

LECTURE 28: SPECTRAL GEOMETRY

Spectral geometry is the branch of differential geometry that studies the relations between the spectrum of the Laplace-type operator and the underline geometry. There are many many beautiful results that have been proved, and at the meantime there are also many many open problems to be studied in the future. In this last lecture, we apply Bochner formula to spectral geometry.

1. SPECTRAL GEOMETRY

¶ Eigenvalues and eigenfunctions.

In spectral geometry there are three typical spectral problems:

- (1) **Closed setting** Let (M, g) be a closed connected Riemannian manifold.

We call λ an *eigenvalue* of Δ if there exists smooth function $u \neq 0$ so that¹

$$\Delta u + \lambda u = 0.$$

- (2) Let (Ω, g) be a compact connected Riemannian manifold with boundary $\partial\Omega$.

- (a) **Dirichlet setting** We call a number λ a *Dirichlet eigenvalue* of Δ if there exists a smooth function $u \neq 0$ so that

$$\begin{cases} \Delta u + \lambda u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

- (b) **Neumann setting** We call a number λ a *Neumann eigenvalue* of Δ if there exists a smooth function $u \neq 0$ so that

$$\begin{cases} \Delta u + \lambda u = 0, & \text{in } \Omega, \\ \partial_\nu u = 0, & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν represents the outer normal derivative.

We have seen from PSet 1 that

- All eigenvalues of Δ are non-negative real numbers.
- $\lambda = 0$ is always an eigenvalue for the closed problem and the Neumann eigenvalue problem (with eigenfunctions the constant functions), and $\lambda = 0$ is not an eigenvalue of the Dirichlet problem.
- If u and v are eigenfunctions of different eigenvalues, then $\langle u, v \rangle_{L^2} = 0$.

According to the standard spectral theory in functional analysis, one can prove

¹Here we use $\Delta = \operatorname{div}\nabla = \operatorname{Tr}(\nabla^2)$. If one uses $\Delta = -\operatorname{div}\nabla = -\operatorname{Tr}(\nabla^2) = d\delta + \delta d$, then the equation should be $\Delta u = \lambda u$.

Theorem 1.1. *In all three settings above, each eigenvalue has finite multiplicity and the eigenvalues of Δ form an increasing sequence that tends to ∞ , namely*

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \rightarrow \infty$$

for the closed eigenvalues and the Neumann eigenvalues, and

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \rightarrow \infty$$

for the Dirichlet eigenvalues. Moreover, one can choose an eigenbasis so that they form a complete orthonormal basis of $L^2(M)$ or $L^2(\Omega)$.

The simplest example being

Example. For S^1 , the Laplacian eigenvalues are the squares $0, 1, 1, 4, 4, 9, 9, \dots$, with eigenfunctions $\cos(kx)$ and $\sin(kx)$. Since these functions already form an orthonormal basis, there are no other eigenvalues/eigenfunctions.

Similarly for $\mathbb{T}^m = S^1 \times \cdots \times S^1$, equipped with the standard flat metric, the eigenvalues are numbers of the form $k_1^2 + \cdots + k_m^2$, with eigenfunctions $\cos(k \cdot x)$ and $\sin(k \cdot x)$, and again they form an orthonormal basis. [Note that in this case, the multiplicity is complicated since there may be many different ways to represent a given positive integer as the sum of m squares.]

Example. One can show that the eigenvalues of the standard sphere S^m are $k(k + m - 1)$ ($k = 0, 1, 2, \dots$), with multiplicity $n_k = \binom{m+k}{m} - \binom{m+k-2}{m}$.

Unfortunately, other than very few examples like the sphere/the torus/the projective spaces etc (or rectangles/balls/annulus etc in the case of manifold with boundary), for most Riemannian manifolds there is no way to calculate its eigenvalues explicitly. There are two major problems in spectral geometry:

- **The direct problem** Given information of (M, g) or (Ω, g) , what can we say about these eigenvalues/eigenfunctions?
- **The inverse problem** Given the sequence of eigenvalues, what can we say about the geometry of (M, g) or (Ω, g) ?

¶ The first eigenvalue λ_1 .

The first non-zero eigenvalue λ_1 is very important and has received much attention. Although in general one can't calculate it explicitly, we do have a variational characterization as follows. Given any smooth function $\varphi \neq 0$, we call

$$R(\varphi) = \frac{\int_M |\nabla \varphi|^2 dV_g}{\int_M \varphi^2 dV_g}$$

the *Rayleigh quotient* of φ . Then

Theorem 1.2 (Variational characterization of λ_1). *For the closed or the Neumann eigenvalue problem,*

$$\lambda_1 = \inf\{R(\varphi) \mid 0 \neq \varphi \in H^1(M), \int_M \varphi = 0.\},$$

while for the Dirichlet eigenvalue problem,

$$\lambda_1 = \inf\{R(\varphi) \mid 0 \neq \varphi \in H_0^1(M).\}$$

Proof. For any φ in the given space, we may expand $\varphi = \sum_{k=1}^{\infty} c_k u_k$. We may assume $\|\varphi\|_{L^2} = 1$, i.e. $\sum c_k^2 = 1$. Then

$$R(\varphi) = \sum_{k \geq 1} \lambda_k c_k^2 \geq \sum_{k \geq 1} \lambda_1 c_k^2 = \lambda_1.$$

On the other hand, if we take $\varphi = u_1$, then $R(\varphi) = R(u_1) = \lambda_1$. \square

Remark. For example, given bounded domain Ω , the Poincaré inequality states that there exists constant C so that

$$\int_{\Omega} |u|^2 dV_g \leq C \int_{\Omega} |\nabla u|^2 dV_g, \quad \forall u \in H_0^1(\Omega).$$

Now in view of the above theorem, the smallest (=the best) constant C for the Poincaré inequality to be true is the reciprocal of the first Dirichlet eigenvalue of Ω .

Remark. One also has variational characterization of higher eigenvalues λ_k for all k .

2. SOME RESULTS ON THE FIRST EIGENVALUE λ_1

Now suppose (M, g) is closed and we focus on the first nonzero eigenvalue.

¶ Lichnerowitz estimate for λ_1 .

Now we apply Bochner formula to prove a lower bound estimate for λ_1 .

Theorem 2.1 (Lichnerowitz). *Let (M, g) be a closed Riemannian manifold with $Ric \geq (m-1)k$ for some $k > 0$. Then the first eigenvalue*

$$\lambda_1 \geq mk.$$

Proof. According to Corollary 1.3 in Lecture 28, for any $u \in C^\infty(M)$,

$$\frac{1}{2} \Delta(|\nabla u|^2) \geq \frac{1}{m} (\Delta u)^2 + \langle \nabla u, \nabla(\Delta u) \rangle + Rc(\nabla u, \nabla u).$$

So if we take u be an eigenfunction, i.e. $\Delta u + \lambda u = 0$, then we get

$$(1) \quad \frac{1}{2} \Delta|\nabla u|^2 \geq -\frac{\lambda}{m} u \Delta u - \lambda \langle \nabla u, \nabla u \rangle + Rc(\nabla u, \nabla u).$$

Integrate over M and apply the Green's formula

$$-\int_M u \Delta u dx = \int_M |\nabla u|^2 dx$$

we get

$$0 \geq \int_M \left(\frac{\lambda}{m} - \lambda + (m-1)k \right) |\nabla u|^2 dx.$$

This implies

$$\frac{\lambda}{m} - \lambda + (m-1)k \leq 0,$$

i.e. $\lambda \geq mk$. □

¶ Obata's λ_1 rigidity theorem.

One can prove that the first eigenvalue of the standard sphere S^m is m . In fact, this is the only case where $\lambda_1 = m$ if (M, g) satisfies the conditions in Theorem 2.1.

Theorem 2.2 (Obata). *Let (M, g) be a closed Riemannian manifold with $\text{Ric} \geq (m-1)k$ for some $k > 0$. If $\lambda_1 = mk$, then (M, g) is isometric to the sphere S_k^m .*

Proof. Without loss of generality we may assume $k = 1$. If $\lambda_1 = m$, then from the proof above we see

$$\text{Rc}(\nabla u, \nabla u) = (m-1)|\nabla u|^2.$$

Since $\Delta(u^2) = 2u\Delta u + 2|\nabla u|^2$ (see PSet 1 for $-\Delta$), from (1) we get

$$\frac{1}{2}\Delta(|\nabla u|^2 + u^2) \geq -u\Delta u - m|\nabla u|^2 + (m-1)|\nabla u|^2 + u\Delta u + |\nabla u|^2 = 0.$$

It follows $\Delta(|\nabla u|^2 + u^2) \equiv 0$ since its integral over M is 0. In other words,

$$|\nabla u|^2 + u^2 = \text{constant}.$$

We normalize u so that $\max_M u^2 = 1$. Since $\nabla u = 0$ at the maximum/minimum points of u , we get

$$|\nabla u|^2 + u^2 = 1 \quad \text{and} \quad \max_M u = -\min_M u = 1.$$

Now let $p, q \in M$ be points such that $u(p) = -1, u(q) = 1$. Let $l = d(p, q)$ and let $\gamma : [0, l] \rightarrow M$ be a normal geodesic from p to q . Let $f(t) = u(\gamma(t))$. Then

$$\frac{f'(t)}{\sqrt{1-f^2(t)}} \leq \frac{|\nabla u(\gamma(t))|}{\sqrt{1-u(\gamma(t))^2}} = 1.$$

Integrating both sides we get

$$\pi = \int_0^l \frac{f'(t)}{\sqrt{1-f^2(t)}} dt \leq \int_0^l dt = l = d(p, q).$$

So $\text{diam}(M, g) \geq \pi$. But by Bonnet-Meyer, $\text{diam}(M, g) \leq \pi$. So $\text{diam}(M, g) = \pi$. Finally by Cheng's maximal diameter theorem, (M, g) is isomorphic to S^m . □

¶ Reilly's formula.

Let Ω be a compact smooth manifold with smooth boundary $M = \partial\Omega$. Then one can define the second fundamental form of M (as a Riemannian submanifold of Ω) as follows: For any $p \in M$, the vector-valued second fundamental form II at p is a symmetric bilinear map

$$\text{II} : T_p M \times T_p M \rightarrow N_p M, \quad (X, Y) \mapsto (\nabla_{\bar{X}}^{\Omega} \bar{Y})^{\perp},$$

where \bar{X}, \bar{Y} are smooth vector fields whose value at p are X and Y respectively [According to PSet 2, $\text{II}(X, Y)$ is well-defined and is symmetric]. Since in the hypersurface case there is only one normal dimension, we may study the (scalar-valued) second fundamental form

$$h : T_p M \times T_p M \rightarrow \mathbb{R}, \quad (X, Y) \mapsto h(X, Y) := -\langle \text{II}(X, Y), \nu \rangle.$$

If we pick a local orthonormal coordinate system $\{e_i\}$ near $p \in M$, where e_{m+1} is the out normal direction, then for any $X = X^i e_i, Y = Y^j e_j \in T_p M$, one has

$$h(X, Y) = \sum_{i,j=1}^m h_{ij} X^i Y^j,$$

where $h_{ij} = -\langle \nabla_{e_i} e_j, e_{m+1} \rangle = \langle \nabla_{e_i} e_{m+1}, e_j \rangle$. The trace of h ,

$$H := \text{Tr}(h) = \sum_i h_{ii},$$

is known as the *mean curvature* of M at p .

By integrating Bochner formula, one can prove the following useful formula obtained by R. Reilly in 1977:

Theorem 2.3 (Reilly's formula). *Let Ω be a compact Riemannian manifold of dimension $m + 1$, with smooth boundary $M = \partial\Omega$. Then for any $f \in C^\infty(\Omega)$,*

$$\frac{m}{m+1} \int_{\Omega} (\Delta^{\Omega} f)^2 \geq \int_M H f_{\nu}^2 + 2 \int_M f_{\nu} \Delta^M f + \int_M h(\nabla^M f, \nabla^M f) + \int_{\Omega} R c^{\Omega}(\nabla f, \nabla f).$$

Moreover, the equality holds if and only if $f_{ij} = \frac{\Delta^{\Omega} f}{m+1} \delta_{ij}$, i.e. $\nabla^2 f = \frac{\Delta^{\Omega} f}{m+1} \text{Id}$.

Proof. For simplicity we write $\Delta^{\Omega} f = g$, and write $f|_{\partial\Omega} = u$. So in what follows we may abbreviate $\Delta^{\Omega} f = \Delta f, \nabla^{\Omega} f = \nabla f$ and $\Delta^M u = \Delta u, \nabla^M u = \nabla u$. By Bochner formula we have

$$\frac{1}{2} \Delta(|\nabla f|^2) \geq \frac{1}{m+1} g^2 + \langle \nabla f, \nabla g \rangle + R c^{\Omega}(\nabla f, \nabla f),$$

with equality if and only if $\nabla^2 f = \frac{\Delta f}{m+1} \text{Id}$. Integrate and in view of Green's formula

$$\int_{\Omega} \langle \nabla f, \nabla g \rangle = - \int_{\Omega} g \Delta f + \int_{\partial\Omega} g f_{\nu}$$

we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \Delta(|\nabla f|^2) &\geq \frac{1}{m+1} \int_{\Omega} g^2 + \int_{\Omega} \langle \nabla f, \nabla g \rangle + \int_{\Omega} Rc^{\Omega}(\nabla f, \nabla f) \\ &= \frac{-m}{m+1} \int_{\Omega} g^2 + \int_M g f_{\nu} + \int_{\Omega} Rc^{\Omega}(\nabla f, \nabla f). \end{aligned}$$

In what follows we will prove

$$(\star) \quad \frac{1}{2} \int_{\Omega} \Delta(|\nabla f|^2) = - \int_M H f_{\nu}^2 + \int_M g f_{\nu} - 2 \int_M f_{\nu} \Delta u - \int_M h(\nabla u, \nabla u)$$

from which the theorem follows.

We choose orthonormal frame near p so that $e_{m+1}(p) = \nu(p)$ and $\nabla_{e_{m+1}} e_{m+1} = 0$. Cover M by such coordinate neighborhoods and let ρ_{α} be a partition of unity subordinate to this covering (together with the open set $\Omega \setminus M$). As we have seen in Lecture 4, $\sum_{\alpha} \rho_{\alpha} \operatorname{div}(X) = \sum_{\alpha} \operatorname{div}(\rho_{\alpha} X)$. So

$$\frac{1}{2} \int_{\Omega} \Delta(|\nabla f|^2) = \frac{1}{2} \sum_{\alpha} \int_{\Omega} \rho_{\alpha} \Delta(|\nabla f|^2) = \frac{1}{2} \sum_{\alpha} \int_{\Omega} \operatorname{div}(\rho_{\alpha} \nabla(|\nabla f|^2)).$$

So by divergence theorem,

$$\frac{1}{2} \int_{\Omega} \Delta(|\nabla f|^2) = \frac{1}{2} \sum_{\alpha} \int_M \langle \rho_{\alpha} \nabla(|\nabla f|^2), \nu \rangle = \frac{1}{2} \sum_{\alpha} \int_M \rho_{\alpha} \partial_{\nu}(|\nabla f|^2).$$

Thus we may compute in the above local coordinates. Since $|\nabla f|^2 = \sum_i (e_i f)^2$,

$$\begin{aligned} \frac{1}{2} \int_M \rho_{\alpha} \partial_{\nu}(|\nabla f|^2) &= \int_M \rho_{\alpha} \sum_{i=1}^{m+1} (e_i f)(e_{m+1} e_i f) \\ &= \int_M \rho_{\alpha} [(e_{m+1} f)(e_{m+1} e_{m+1} f) + \sum_{i=1}^m (e_i f)(e_{m+1} e_i f)]. \end{aligned}$$

Note that $\nabla_{e_{m+1}} e_{m+1} = 0$ in the neighborhood. So we get, at all $x \in M$,

$$\begin{aligned} e_{m+1} e_{m+1} f &= \sum_{i=1}^{m+1} [e_i e_i f - (\nabla_{e_i} e_i) f] - \sum_{i=1}^m [e_i e_i f - (\nabla_{e_i} e_i) f] \\ &= \sum_{i=1}^{m+1} [e_i e_i f - (\nabla_{e_i} e_i) f] - \sum_{i=1}^m [e_i e_i f - (\nabla_{e_i} e_i)^T f] + \sum_{i=1}^m (\nabla_{e_i} e_i)^{\perp} f \\ &= \Delta f - \Delta u + \sum_{i=1}^m \langle \nabla_{e_i} e_i, e_{m+1} \rangle f_{\nu} \\ &= g - \Delta u - H f_{\nu}. \end{aligned}$$

For the second term we use

$$\begin{aligned}
e_{m+1}e_i f &= e_i e_{m+1} f + (\nabla_{e_{m+1}} e_i) f - (\nabla_{e_i} e_{m+1}) f \\
&= e_i e_{m+1} f + \sum_{j=1}^{m+1} \langle \nabla_{e_{m+1}} e_i, e_j \rangle f_j - \sum_{j=1}^{m+1} \langle \nabla_{e_i} e_{m+1}, e_j \rangle f_j \\
&= e_i e_{m+1} f + \sum_{j=1}^m \langle \nabla_{e_{m+1}} e_i, e_j \rangle f_j - \sum_{j=1}^m h_{ij} f_j,
\end{aligned}$$

where in the last step we used the facts $\langle \nabla_{e_{m+1}} e_i, e_{m+1} \rangle = -\langle e_i, \nabla_{e_{m+1}} e_{m+1} \rangle = 0$, $\langle \nabla_{e_i} e_{m+1}, e_{m+1} \rangle = \frac{1}{2} e_i |e_{m+1}|^2 = 0$ and $h_{ij} = \langle \nabla_{e_i} e_{m+1}, e_j \rangle$. So we get three terms. For the first one we have

$$\sum_{\alpha} \int_M \rho_{\alpha} \sum_{i=1}^m (e_i f) (e_i e_{m+1} f) = \sum_{\alpha} \int_M \rho_{\alpha} \langle \nabla^M f, \nabla^M f_{\nu} \rangle = - \int_M (\Delta u) f_{\nu}.$$

For the second term, we have $\sum_{i,j=1}^m (e_i f) \langle \nabla_{e_{m+1}} e_i, e_j \rangle f_j = 0$ since

$$S := \sum_{i,j=1}^m (e_i f) \langle \nabla_{e_{m+1}} e_i, e_j \rangle f_j = \sum_{i,j} \langle \nabla_{e_{m+1}} e_i, e_j \rangle f_i f_j = - \sum_{i,j=1}^m \langle e_i, \nabla_{e_{m+1}} e_j \rangle f_i f_j = -S.$$

Finally for the last term,

$$\sum_{i,j=1}^m (e_i f) \langle \nabla_{e_i} e_{m+1}, e_j \rangle f_j = \sum_{i,j=1}^m h_{ij} f_i f_j = h(\nabla u, \nabla u).$$

So we get the desired equality (\star). \square

Remark. If we don't apply Cauchy-Schwartz inequality at the first step, then

$$\int_{\Omega} ((\Delta f)^2 - |\nabla^2 f|^2) = \int_M (H f_{\nu}^2 + 2 f_{\nu} \Delta^M f + h(\nabla^M f, \nabla^M f)) + \int_{\Omega} Rc(\nabla f, \nabla f).$$

¶ Yau's conjecture.

A Riemannian submanifold M^m of N is called minimal if it has mean curvature $H = 0$. Minimal submanifolds are very important objects in Riemannian geometry, especially the branch "submanifold geometry". As an application we prove

Theorem 2.4 (Choi-Wang, 1984). *Let M^m be a compact connected embedded oriented minimal hypersurface in a compact oriented Riemannian manifold N^{m+1} . Suppose N has Ricci curvature $Ric^N \geq mk > 0$, then $\lambda_1(M) \geq \frac{mk}{2}$.*

Proof. Since $Ric^N > 0$, by Bochner theorem (c.f. Corollary 1.7 in Lecture 28), $b_1(N) = 0$. Let Ω be a tubular neighborhood of M . Then the Mayer-Vietories sequence of de Rham cohomologies,

$$0 \rightarrow H^0(N) \rightarrow H^0(N \setminus M) \oplus H^0(\Omega) \rightarrow H^0((N \setminus M) \cap \Omega) \rightarrow H^1(N)$$

becomes

$$0 \rightarrow \mathbb{R} \rightarrow H^0(N \setminus M) \oplus \mathbb{R} \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow 0.$$

It follows that $H^0(N \setminus M) \cong \mathbb{R} \oplus \mathbb{R}$, i.e. $N \setminus M$ contains exactly two connected components. We denote

$$N \setminus M = \Omega_1 \cup \Omega_2, \quad \partial\Omega_1 = \partial\Omega_2 = M.$$

Now let $u \in C^\infty(M)$ be an eigenfunction associated to $\lambda_1 = \lambda_1(M)$, i.e. $\Delta_M u + \lambda_1 u = 0$. Without loss of generality, we assume

$$\int_M h(\nabla u, \nabla u) dV_g \geq 0,$$

where we regard M as $\partial\Omega_1$. [If this inequality is not true, then the analogue inequality for Ω_2 holds and we proceed with Ω_2 instead of Ω_1 .] Let f be a solution to

$$\begin{cases} \Delta^N f = 0, & \text{in } \Omega_1 \\ f = u, & \text{on } M = \partial\Omega_1. \end{cases}$$

By Reilly's formula,

$$0 \geq -2\lambda \int_M u f_\nu + \int_M h(\nabla u, \nabla u) + mk \int_{\Omega_1} |\nabla f|^2 \geq -2\lambda \int_M u f_\nu + mk \int_{\Omega_1} |\nabla f|^2.$$

Since $\Delta f = 0$, by Green's formula we get

$$\int_M u f_\nu = \int_{\partial\Omega_1} f f_\nu = \int_{\Omega_1} |\nabla f|^2,$$

thus

$$0 \geq (-2\lambda_1 + mk) \int_{\Omega_1} |\nabla f|^2.$$

It follows $\lambda_1 \geq mk/2$. □

In particular, if we take $N^{m+1} = S^{m+1}$ we get

$$\lambda_1(M) \geq \frac{m}{2}.$$

This lower bound is half of the conjectured bound by Yau in 1982:

Conjecture (Yau). *For any embedded minimal hypersurface M of S^{m+1} , one has*

$$\lambda_1(M) \geq m.$$

Dedicated to Yau for his 75th Birthday