LECTURE 27: THE SPHERE THEOREM

1. Critical Point Theory of Distance Functions

¶ A glimpse into Morse theory.

As we have mentioned in Lecture 20, Morse theory is a basic tool in differential topology relates the topology of M to the critical points of a Morse function on M, and the theory has many applications in Riemannian geometry.

On major theme in Morse theory is to study the change of topology of sub-level sets $M_a = \{x \mid f(x) \le a\}$ as a varies. Two crucial facts in Morse theory are

Theorem A (Isotopy lemma)**.** *Suppose $f \in C^\infty(M)$, $f^{-1}([a,b])$ is compact, and $f^{-1}([a,b]) \cap \mathrm{Crit}(f) = \emptyset$. Then M_a is diffeomorphic [and is a deformation retract] to M_b.*

Idea of proof. "Push" M_b down to M_a along trajectories of $\frac{\nabla f}{|\nabla f|}$ (which are of constant speed and are perpendicular to each level set $f = c$). The topology is not changed during this procedure. [See my notes on smooth manifolds for detail] □

One can show that on any smooth manifold, there are lots of "good Morse functions" [i.e., the critical points are disjoint, non-degenerate and take different values].

Theorem B. *Suppose $f \in C^\infty(M)$, p is a non-degenerate critical point of f, $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact, and $f^{-1}([c-\varepsilon, c+\varepsilon]) \cap \mathrm{Crit}(f) = \{p\}$. Then $M_{c+\varepsilon}$ is homotopy equivalent to "$M_{c-\varepsilon}$ with a λ-cell attached", where λ is the index of p.*

As a result, one can detect the homotopy type of M from a good Morse function. A useful theorem in differential topology that can be used to produce a sphere is

Theorem C (Brown)**.** *If M is a compact manifold, $M = U_1 \cup U_2$, and U_1, U_2 are both homeomorphic to $\mathbb{R}^m$, then M is homeomorphic to S^m.*

As a consequence, one has

Theorem D (Reeb)**.** *If M is compact, $f \in C^\infty(M)$ is a Morse function that has only two critical points, then M is homeomorphic to S^m.*

Proof. The two critical points have to be the maximum/minimum of f. Take a close to the minimal value of f and b closed to the maximal value of f, so that both $f^{-1}((-\infty, a))$ and $f^{-1}((b, +\infty))$ are homeomorphic to $\mathbb{R}^m$. Take b' between b and the maximal value of f. By Theorem A $f^{-1}((-\infty, a))$ is homeomorphic to $f^{-1}((-\infty, b'))$. Since $M = f^{-1}((-\infty, b')) \cup f^{-1}((b, +\infty))$,by Brown's theorem, M is homeomorphic to S^m. □

Note that in this proof we avoided the use of Theorem B.

¶ Critical points of the distance function.

Now let (M, g) be a Riemannian manifold, and $p \in M$ be a point. In some sense the distance functions d_p's are the most natural functions that are defined on M. Although $d_p \notin C^\infty(M)$, Grove and Shiohama succeeded in developing a Morse theory for d_p in 1977 which played an important role in studying the topology of Riemannian manifolds. To get an idea let's examine the behavior of d_p on (M, g):

- As we have seen, the distance function d_p is smooth at any $q \notin \mathrm{Cut}(p) \cup \{p\}$, with $(\nabla d_p)_q = \dot\gamma(d(p, q))$, where γ is the unique minimizing normal geodesic from p to q. In particular, $|\nabla d_p| = 1$ at any $q \notin \mathrm{Cut}(p) \cup \{p\}$. As a result, these points are not critical points of d_p.
- The singularity of d_p at the point p is not too bad, since it is the only minimum of d_p, and the change of topology near p is well-understood. This point can be regarded as a "trivial critical point" of d_p.
- So we are more interested in those points $q \in \mathrm{Cut}(p)$. They are candidates of critical points for d_p. To get a better idea, let's take a closer look at the example $S^1 \times \mathbb{R}$: given any $p = (e^{i\theta}, z_0) \in S^1 \times \mathbb{R}$, $\mathrm{Cut}(p) = \{(e^{-i\theta}, z) \mid z \in \mathbb{R}\}$. When will the topology of $M_a = \{q \mid d(p, q) < a\}$ change as a varies? Obviously
  - the topology of M_a will not change for $a < \pi$ [no critical points there],
  - the topology of M_a will change when a pass the value π, i.e. pass the cut point $\tilde p = (e^{-i\theta}, z_0)$,
  - the topology of M_a will not change for $a > \pi$, although there are two non-smooth points of d_p for each such a.

  Why the topology for M_a will not change for $a > \pi$? Because although there are two minimizing geodesics meeting at one point $q \in \mathrm{Cut}(p)$ with $d_p(q) = a$, their directions at q lie in the same open half space. As a result, there is one direction that one can "flow-out" M_a to M_b $(b > a)$, and that is a direction whose angles with both geodesics are obtuse. Why such "flow-out" argument fails for $a = \pi$, i.e. at $\tilde p = (e^{-i\theta}, z_0)$? Because for the two geodesics meeting at $\tilde p$, one can't find such a direction whose angles with both geodesics are obtuse!

We are thus led to the following definition:

Definition 1.1. A point $q \neq p$ is called a *critical point* of d_p [or a critical point of p] if for all $X_q \in T_qM$, there exists a minimizing geodesic γ from $q = \gamma(0)$ to p so that

$$\langle \dot\gamma(0), X_q \rangle \geq 0,$$

i.e. the angle α between $\dot\gamma(0)$ and X_q is no moore than $\frac{\pi}{2}$.

The set of all such critical points of of d_p will be denoted as $\mathrm{CP}(p)$. Note that if q is not a critical point of d_p, then the tangent vector of all minimizing geodesic from q to p lie in an open half space of T_qM.

¶ Examples of critical points of the distance function.

Example. Here are some immediate examples:

- $M = S^2$ the standard sphere: the only critical point of p is its antipodal $\bar{p}$.
- $M = S^1 \times \mathbb{R}^1$ the cylinder: the only critical point of $(e^{i\theta}, z)$ is $(e^{-i\theta}, z)$.
- $M = S^1 \times S^1$ the flat torus with fundamental domain a square centered at p: the critical points are the two midpoints of the sides and the corner point.
- If γ is a closed geodesic of length $2l$ so that both $\gamma|_{[0,l]}$ and $\gamma|_{[l,2l]}$ are minimal, then $\gamma(l)$ is a critical point of $\gamma(0)$.

Recall that for any point $q \notin \mathrm{Cut}(p)$, there is a unique minimizing geodesic joining p to q. So $\exp_p$ is injective on an open ball $B^p(0,r) \subset T_pM$ if $B(p,r) \subset M \setminus \mathrm{Cut}(p)$. Moreover, for most points in $\mathrm{Cut}(p)$, there exists at least two minimizing normal geodesic to p (c.f. PSet 3). So we conclude

$$\mathrm{inj}_p(M,g) = \mathrm{dist}(p, \mathrm{Cut}(p)) \qquad \text{and} \qquad \mathrm{inj}(M,g) = \inf_{p \in M} \mathrm{dist}(p, \mathrm{Cut}(p)).$$

Proposition 1.2. *If $q \in \mathrm{Cut}(p)$ is not conjugate to p and*

$$d(p,q) = d(p, \mathrm{Cut}(p)),$$

then there are exactly two minimizing normal geodesic γ and σ from p to q, and $\dot{\sigma}(l) = -\dot{\gamma}(l)$. In particular, q is a critical point of p.

Proof. We have seen in Theorem 1.7 in Lecture 21 that there are at least two minimizing normal geodesic γ, σ from p to q. We shall prove $\dot{\sigma}(l) = -\dot{\gamma}(l)$. Suppose not, then there exists $X_q \in T_qM$ with $|X_q| = 1$, such that

$$\langle X_q, \dot{\gamma}(l) \rangle < 0 \quad \text{and} \quad \langle X_q, \dot{\sigma}(l) \rangle < 0.$$

Since q is not conjugate to p along γ, there exists $U \ni l\dot{\gamma}(0)$ such that $\exp_p|_U$ is a diffeomorphism. Let

$$\xi(s) = (\exp_p|_U)^{-1} \exp_q(sX_q).$$

Then $\gamma_s(t) = \exp_p(\frac{t}{l}\xi(s))$ is a geodesic variation of γ. By the first variation formula,

$$\frac{d}{ds}|_{s=0} L(\gamma_s) = \langle X_q, \dot{\gamma}(l) \rangle < 0.$$

So for $s > 0$ small enough, $L(\gamma_s) < L(\gamma) = l$.

Similarly one can construct a geodesic variation

$$\sigma_s(t) = \exp_p(\frac{s}{l}\eta(s)), \quad \text{where } \eta(s) = (\exp_p|_V)^{-1} \exp_q(sX_q)$$

of the minimizing geodesic σ so that $L(\sigma_s) < L(\sigma) = l$ for $s > 0$ small enough. Note that for each s, both γ_s and σ_s are geodesics from p to $\exp_q(sX_q)$. Moreover, for $s > 0$ small enough,

$$l_s := d(p, \exp_q(sX_q) \le L(\gamma_s) \le l.$$

So $\exp_p$ is NOT injective on $B^p(0, \frac{l_s + l}{2})$, which contradicts with the fact that $\exp_p$ is a diffeomorphism on $B^p(0,l)$, since $l = d(p, \mathrm{Cut}(p))$. $\square$

¶ The isotopy lemma for d_p.

As in the usual Morse theory the following fact is crucial in all applications.

Theorem 1.3 (The Isotopy Lemma)**.** *Suppose (M,g) is complete, $b>a>0$, and $d_p^{-1}([a,b])\cap \mathrm{CP}(d)=\emptyset$. Then M_a is diffeomorphic [and is a deformation retract] to M_b.*

Proof. For any point $q\notin \mathrm{CP}(p)$, then there exists $X_q\in T_qM$ so that for any minimizing geodesic γ from q to p, the angle

$$\angle(X_q,\dot\gamma(0))<\frac{\pi}{2}.$$

Extend the vector X_q to a vector field X^q defined on a neighborhood U_q of q so that for any $\bar q\in U_q$ and any minimizing geodesic $\bar\gamma$ from $\bar q$ to p,

$$\angle(X^q(\bar q),\dot{\bar\gamma}(0))<\frac{\pi}{2}.$$

Take a locally finite covering $\{U_{q_i}\}$ of $\overline{B(p,b)}\setminus B(p,a)$ using such neighborhoods, and a smooth partition of unity $\{\rho_i\}$ subordinate to this covering. Let $X=\sum \rho_i X^{q_i}$. Clearly X is a smooth non-vanishing vector field on $\overline{B(p,b)}\setminus B(p,a)$, since

$$\langle X(\bar q),\dot{\bar\gamma}(0)\rangle=\sum \rho_i\langle X^{q_i}(\bar q),\dot{\bar\gamma}(0)\rangle>0,\quad \forall \bar q\in \overline{B(p,b)}\setminus B(p,a).$$

We normalize X so that $|X(\bar q)|=1$ at each $\bar q$, and then repeat the proof of Theorem A. More precisely, for any $\bar q\in\overline{B(p,b)}\setminus B(p,a)$ we let $\sigma^{\bar q}$ be the integral curve of X passing $\bar q$, and for any $\sigma^{\bar q}(t)\in \overline{B(p,b)}\setminus B(p,a)$ we let $\bar\gamma_t$ be the minimizing geodesic from $\sigma^{\bar q}(t)$ to p. Then by the first variation formula,

$$\frac{d}{dt}(d_p(\sigma^{\bar q}(t))=\frac{d}{dt}L(\bar\gamma_t)=-\langle X(\sigma^{\bar q}(t)),\dot{\bar\gamma}_t(0)\rangle.$$

Fix $t_1<t_2$ so that $\sigma^{\bar q}([t_1,t_2])\subset\overline{B(p,t_2)}\setminus B(p,t_1)$. By compactness, $\exists\varepsilon>0$ so that

$$-\langle X(\sigma^{\bar q}(t)),\dot{\bar\gamma}_t(0)\rangle\le-\cos(\frac{\pi}{2}-\varepsilon)<0$$

for all $t\in[t_1,t_2]$. It follows

$$d_p(\sigma^{\bar q}(t_2))-d_p(\sigma^{\bar q}(t_1))=\int_{t_1}^{t_2}\frac{d}{dt}(d_p(\sigma^{\bar q}(t)))dt\le-(t_2-t_1)\cos(\frac{\pi}{2}-\varepsilon)<0.$$

So as t increases, d_p is strictly decreasing along the integral curves $\sigma^{\bar q}(t)$ of X inside $\overline{B(p,t_2)}\setminus B(p,t_1)$ as. So the flow of X gives the desired diffeomorphism. □

Since the topology changes after the "farthest point", we get

Corollary 1.4. *Let (M,g) be a compact Riemannian manifold, $p\in M$, and q is a farthest point from p, then q is a critical point of p.*

In particular, if $d(p,q)=\mathrm{diam}(M,g)$, then for any $X_p\in T_pM$, there is a minimal geodesic γ from $p=\gamma(0)$ to q so that $\langle\dot\gamma(0),X_p\rangle\ge 0$.

¶ The Reeb theorem for d_p.

Although there is no Morse lemma and there is no index for the critical points of a distance function, near the trivial critical point p of d_p the sub-level set is still an m-ball. Similar phenomena happens near a "non-degenerate (=discrete) farthest point". So it is not amazing that we still have the following analogue to the Reeb theorem for d_p:

Corollary 1.5. *Let (M, g) be a compact Riemannian manifold and $p \in M$. If d_p has only one nontrivial critical point $q \neq p$, then M is homeomorphic to S^m.*

Proof. According to Corollary 1.4, q has to be the only farthest point of p. Take r_1 small so that both $B(p, r_1)$ and $B(q, r_1)$ are homeomorphic to $\mathbb{R}^m$. Take $r_2 \in (r_1, d(p, q))$ large so that $B(p, r_2) \cup B(q, r_1) = M$. Then d_p has no critical point in $\overline{B(p, r_2)} \setminus B(p, r_1)$. By the isotopy lemma, $B(p, r_2)$ is homeomorphic to $B(p, r_1)$, and thus is homeomorphic to $\mathbb{R}^m$. By Brown's theorem, M is homeomorphic to S^m. □

2. Some sphere theorems

¶ The diameter sphere theorem of Grove-Shiohama.

As a first application of the critical point theory of distance function, we shall prove the following diameter sphere theorem:

Theorem 2.1 (Grove-Shiohama). *Let (M, g) be a complete connected Riemannian manifold with*

$$K > \frac{1}{4} \quad \text{and} \quad \operatorname{diam}(M, g) \geq \pi,$$

then M is homeomorphic to S^m.

Proof. Since M is compact, there exists $k > \frac{1}{4}$ so that $K \geq k$. By Bonnet-Meyer, $\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}$. In view of Cheng's maximal diameter theorem, we may assume

$$\operatorname{diam}(M, g) = l < \frac{\pi}{\sqrt{k}}$$

Let $p, q \in M$ so that $d(p, q) = l = \operatorname{diam}(M, g)$. By Corollary 1.4, q is a critical point of p. By Corollary 1.5, it is enough to prove that p has no other critical points.

Suppose to the contrary, $\bar{q} \neq q$ is a critical point of p. Denote $l' = d(p, \bar{q})$ and $l'' = d(q, \bar{q})$. Let γ be a minimizing normal geodesic from $q = \gamma(0)$ to $\bar{q} = \gamma(l'')$. By definition of critical points, there exists a minimizing normal geodesic σ from $\bar{q} = \sigma(0)$ to $p = \sigma(l')$ so that

$$\alpha = \angle(-\dot{\gamma}(l), \dot{\sigma}(0)) \leq \frac{\pi}{2}.$$

Apply the Toponogov comparison theorem (triangle version), we conclude that there is a geodesic triangle in $S^m(\frac{1}{\sqrt{k}})$ whose sides have lengths l, l', l'', so that the opposite angle of l is $\tilde{\alpha} \le \frac{\pi}{2}$. Since $\pi \le l < \frac{\pi}{\sqrt{k}}$, we get

$$\frac{\pi}{2} \le \frac{l}{2} < \sqrt{k}l < \pi \qquad \text{and} \quad 0 < \sqrt{k}l', \sqrt{k}l'' < \pi.$$

Thus by the cosine law in $S^m(\frac{1}{\sqrt{k}})$,

$$\begin{aligned} 0 > \cos(\sqrt{k}l) &= \cos(\sqrt{k}l')\cos(\sqrt{k}l'') + \sin(\sqrt{k}l')\sin(\sqrt{k}l'')\cos(\tilde{\alpha}) \\ &\ge \cos(\sqrt{k}l')\cos(\sqrt{k}l''), \end{aligned}$$

which implies that exactly one of l' and l'' is strictly greater than $\frac{\pi}{2\sqrt{k}}$, and the other is strictly smaller than $\frac{\pi}{2\sqrt{k}}$. Without loss of generality, assume $0 < l'' < \frac{\pi}{2\sqrt{k}}$. Then

$$\cos(\sqrt{k}l) \ge \cos(\sqrt{k}l')\cos(\sqrt{k}l'') > \cos(\sqrt{k}l').$$

In other words, $l < l'$. This contradicts with the fact $l = \text{diam}(M, g)$. □

¶ The topological sphere theorem.

In 1940s, Hopf asked the followng question:

> If a simply connected complete Riemannian manifold (M, g) has sectional curvature close to 1, is it homeomorphic to or even diffeomorphic to the sphere S^m?

In 1951 Rauch gave a positive answer: he proved that (M, g) is homeomorphic to S^m if it has sectional curvature $\delta \le K \le 1$, where $\delta \approx 0.75$ is the solution to $\sin(\sqrt{\delta}\pi) = \sqrt{\delta}/2$. The pinching constant δ was improved to 1/4 by Berger (1960, for m even) and Klingenberg (1961, for m odd).

Theorem 2.2 (Topological sphere theorem, Rauch-Berger-Klingenberg)**.** *Let (M, g) be a complete simply connected Riemannian manifold with $\frac{1}{4} < K \le 1$. Then M is homeomorphic to S^m.*

Remark. The constant 1/4 is sharp, since $\mathbb{CP}^m$ has sectional curvature $1/4 \le K \le 1$. In fact, suppose (M, g) is a complete and simply connected, then

(1) (Berger 1983) If m is even, then there exists $\varepsilon(m) > 0$ so that if $\frac{1}{4} - \varepsilon(m) \le K \le 1$, then M is either homeomorphic to S^m or diffeomorphic to one of the CROSSes: $\mathbb{CP}^{m/2}, \mathbb{HP}^{m/4}, \text{Ca}\mathbb{P}^2$.
(2) (Abresch-Meyer, 1994) If m is odd, then there exists $\varepsilon > 0$ so that if $\frac{1}{4} - \varepsilon \le K \le 1$, then M is homeomorphic to S^m.

Remark. The other half of Hopf's question was proved by Brendle and Schoen:

> **Theorem 2.3** (Differential sphere theorem, Brendle-Schoen 2009)**.** *Let (M, g) be complete and simply connected, such that any $p \in M$, $0 < \sup_{\Pi_p} K(\Pi_p) < 4 \inf_{\Pi_p} K(\pi_p)$, then M is diffeomorphic to S^m.*

Remark (Sphere theorem in lower dimensions).

(1) For $m = 2$: let M be an oriented compact surface with $K > 0$, then by the Gauss-Bonnet formula M is diffeomorphic to S^2.
(2) For $m = 3$, by introducing the method of Ricci flow, R.Hamilton proved in 1982 that if (M, g) is a 3 dimensional compact Riemannian manifold with $\mathrm{Ric} > 0$, then (M, g) is diffeomorphic to S^3.
(3) For $m = 4$ there is a very interesting conformal sphere theorem:

   Theorem 2.4 (Chang-Gursky-Yang 2003)**.** *Let (M, g) be a compact 4-manifold whose Yamabe invariant is positive. Suppose $\int_M |W|^2 dv < 16\pi^2\chi(M)$, then M is diffeomorphic to S^4 or $\mathbb{RP}^4$.*

In view of Grove-Shiohama's diameter sphere theorem, to prove topological sphere theorem it is enough to prove

Theorem 2.5 (Klingenberg injectivity radius estimate)**.** *Let (M, g) be a complete simply connected Riemannian manifold with $1/4 < K \le 1$, then* $\mathrm{inj}(M, g) \ge \pi$.

In what follows we will prove Klingenberg's injectivity radius estimate for m even. The odd case is more involved.

¶ Klingenberg lemma.

We need

Lemma 2.6 (Klingenberg lemma)**.** *Let (M, g) be a compact Riemannian manifold whose sectional curvature satisfies $K \le C$ for some constant C. Then either*

$$\mathrm{inj}(M, g) \ge \frac{\pi}{\sqrt{C}}$$

or there exists a closed geodesic γ in M whose length is minimum among all closed geodesics, such that

$$\mathrm{inj}(M, g) = \frac{1}{2}L(\gamma).$$

Proof. Take $p \in M$ and $q \in \mathrm{Cut}(p)$ so that $\mathrm{dist}(p, q) = \mathrm{inj}(M, g)$. If q is conjugate to p along some minimizing geodesic, then by Corollary 2.2 in Lecture 23,

$$\mathrm{inj}(M, g) = \mathrm{dist}(p, q) \ge \frac{\pi}{\sqrt{C}}.$$

If q is not conjugate to p, then by Proposition 1.2, there exists two minimizing normal geodesics σ, τ joining p to q so that $\dot\sigma(l) = -\dot\tau(l)$, where $l = \mathrm{dist}(p, q)$. Since p is also a cut point of q, and by definition p realize the distance from q to $\mathrm{Cut}(q)$. It follows that $\dot\sigma(0) = -\dot\tau(0)$. So σ and τ together form a closed geodesic. If we denote this closed geodesic by γ, then

$$\mathrm{inj}(M, g) = \frac{1}{2}L(\gamma).$$

Finally we prove γ has minimal length among all closed geodesics: Otherwise if there is another closed geodesic γ' with length $L(\gamma') < L(\gamma)$, and let p', q' be two "antipodal" points on γ', i.e. $\mathrm{dist}(p',q') = \frac{1}{2}L(\gamma')$, then by definition there is a point q'' on γ' which lies in $\mathrm{Cut}(p')$, and $\mathrm{dist}(p',q'') \le \frac{1}{2}L(\gamma') < \mathrm{inj}(M,g)$. Contradiction. □

¶ Proof of Klingenberg injectivity radius estimate, m even.

[In what follows we only need to assume M is orientable (which implies M is simply connected by Sygne's theorem).]

By Bonnet-Myers' theorem, M is compact. So there exists $p \in M$ and $q \in \mathrm{Cut}(p)$ so that $\mathrm{dist}(p,q) = \mathrm{inj}(M,g) =: l$. Suppose the theorem fails, i.e. $l < \pi$. Then by Corollary 2.2 in lecture 23, q is not conjugate to p. So according to Klingenberg lemma, there exists a closed normal geodesic γ in M passing $p = \gamma(0)$ and $q = \gamma(l)$ whose length is $L(\gamma) = 2l < 2\pi$.

Since M is of even-dimension and is oriented, by repeating the proof of Synge's theorem, we can find a vector field $X(t)$ parallel along γ with

$$X(2l) = X(0) = X_p \in \dot\gamma(0)^\perp,$$

so that the variation of γ with variation field X satisfies

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s) = -\int R(\dot\gamma, X, \dot\gamma, X)dt < 0.$$

In other words, $L(\gamma_s) < L(\gamma)$ for all small $s \neq 0$.

Denote $p_s = \gamma_s(0)$ and let $q_s = \gamma_s(l_s)$ be the point on γ_s which is farthest to p_s. Then

$$\mathrm{dist}(p_s, q_s) < l = \mathrm{inj}(M,g),$$

so there exists a unique normal minimizing geodesic σ_s joint $q_s = \sigma_s(0)$ to p_s. Since $\lim_{s\to 0} q_s = q$, there exists a sequence $s_i \to 0$ so that $\dot\sigma_{s_i}(0)$ converges to a unit vector $Y_q \in T_qM$. By continuity, $\sigma(t) = \exp_q(tY_q)$ is a minimizing normal geodesic connecting q to p. In what follows we will show $\dot\sigma(0) \perp \dot\gamma(l)$, so that σ is not one of the two parts of γ. As a consequence, we get three minimizing geodesic from q to p. This contradicts with Proposition 1.2.

It remains to prove $\dot\sigma(0) \perp \dot\gamma(l)$. We let $\sigma_{s,t}$ be the minimizing normal geodesic from $p_s = \gamma_s(0)$ to $\gamma_s(t)$ for $\gamma_s(t)$ close to $q_s = \gamma_s(l_s)$. Then $\sigma_{s,t}$ is a variation of $\sigma_s = \sigma_{s,l_s}$. By the choice of q_s, $L(\sigma_{s,t}) \le L(\sigma_s)$. So according to the first variation formula,

$$0 = \left.\frac{d}{dt}\right|_{t=l_s} E(\sigma_{s,t}) = -\langle \dot\sigma_s(0), \dot\gamma_s(l_s)\rangle.$$

It follows that $\dot\sigma_s(0) \perp \dot\gamma_s(l_s)$. Passing to the subsequence s_i and taking limit, we get $\dot\sigma(0) = Y_q \perp \dot\gamma(l)$.

Remark. By checking the prove above one can see that for the case $m = \dim M$ even, it's enough to assume that (M,g) is oriented and satisfies the weaker curvature condition that there exists ε such that $0 < \varepsilon < K \le 1$.