mathrm{Ric} \ge (m-1)k$, *and $p \in M$ is an arbitrary point. Let $S_k(r)$ and $B_k(r)$ be the metric sphere and the metric ball of radius r in M_k^m. Then the functions*

$$\frac{\mathrm{Area}(S(p,r))}{\mathrm{Area}(S_k(r))} \text{ and } \frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}(B_k(r))}$$

are non-increasing in r, and both tends to 1 as $r \to 0+$. Moreover, the quotient is a constant for $r \in [r_1, r_2]$ if and only if $B(p,r_2)$ is isometric to $B_k(r_2)$.

Proof. By definition

$$\mathrm{Area}(S(p,r)) = \int_{S^{m-1}} \mu_p(r,\Theta)d\Theta, \quad \mathrm{Vol}(B(p,r)) = \int_0^r \int_{S^{m-1}} \mu_p(r,\Theta)d\Theta dr.$$

Thus if we denote the surface area of the sphere $S^{m-1} \subset \mathbb{R}^m$ by ω_{m-1}, then

$$\frac{\mathrm{Area}(S(p,r))}{\mathrm{Area}(S_k(r))} = \frac{\int_{S^{m-1}} \mu_p(r,\Theta)d\Theta}{\int_{S^{m-1}} \mu_k(r)d\Theta} = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \frac{\mu_p(r,\Theta)}{\mu_k(r)} d\Theta$$

is non-increasing and tends to 1 as $r \to 0$. As a consequence,

$$\begin{aligned}\frac{d}{dr}\left(\log\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}(B_k(r))}\right) &= \frac{\mathrm{Area}(S(p,r))}{\mathrm{Vol}(B(p,r))} - \frac{\mathrm{Area}(S_k(r))}{\mathrm{Vol}(B_k(r))} \\ &= \frac{\int_0^r (\mathrm{Area}(S(p,r))\mathrm{Area}(S_k(t)) - \mathrm{Area}(S_k(r))\mathrm{Area}(S(p,t)))dt}{\mathrm{Vol}(B(p,r))\mathrm{Vol}(B_k(r))} \\ &\le 0\end{aligned}$$

and thus $\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}(B_k(r))}$ is also non-increasing in r, and tends to 1 as $t \to 0$. □

Corollary 2.5. *We have*

$$\mathrm{Area}(S(p,r)) \le \mathrm{Area}(S_k(r)), \textit{and} \quad \mathrm{Vol}(B(p,r)) \le \mathrm{Vol}(B(p,r))$$

for all $r \ge 0$. Moreover, equality holds if and only if $B(p,r)$ is isometric to $B_k(r)$.