

## LECTURE 28: BOCHNER'S TECHNIQUE AND APPLICATIONS

In studying the relation between the curvatures of a Riemannian manifold and its geometry/topology, another very useful method is the so called Bochner technique.

### 1. BOCHNER'S FORMULA

#### ¶ Bochner's formula.

We start with

**Theorem 1.1.** *Let  $(M, g)$  be a Riemannian manifold, and  $X \in \Gamma^\infty(TM)$ .*

(1) *If  $\nabla X$  is symmetric, i.e.  $\langle \nabla_u X, v \rangle = \langle \nabla_v X, u \rangle$  for all  $u, v \in T_x X$ , then*

$$\frac{1}{2} \Delta(|X|^2) = |\nabla X|^2 + \langle X, \nabla(\operatorname{div} X) \rangle + Rc(X, X).$$

(2) *If  $\nabla X$  is anti-symmetric, i.e.  $\langle \nabla_u X, v \rangle = -\langle \nabla_v X, u \rangle$  for all  $u, v \in T_x X$ , then*

$$\frac{1}{2} \Delta(|X|^2) = |\nabla X|^2 - Rc(X, X).$$

*Proof.* (1) With Riemannian normal coordinates centered at  $x$ , we have

$$\nabla_{\partial_i} \partial_j(x) = 0, \quad \forall i, j.$$

Recall at  $x$  one can write

$$(\nabla^2 f)_x(\partial_i, \partial_j) = (\partial_i \partial_j f)(x) \quad \text{and} \quad (\Delta f)(x) = \sum (\partial_i \partial_i f)(x).$$

It follows that at  $x$ ,

$$\begin{aligned} \frac{1}{2} \Delta(|X|^2) &= \frac{1}{2} \sum_i \partial_i \partial_i \langle X, X \rangle = \sum_i \partial_i \langle \nabla_{\partial_i} X, X \rangle \\ &\stackrel{\star}{=} \sum_i \partial_i \langle \nabla_X X, \partial_i \rangle \\ &= \sum_i \langle \nabla_{\partial_i} \nabla_X X, \partial_i \rangle \\ &= \sum_i (\langle \nabla_X \nabla_{\partial_i} X, \partial_i \rangle - \langle \nabla_{[X, \partial_i]} X, \partial_i \rangle - \langle R(X, \partial_i) X, \partial_i \rangle) \\ &= \sum_i (\langle \nabla_X \nabla_{\partial_i} X, \partial_i \rangle - \langle \nabla_{[X, \partial_i]} X, \partial_i \rangle + Rm(X, \partial_i, X, \partial_i)) \\ &= \sum_i (\langle \nabla_X \nabla_{\partial_i} X, \partial_i \rangle - \langle \nabla_{[X, \partial_i]} X, \partial_i \rangle) + Rc(X, X). \end{aligned}$$

Note that if we write  $X = X^i \partial_i$ , then

$$\operatorname{Tr}(\nabla X) = \sum_i \langle \nabla_{\partial_i} X, \partial_i \rangle \stackrel{\clubsuit}{=} \sum_i \partial_i \langle X, \partial_i \rangle = \sum_i \partial_i X^i = \operatorname{div} X.$$

it follows

$$\sum_i \langle \nabla_X \nabla_{\partial_i} X, \partial_i \rangle = \sum_i X \langle \nabla_{\partial_i} X, \partial_i \rangle = X(\operatorname{div} X) = \langle X, \nabla \operatorname{div} X \rangle.$$

On the other hand, since  $[\partial_i, X] = \nabla_{\partial_i} X - \nabla_X \partial_i = \nabla_{\partial_i} X$ ,

$$\begin{aligned} - \sum_i \langle \nabla_{[X, \partial_i]} X, \partial_i \rangle &= \sum_i \langle \nabla_{\nabla_{\partial_i} X} X, \partial_i \rangle \\ &\stackrel{\star}{=} \sum_i \langle \nabla_{\partial_i} X, \nabla_{\partial_i} X \rangle \\ &= |\nabla X|^2. \end{aligned}$$

So the conclusion follows.

(2) If  $\nabla X$  is anti-symmetric, then there will be a negative sign at the right hand side of the two  $\star$ , and we will get 0 after the  $\clubsuit$ . So the conclusion follows.  $\square$

### ¶ Bochner's formula for smooth functions.

In particular, if  $u \in C^\infty(M)$ , then  $X = \nabla u$  is smooth and  $\nabla X = \nabla^2 u$  is symmetric. Moreover,  $\operatorname{div} X = \operatorname{div} \nabla u = \Delta u$ . It follows

**Theorem 1.2.** *For any  $u \in C^\infty(M)$ ,*

$$\frac{1}{2} \Delta(|\nabla u|^2) = |\nabla^2 u|^2 + \langle \nabla u, \nabla(\Delta u) \rangle + \operatorname{Ric}(\nabla u, \nabla u).$$

Sometimes one need to replace the Hessian term  $|\nabla^2 u|^2$  by a simpler one. Note that by Cauchy-Schwartz inequality, for any  $A = (a_{ij})$ ,

$$|A|^2 = \sum_{ij} |a_{ij}|^2 \geq \sum_I a_{ii}^2 \geq \frac{1}{m} \left( \sum_i a_{ii} \right)^2 = \frac{1}{m} (\operatorname{Tr} A)^2.$$

As a result, we get  $|\nabla^2 u|^2 \geq \frac{1}{m} (\Delta u)^2$  and thus

**Corollary 1.3.** *For any  $u \in C^\infty(M)$ ,*

$$\frac{1}{2} \Delta(|\nabla u|^2) \geq \frac{1}{m} (\Delta u)^2 + \langle \nabla u, \nabla(\Delta u) \rangle + \operatorname{Ric}(\nabla u, \nabla u).$$

*Remark.* In particular, if  $\operatorname{Ric} \geq (m-1)k$ , then for any  $u \in C^\infty(M)$ ,

$$(*) \quad \frac{1}{2} \Delta(|\nabla u|^2) \geq \frac{1}{m} (\Delta u)^2 + \langle \nabla u, \nabla(\Delta u) \rangle + (m-1)k |\nabla u|^2.$$

Conversely, if this inequality holds for any  $u \in C^\infty(M)$ , then we must have  $Ric \geq (m-1)k$ . To see this, given any  $x_0 \in M$  and any  $X_0 \in T_{x_0}M$ , we take  $u \in C^\infty(M)$  so that  $\nabla u(x_0) = X_0$  and  $(\nabla^2 u)_{x_0} = \text{cId}$ . Then by Bochner formula and (\*),

$$|\nabla^2 u|^2 + Rc(X_0, X_0) = \frac{1}{2}\Delta(|\nabla u|^2) - \langle \nabla u, \nabla(\Delta u) \rangle \geq \frac{1}{m}(\Delta u)^2 + (m-1)k|X_0|^2,$$

On the other hand, our choice of  $u$  implies  $|\nabla^2 u|^2 = \frac{1}{m}(\Delta u)^2$ , so we get

$$Ric \geq (m-1)k.$$

The condition (\*) is used in discrete geometric analysis (on graphs one can define  $\nabla$  and  $\Delta$  but not curvature tensor) as a definition of " $Ric \geq (m-1)k$ ".

### ¶ Bochner formula for closed 1-forms.

Recall from Lecture 2 that given any smooth 1-form  $\omega \in \Omega^1(M)$ , the musical isomorphism produce a smooth vector field  $X = \sharp\omega$ . It is not hard to check

$$|X| = |\omega|, \quad |\nabla X|^2 = |\nabla\omega|^2.$$

Now suppose  $\omega \in \Omega^1(M)$  is a closed 1-form. Then locally  $\omega$  is exact, i.e. locally of the form  $\omega = du$ . As a result,  $X = \sharp\omega = \nabla u$ , and  $\nabla X$  is symmetric<sup>1</sup>. So we may apply part (1) of Theorem 1.1 to get

$$\frac{1}{2}\Delta(|\omega|^2) = |\nabla\omega|^2 + \langle \sharp\omega, \nabla \text{div}\sharp\omega \rangle + Rc(\sharp\omega, \sharp\omega).$$

To proceed let's do some local computation. Suppose  $\omega = \omega_i dx^i$ . Then

$$\sharp\omega = \sum_i \omega_i \partial_i, \quad \text{div}\sharp\omega = \sum_i \partial_i \omega_i, \quad \nabla \text{div}\sharp\omega = \sum_{i,j} \partial_j \partial_i \omega_i \partial_j.$$

and thus

$$\langle \sharp\omega, \nabla \text{div}\sharp\omega \rangle = \sum_{i,j} \omega_j \partial_j \partial_i \omega_i.$$

On the other hand, as we have seen in Lecture 4, for any smooth function  $f$ ,

$$\text{div}(f\sharp\omega) = f \text{div}\sharp\omega + \langle \nabla f, \sharp\omega \rangle.$$

Now we assume  $M$  is compact. After integration we get

$$0 = \int_M \text{div}(f\sharp\omega) dV_g = \int_M (f \text{div}\sharp\omega + \langle \nabla f, \sharp\omega \rangle) dV_g = \int_M (f \text{div}\sharp\omega + \langle df, \omega \rangle) dV_g.$$

It follows that

$$\langle df, \omega \rangle_{L^2} = \int_M \langle df, \omega \rangle dV_g = \int_M f(-\text{div}\sharp\omega) dV_g = \langle f, -\text{div}\sharp\omega \rangle_{L^2}.$$

So if we define  $\delta\omega = -\text{div}\sharp\omega$ . Then  $\delta : \Omega^1(M) \rightarrow C^\infty(M)$  is the  $L^2$ -dual of  $d$ ,

$$\langle df, \omega \rangle_{L^2} = \langle f, \delta\omega \rangle_{L^2}, \quad \forall f \in C^\infty(M), \omega \in \Omega^1(M).$$

<sup>1</sup>In fact one can show that  $\nabla X$  is symmetric if and only if  $\omega = \flat X$  is closed.

For any closed 1-form  $\omega$  we define  $\Delta\omega = d\delta\omega$ . Then locally

$$\Delta\omega = d\left(-\sum_i \partial_i \omega_i\right) = -\sum_{i,j} \partial_j \partial_i \omega_i$$

and thus

$$\langle \omega, \Delta\omega \rangle = -\sum_{i,j} \omega_j \partial_j \partial_i \omega_i = -\langle \sharp\omega, \nabla \operatorname{div} \sharp\omega \rangle.$$

So we end with

**Theorem 1.4** (Bochner's formula for closed 1-form). *Let  $(M, g)$  be a compact Riemannian manifold, then for any closed 1-form  $\omega \in \Omega^1(M)$ ,*

$$\frac{1}{2} \Delta(|\omega|^2) = |\nabla\omega|^2 - \langle \omega, \Delta\omega \rangle + \operatorname{Ric}(\sharp\omega, \sharp\omega).$$

### ¶ Harmonic $k$ -form.

More generally, one can define  $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$  so that

$$\langle \omega, d\eta \rangle_{L^2} = \langle \delta\omega, \eta \rangle_{L^2}, \quad \forall \omega \in \Omega^k(M), \eta \in \Omega^{k-1}(M),$$

and define the *Hodge Laplacian* on all smooth  $k$ -forms to be

$$\Delta := d\delta + \delta d : \Omega^k(M) \rightarrow \Omega^k(M).$$

One can check that when  $k = 0$  this definition coincides with the Laplace-Beltrami operator  $\Delta$  on smooth functions (and thus differed with  $\Delta = \operatorname{tr} \nabla^2$  by a negative sign). A differential form  $\omega \in \Omega^k(M)$  is called a *harmonic  $k$ -form* if  $\Delta\omega = 0$ . In view of the fact

$$\langle \omega, \Delta\omega \rangle_{L^2} = \langle \omega, dd\omega \rangle_{L^2} + \langle \omega, \delta d\omega \rangle_{L^2} = \langle d\omega, d\omega \rangle_{L^2} + \langle \delta\omega, \delta\omega \rangle_{L^2}$$

and the definition of  $\Delta$ , we have

**Proposition 1.5.**  $\omega \in \Omega^k(M)$  is harmonic if and only if  $d\omega = 0$  and  $\delta\omega = 0$ .

Now suppose  $\omega$  is a harmonic 1-form on compact Riemannian manifold  $(M, g)$ . Then  $\omega$  is closed, and thus by Theorem 1.4,

$$0 = \int_M \frac{1}{2} \Delta(|\omega|^2) = \int_M |\nabla\omega|^2 + \int_M \operatorname{Ric}(\sharp\omega, \sharp\omega).$$

So we get

**Theorem 1.6** (Bochner). *Let  $(M, g)$  be a compact Riemannian manifold, then*

- (1) *Suppose  $\operatorname{Ric} \geq 0$ . If  $\Delta\omega = 0$ , then  $\nabla\omega = 0$ .*
- (2) *Suppose  $\operatorname{Ric} \geq 0$ , and  $\operatorname{Ric} > 0$  at one point. If  $\Delta\omega = 0$ , then  $\omega = 0$ .*

According to the famous Hodge theory, the space of harmonic  $k$ -forms is isomorphic to the de Rham cohomology group  $H_{dR}^k(M)$ . So we conclude

**Corollary 1.7.** *Let  $(M, g)$  be a closed oriented Riemannian manifold,  $\operatorname{Ric} \geq 0$ , and  $\operatorname{Ric} > 0$  at one point. Then  $b_1(M) = 0$ .*

*Remarks.* There is a Bochner-Weitzenböck formula that generalize the Bochner formula above to  $k$ -forms using which one can prove: If  $(M, g)$  is a closed Riemannian manifold with nonnegative curvature operator, then all harmonic forms of order  $1 \leq k \leq m - 1$  on  $M$  are parallel.

### ¶ Bochner formula for Killing forms.

Now let's turn to part (2) in Theorem 1.1. As we have seen in PSet 1,  $\nabla X$  is anti-symmetric if and only if  $X$  is a Killing field on  $(M, g)$ . So we get

**Corollary 1.8.** *For any Killing vector field on  $M$ ,*

$$\frac{1}{2}\Delta(|X|^2) = |\nabla X|^2 - Rc(X, X).$$

As a result,

**Theorem 1.9** (Bochner, 1946). *Any Killing vector field of a compact Riemannian manifold with negative Ricci curvature must be zero.*

Since the space of all Killing vector fields on  $(M, g)$  is the Lie algebra of the isometry group  $\text{Iso}(M, g)$  [which is a Lie group], and the isometry group of any compact Riemannian manifold is compact, we conclude that the isometry group of any compact Riemannian manifold with negative Ricci curvature must be a finite group.

## 2. CHEEGER-GROMOLL SPLITTING THEOREM

### ¶ Cheeger-Gromoll splitting theorem.

Let  $(M, g)$  be a complete non-compact connected Riemannian manifold. Recall that a line in  $M$  is a normal geodesic  $\gamma : \mathbb{R} \rightarrow M$  so that

$$d(\gamma(a), \gamma(b)) = |a - b|, \quad \forall a, b \in \mathbb{R}.$$

Unlike the case of rays, it is possible that there is no ray in a complete non-compact Riemannian manifold. For example, a cylinder  $\mathbb{R} \times S^1$  admits many lines, while the paraboloid  $z = x^2 + y^2$  admits no line at all.

As another application of Bochner formula, we prove the following structure theorem for Riemannian manifolds with positive Ricci curvature that admit lines:

**Theorem 2.1** (Cheeger-Gromoll, 1971). *Let  $(M, g)$  be a complete non-compact Riemannian manifold with  $Ric \geq 0$ . Suppose there exists a line in  $M$ . Then  $(M, g)$  is isometric to  $\mathbb{R} \times N$ , where  $N$  is an  $(m - 1)$ -dimensional complete Riemannian manifold with  $Ric \geq 0$ .*

The idea is to construct a function on  $M$  which behaves like the function  $f(x, r) = r$  on  $N \times \mathbb{R}$ , so that the level sets of  $f$  gives the desired component  $N$ . So what is the speciality of the function  $f(x, r) = r$ ? It is smooth, with gradient  $\nabla f = \partial_r$  which has length 1, and has Hessian  $\nabla^2 f = 0$  (so that  $\nabla f$  is parallel). It is in the proof of " $\nabla^2 f = 0$ " that we need Bochner's formula.

To apply Bochner's formula one needs the function to be smooth. However, the construction below uses the distance, and thus the function is only Lipschitz. To solve this problem we will apply theories on PDEs in the barrier sense. More precisely, we need the Hopf-Calabi strong maximum principle,

**Theorem 2.2** (Hopf-Calabi strong maximum principle). *Let  $\Omega \subset M$  be a connected open set. Suppose  $\Delta f \leq 0$  in  $M$  in the barrier sense, and  $f$  attains an interior minimum, then  $f$  is constant on  $\Omega$ .*

We also need the well-known Weyl lemma to increase regularity:

**Theorem 2.3** (Weyl Lemma). *If  $\Delta f = 0$  in the barrier sense, then  $f$  is smooth.*

### ¶ Busemann function.

The function that we need is the so-called Busemann function (introduced by Busemann in 1955), defined as follows.

Since  $(M, g)$  is complete and non-compact, for any ray  $\gamma : [0, +\infty) \rightarrow M$ , let

$$b_\gamma^t : M \rightarrow \mathbb{R}, \quad b_\gamma^t(x) = t - d(x, \gamma(t)).$$

By triangle inequality, it is easy to see

- $b_\gamma^t(x) \leq d(\gamma(0), x)$ ,
- for any  $t < s$ , one has  $b_\gamma^s(x) - b_\gamma^t(x) = (s - t) + d(x, \gamma(t)) - d(x, \gamma(s)) \geq 0$ ,
- $|b_\gamma^t(x) - b_\gamma^t(y)| \leq d(x, y)$ .

As a result, the limit

$$b_\gamma(x) := \lim_{t \rightarrow +\infty} (t - d(x, \gamma(t)))$$

is well-defined and is Lipschitz with Lipschitz constant 1. We call the function  $b_\gamma : M \rightarrow \mathbb{R}$  the *Busemann function* associated with  $\gamma$ .

By Laplacian comparison theorem, formally we have

$$\Delta b_\gamma(x) \geq - \lim_{t \rightarrow +\infty} \frac{m-1}{d(x, \gamma(t))} = 0.$$

This can be proved rigorously by constructing a lower barrier. Let's admit this:

**Proposition 2.4.** *Let  $b_\gamma$  be the Busemann function associated with a ray  $\gamma$ , then  $\Delta b_\gamma \geq 0$  in the barrier sense.*

Now let  $l : (-\infty, \infty) \rightarrow M$  be a line in  $M$ , and let  $\gamma_+, \gamma_- : [0, \infty) \rightarrow M$  be the two rays in  $l$  defined by  $\gamma_+(t) = l(t)$  and  $\gamma_-(t) = l(-t)$ . Then by Proposition 2.4,

$$\Delta(b_{\gamma_+}(x) + b_{\gamma_-}(x)) \geq 0.$$

On the other hand, by definition we have

$$2t = d(\gamma_-(t), \gamma_+(t)) \leq d(\gamma_-(t), x) + d(\gamma_+(t), x),$$

which implies

$$b_{\gamma_+}(x) + b_{\gamma_-}(x) \leq 0.$$

Note that on the line  $l$ , if we denote  $x = l(s)$ , then

$$b_{\gamma_+}(x) + b_{\gamma_-}(x) = \lim_{t \rightarrow \infty} (2t - d(x, l(t)) - d(x, l(-t))) = \lim_{t \rightarrow \infty} (2t - t - s - t + s) = 0.$$

So we conclude that  $b_{\gamma_+} + b_{\gamma_-}$  is a subharmonic function that achieves its maximum at an interior point. By Theorem 2.2,

$$b_{\gamma_+}(x) + b_{\gamma_-}(x) = 0.$$

In other words,  $b_{\gamma_+}(x) = -b_{\gamma_-}(x)$ , and thus

$$\Delta b_{\gamma_+}(x) = -\Delta b_{\gamma_-}(x) \leq 0.$$

So we arrive at

$$\Delta b_{\gamma_+}(x) = -\Delta b_{\gamma_-}(x) = 0.$$

By Theorem 2.3,  $b_{\gamma_+}, b_{\gamma_-} \in C^\infty(M)$ . Moreover, since  $b_{\gamma_+}$  has Lipschitz constant 1,

$$|\nabla b_{\gamma_+}| \leq 1.$$

Also note that by definition, on the line  $l$  we have  $b_{\gamma_+}(l(s)) = s$  and thus

$$|\nabla b_{\gamma_+}| = 1.$$

### ¶ The proof of Cheeger-Gromoll splitting theorem.

Now we apply Bochner formula to the function  $b_{\gamma_+}$ , to get

$$\frac{1}{2} \Delta (|\nabla b_{\gamma_+}|^2) = |\nabla^2 b_{\gamma_+}|^2 + \text{Ric}(\nabla b_{\gamma_+}, \nabla b_{\gamma_+}) \geq 0.$$

So again,  $|\nabla b_{\gamma_+}|^2$  is a sub-harmonic function that achieves its interior maximum. So by the Hopf strong maximum principle again,

$$|\nabla b_{\gamma_+}|^2 = 1.$$

It follows that  $|\nabla b_{\gamma_+}| = 1$  and in particular,  $\nabla b_{\gamma_+}$  is a complete vector field. Moreover, it follows that

$$\Delta (|\nabla b_{\gamma_+}|) = 0,$$

and thus  $|\nabla^2 b_{\gamma_+}|^2 = 0$ , i.e.

$$\nabla^2 b_{\gamma_+} = 0.$$

Finally we construct the splitting. Let  $M_t = b_{\gamma_+}^{-1}(t)$ . Since  $|\nabla b_{\gamma_+}| = 1$ , any  $t \in \mathbb{R}$  is a regular value of  $b_{\gamma_+}$ . So  $M_t$  is a smooth submanifold of  $M$  of dimension  $m - 1$ .

Denote  $N = M_0$ . Let  $\varphi_s : M \rightarrow M$  be the flow of the vector field  $\nabla b_{\gamma_+}$ . Then  $\varphi_s$  is a diffeomorphism. Moreover, for any  $s \in \mathbb{R}$  and any  $x \in N = M_0$ , we have  $\varphi_s(x) \in M_s$ . So we get a smooth map

$$\Phi : \mathbb{R} \times N \rightarrow M, \quad \Phi(s, p) := \varphi_s(p)$$

which is bijective, and whose inverse

$$\Phi^{-1} : M \rightarrow \mathbb{R} \times N, \quad x \mapsto (b_{\gamma_+}(x), \varphi_{-b_{\gamma_+}(x)}(x))$$

is smooth. So  $\Phi$  is a diffeomorphism.

It remains to prove that  $\Phi$  is an isometry. Note that if we let  $\gamma_p(s) = \varphi_s(p)$  be the integral curve passing  $p$ , then

$$\dot{\gamma}_p = \nabla b_{\gamma_+} \implies \nabla \dot{\gamma}_p = \nabla^2 b_{\gamma_+} = 0$$

which implies that  $\gamma_p$  is the geodesic  $\gamma_p(s) = \exp_p(sX_p)$ , where  $X_p = \nabla b_{\gamma_+}(p)$ . As a result, we have

- $\Phi$  is a radial isometry: we have  $|\partial_s| = 1$  and

$$|d\Phi_{(s,p)}(\partial_s)| = |\dot{\gamma}_p| = |\nabla b_{\gamma_+}| = 1.$$

- $\Phi$  maps “vectors orthogonal to radial direction  $\partial_s$ ” to “vectors orthogonal to radial direction  $d\Phi_{(s,p)}(\partial_s)$ ”: For any  $X_0 \in T_p N = T_p M_0$ , we have  $(d\Phi)_{(s,p)}(0, X_0) = (d\varphi_s)_p(X_0) \in T_{\varphi_s(p)} M_s \perp \nabla b_{\gamma_+}(\varphi_s(p)) = d\Phi_{(s,p)}(\partial_s)$ .
- $\Phi$  preserves the length (and thus the inner product by polarization) of all vectors orthogonal to  $\partial_s$ : For any  $X_0 \in T_p$ , we may extend  $X_0$  to a local coordinate vector field  $\tilde{X}_0$  on  $TN$  such that  $[\partial_s, \tilde{X}_0] = 0$ . Then

$$\nabla_{\dot{\gamma}_p(s)}((d\varphi_s)_p(X_0)) = \nabla_{d\varphi_s(\tilde{X}_0)}(\nabla b_{\gamma_+}) - [(\nabla b_{\gamma_+})(\varphi_s(p)), (d\varphi_s)_p \tilde{X}_0].$$

The first term vanishes since  $\nabla(\nabla b_{\gamma_+}) = 0$ , while the second term vanishes since it equals  $d\varphi_s([\partial_s, \tilde{X}_0]) = 0$ . So we conclude that  $(d\varphi_s)_p(X_0)$  is parallel along  $\gamma_p(s)$ , and thus

$$|d\Phi_{(s,p)}(0, X_p)| = |d\varphi_s(X_0)| = |X_0|.$$

So we conclude that  $(M, g)$  is isometric to  $\mathbb{R} \times N$ . Finally since  $(M, g)$  has non-negative Ricci curvature, and  $N$  is a Riemannian submanifold, and  $K(\partial_s, X_0) = 0$ , we conclude that  $N$  has non-negative Ricci curvature.