理论物理系列讲座之一：经典与量子混沌

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What is chaos?

Chaos ('keios)

1 The disordered formless matter supposed to have existed before the ordered universe.

2. Complete disorder, utter confusion.

3. (Math.) Stochastic behavior occurring in a deterministic system. (1986 Royal Society in London.) ("*Lawless behavior governed entirely by law*")

4. Sensitive dependence on initial condition.

Birth of modern dynamics, chaos theory

- 1887 King Oscar II of Sweden offered a prize of 2,500 crowns for an answer to a fundamental question in astronomy:
  Is the Solar System Stable?

- H Poincare submitted a paper of 270 pages long entitled ``On the Problem of Three Bodies and the Equations of Dynamics", in which he showed that an analytic solution was not possible, even to the simpler problem of only three gravitating bodies, one of which had a mass too small to affect the motion of the other two.

(After Poincare was declared the winner, it was discovered that there was an important error in his proof. The award was made, after the error was corrected.

- In 1892, Poincare made the following observation after studying the 3-body problem: (excerpted from ``*Chance*")

In reaching the conclusion quoted above, Poincare discovered *deterministic chaos*. However, his discovery was overshadowed by the discoveries of *quantum mechanics* and *relativity*. Poincare's discovery was ignored by mainstream physics until 1960's.

What Chaos Theory tells us?

Simple systems can exhibit complex behavior.

(Complexity theory tells us that complex system can exhibit simple `emergent' behavior.)

I. Chaos in Hamiltonian system

- Integrable system
- Application of perturbation theory to perturbed system
- KAM theorem
- Poincare-Birkhoff theorem
- Homoclinic and Heteroclinic points
- General properties of chaos

We discuss here time-independent Hamiltonians, while the general results are usually also valid for time-dependent Hamiltonian systems, with suitable modifications.

1 Nearly-integrable system

Integrable system:

In an integrable system, there exists a set of *f* independent analytical single-valued constants of motion f_i, which are in involution, i.e., their Poisson brackets with each other must vanish $[f_i,f_j]=0$.

For an integrable system, the time-independent Hamilton-Jacobi equation is usually separable into f independent equations and it is possible to introduce the action-angle variables.

Motion in an integrable system is regular, in other words, periodic or quasi-periodic.

Quasi-periodic: periodic in projection to each dimension, but the ratio of frequencies of the motion in the related dimension is incommensurable.

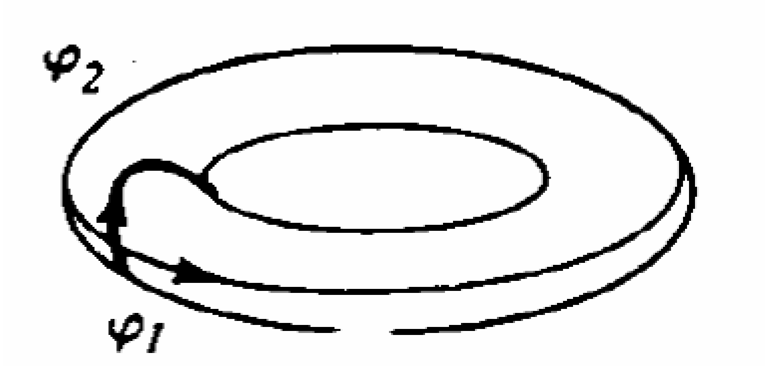
Integrable systems are rare and non-generic.

But they can be used as a starting point in the study of more general systems.

We have the following picture in action-angle variables:

(1) A trajectory in phase space lies in a subspace characterized by the values of the actions I, with I being constants of motion.

(2) Such a subspace is given a name **torus** (f-dimensional) and points on a given torus are labeled by the angle variables. When f=2, it has the shape as shown below, which is the surface of a "doughnut",



Winding number

$p = \omega_1 / \omega_2$

Nearly-integrable systems:

$$H = H_0 + \varepsilon H_1$$

where H_0 is an integrable system and ε is a small quantity.

This system is not far from an integrable system, therefore, we can make use of what we know about the integrable system.

In particular, the action-angle variables of the integrable system can be used as a set of canonical variables in describing the system H.

$$H(I,\theta) = H_0(I) + \epsilon H_1(I,\theta)$$

Since a nearly-integrable system has a small perturbation, we can use perturbation theory in the analytical approach.

As we know, the point is to find out a new set of canonical variables, so that the new Hamiltonian is approximately a function of the new momentum only, then the new momentum are nearly constants of motion and the system can be solved approximately.

By the canonical perturbation theory discussed at the end of part II, we know that this can be done, when it is possible to introduce a generating function in the following way

$$S(\bar{\mathbf{I}}, \theta) = \bar{\mathbf{I}} \cdot \theta + \epsilon i \sum_{\mathbf{m} \neq 0} \frac{H_{1\mathbf{m}}(\bar{\mathbf{I}})}{\mathbf{m} \cdot \omega(\bar{\mathbf{I}})} e^{i\mathbf{m}\cdot\theta} + \cdots .$$

It is seen that such a generating function does not exist, when

$$\mathbf{m} \cdot \omega(\bar{\mathrm{I}}) = 0 \qquad \text{where} \qquad \omega(\bar{\mathrm{I}}) = \frac{\partial H_0(\bar{\mathrm{I}})}{\partial \bar{\mathrm{I}}}$$

That is, we have the famous problem of small denominators.

Let us take f=2, as an example. Then, this equation means that the winding number p is rational, p=r/s.

Since the frequencies are functions of I, when I changes continuously, there is always some I for which the winding number is rational.

The problem becomes serious, when the canonical momenta I are not really constants of motion.

We learn two things from the above analysis by the simple canonical perturbation theory:

(1) Take f=2 as an example. Motions in the system H, which are close to a torus with rational winding number in the system H_0, called rational torus, may be more complicated than those close to irrational torus with irrational winding numbers.

(2) The simple perturbation theory we developed is not enough for a rigorous approach to the problem and a more advanced perturbation theory is needed.

The first problem will be addressed by Poincare-Birkhoff theorem, and the second by the celebrated KAM theorem.

2 KAM theorem: irrational winding numbers

The problem is whether it is possible to find new Hamiltonian satisfying K=K(I') locally for some values of winding numbers, so that for such winding numbers, the tori are just distorted, but not destroyed.

The answer is yes and is given by KAM theorem. For simplicity, let us first discuss f=2 case.

The theorem was proved by Arnold (1961, 1962), for analytic H_1 (all derivatives existing), following a conjecture by Komogorov (1954), and by Moser (1962) for a sufficient number of continuous derivatives. It provides the basis for the existence of invariants in nonlinear coupled systems. The theorem is generally called the KAM theorem in recognition of their work.

KAM proved that, in the system $H=H_0+\varepsilon H_1$, torus of H_0 satisfying the following condition is only distorted,

$$\left|p - \frac{r}{s}\right| > \frac{K(\epsilon)}{s^{2.5}}$$

for all integer r and s, where p is the winding number and K is a constant depending on ε. K<<1, unless the perturbation is strong.

That is, if the winding number p is far enough from any rational number, then, the torus is just distorted, but not destroyed.

The technique used by KAM is to introduce an optimized perturbation theory, other than the canonical perturbation theory we discussed.

The relation between the two perturbation theories is somewhat similar to that between the two rational number approaches to an irrational number discussed at the end of this Lecture, i.e., a direct decimal-truncation approach and the approach using continued fractions.

10.2.1 Superconvergent method,

which plays a fundamental role in the proof of KAM theorem.

In the perturbation theory we discussed, the Hamiltonian

$$H = H_0 + \varepsilon H_1$$

can be transformed by successive canonical transformations that are chosen to increase the order of perturbation by one power in every step. Letting H_n be the untransformed part of the Hamiltonian after the nth transformation,

$$\varepsilon H_1 \rightarrow \varepsilon^2 H_2 \rightarrow \varepsilon^3 H_3 \rightarrow \cdots \varepsilon^n H_n$$

Kolmogorov showed that successive canonical transformations may be chosen such that the order of the perturbation is increased by the square of the preceding one for each step

$$\varepsilon H_1 \rightarrow \varepsilon^2 H_2 \rightarrow \varepsilon^4 H_3 \rightarrow \cdots \varepsilon^{k(n)} H_n, \quad \text{with } k(n)=2^{n-1}$$

This type of convergence is called superconvergence, or sometimes, quadratic convergence.

3 Poincare-Birkhoff theorem

Tori with rational winding numbers and their structure

Historical remark:

Poincaré had stated his theorem in *Sur un théorème de géométrie* in 1912 but could only give a proof in certain special cases.

Birkhoff finally completed the proof.

Poincare-Birkhoff Theorem

- The points where $T^s C_\varepsilon$ and C_ε intersect are fixed points of the map T^s (or periodic points of T with period s).

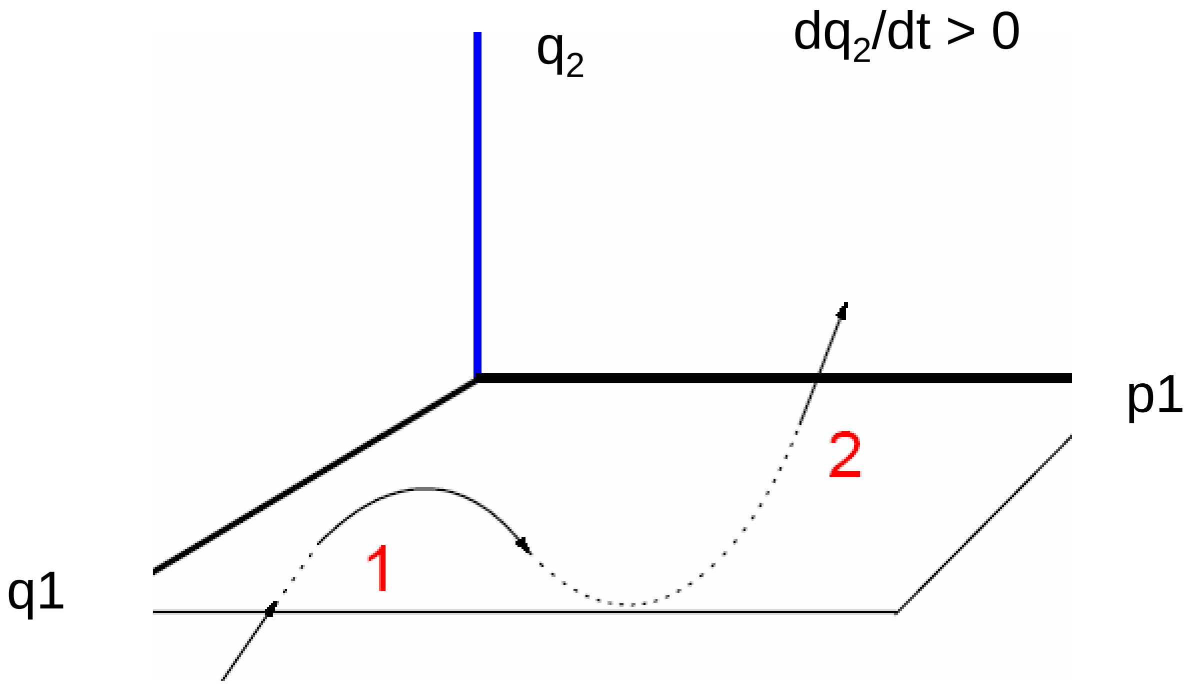
- Since the area under the two curves must be the same, the curves must intersect (except in the trivial case where T^s is the identity).

- There must be an even number of fixed points.

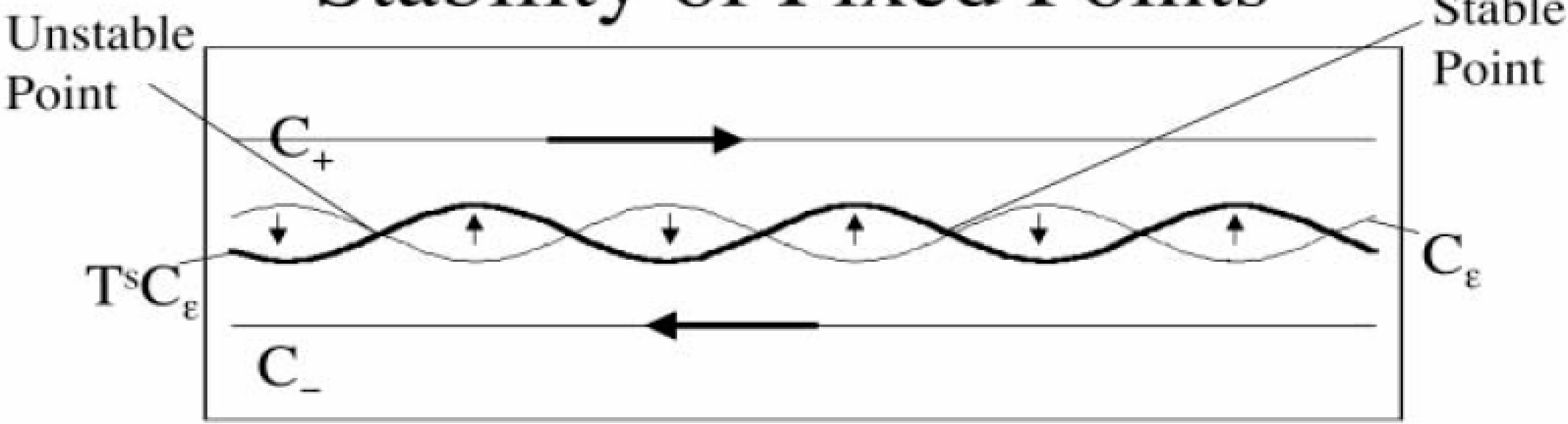
- We will see that half of the fixed points are stable (elliptic) and half are unstable (hyperbolic).

Poincare's method is to set further, e.g., q_2=0 and $dq_2/dt > 0$, and study the change of (q1,p1) at discretized time.

The method is called **Poincare surface of section** (PSOS).



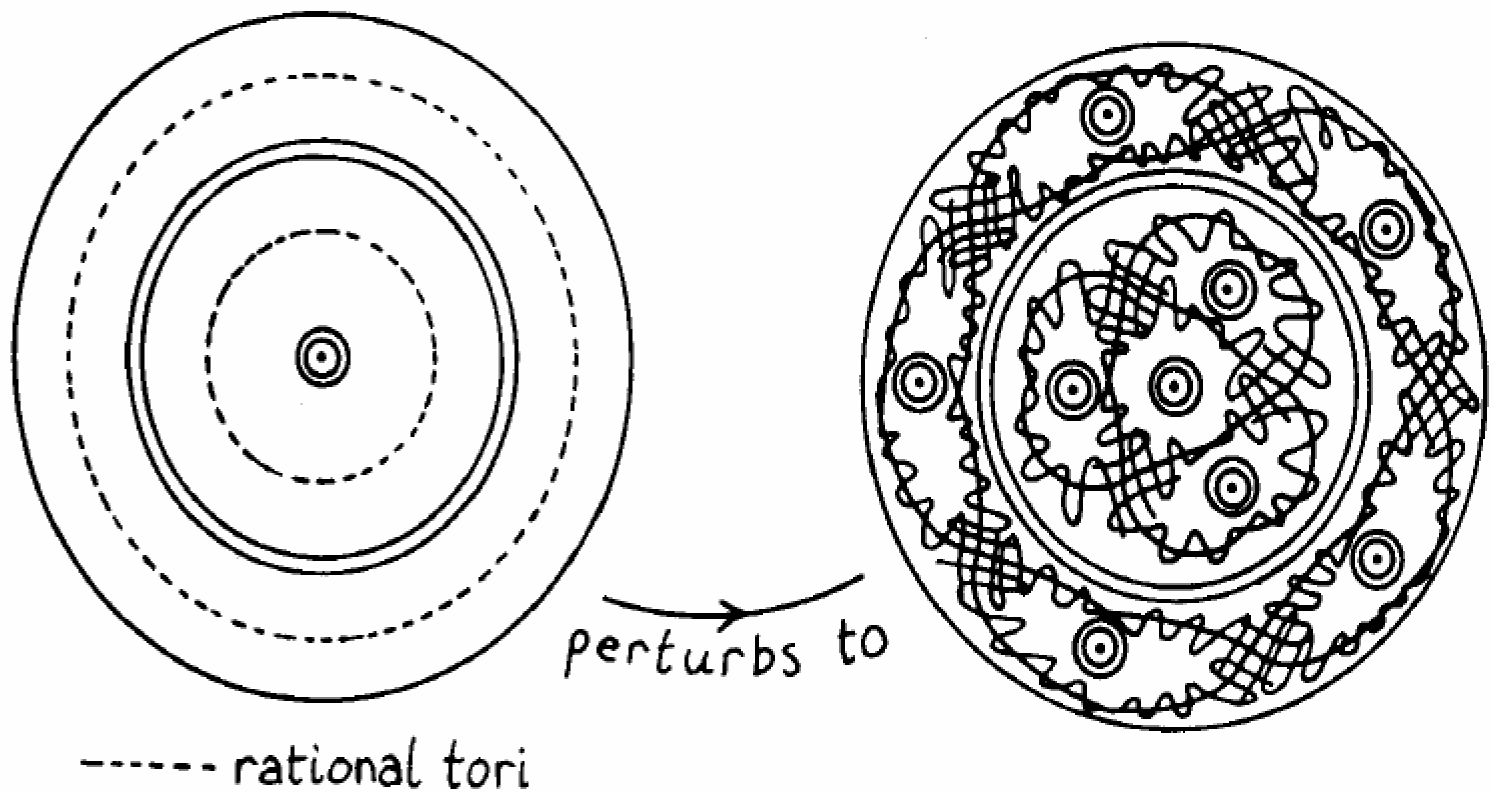
Stability of Fixed Points



Since C_ε is the curve for which there is no winding, the winding direction is to the left below it and to the right above it.

This leads to an alternating series of stable and unstable fixed points.

Combining the above results, we have the following complicated structure in PSOS, with self-similarity, as perturbation increases.



4 Some general properties of chaos

4.1 Lyapunov exponent

------- as a characterization of chaos

Lyapunov exponents play an important role in the theory of both Hamiltonian and dissipative dynamical systems. They provide a computable, quantitative measure of the degree of stochasticity for a trajectory. In addition, there is a close link between the Lyapunov exponents and other measures of randomness such as the Kolmogorov entropy and the capacity.

Roughly speaking, the Lyapunov exponents of a given trajectory characterize the mean exponential rate of divergence of trajectories surrounding it.

Historical remark:

Characterization of the stochasticity of a phase space trajectory in terms of the divergence of nearby trajectories was introduced by Henon and Heiles (1964), and further studied by Zaslavsky and Chirikov (1972), Froeschle and Scheidecker (1973), and Ford (1975).

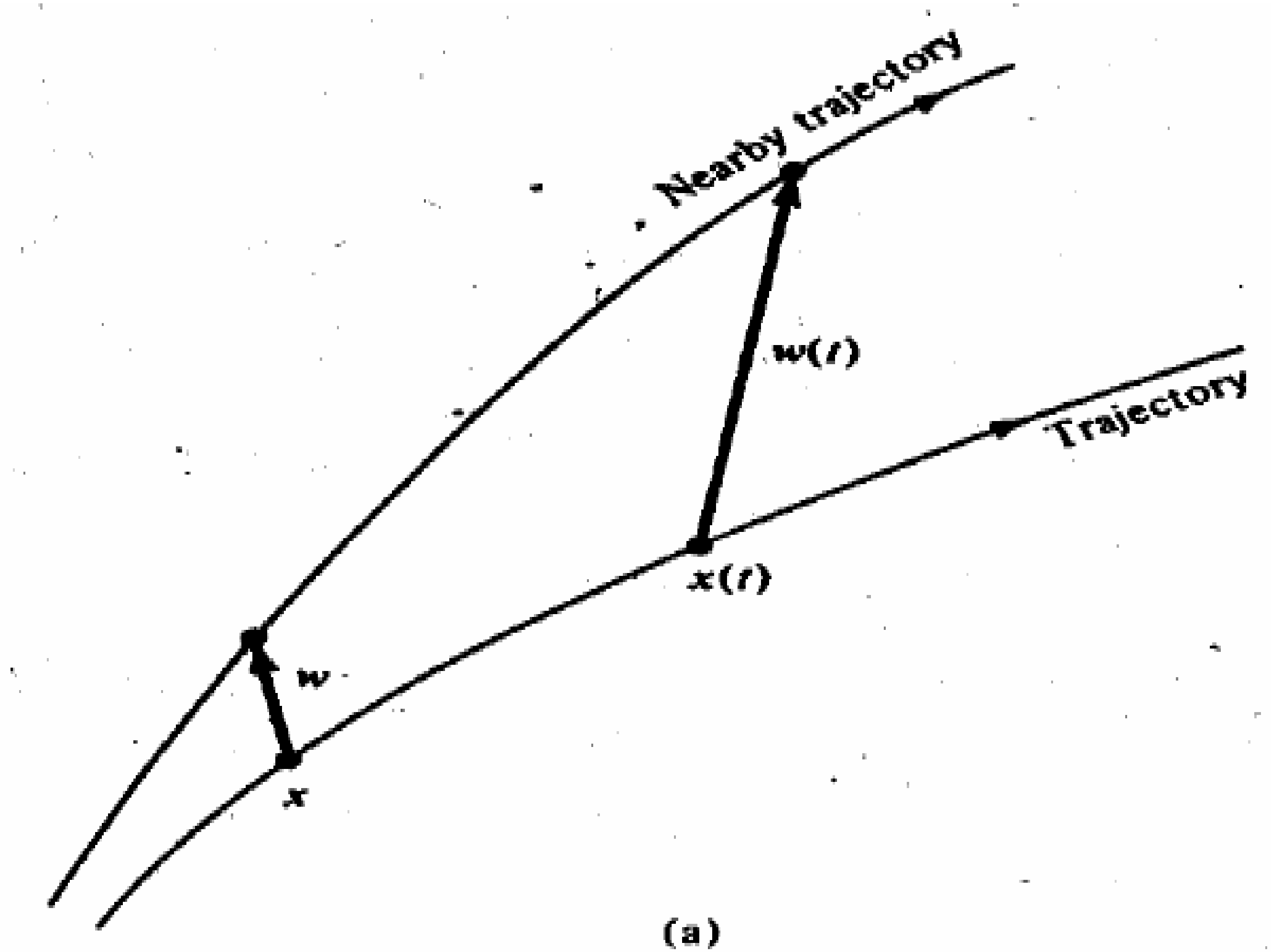
The theory of Lyapunov exponents (Lyapunov 1907) was applied to characterize stochastic trajectories by Oseledec (1968). The connection between the Lyapunov exponents and the exponential divergence was given by Benettin et al (1976) and by Pesin (1977), who also estabilished the connection to Komogorov entropy. The precedure for computing the Lyapunov exponents was developed by Benettin et al (1980).

The purpose is to study the divergence of nearby trajectories.

Let us consider a flow x(t) in a M-dimensional phase space, which is generated by an autonomous first-order system, like the Hamilton's equations in a time-independent Hamiltonian system.

$$\frac{dx_i}{dt} = V_i(\mathbf{x}), \qquad i = 1, 2, \cdots, M$$

Consider a trajectory in M-dimensional phase space and a nearby trajectory with initial conditions $\mathbf{x}_0$ and $\mathbf{x}_0 + \Delta\mathbf{x}_0$, respectively, as shown inthe figure.



The separation changes with time, with a Euclidean norm (distance in the phase space)

$$d(\mathbf{x}_0, t) = ||\Delta \mathbf{x}(\mathbf{x}_0, t)||$$

We now introduce the mean exponential rate of divergence of two initially close trajectories

$$\sigma(\mathbf{x}_0, \mathbf{w}) = \lim_{t\to\infty} \lim_{d(\mathbf{x}_0, t_0)\to 0} \frac{1}{t} \ln \frac{d(\mathbf{x}_0, t)}{d(\mathbf{x}_0, t_0)}$$

The order of limitation is: First take $d(\mathbf{x}_0, t_0)$ to zero, then, take t to infinity.

It can be shown that σ exists and is finite. Furthermore, there is an M-dimensional basis $\{\hat{e}_i\}$ of $\mathbf{w}$ such that for any $\mathbf{w}$, σ takes on one of the M (possibly nondistinct) values

$$\sigma_i(\mathbf{x}_0) = \sigma(\mathbf{x}_0, \hat{e}_i),$$

These are the **Lyapunov (characteristic) exponents**. Note that they are functions of initial position in phase space. They can be ordered by magnitude,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_M.$$

The largest Lyapunov exponent σ_1 is the most important one.

The Lyapunov exponents are independent on the choice of metric for the phase space (Oseledec, 1968).

4.2 Hierarchy of randomness (stochasticity)

1. Ergodic system
2. Mixing system
3. K-system
4. C-system
5. Bernoulli system

Ergodicity: A system is said to be **ergodic**, if the time average of an arbitrary function f(q,p) with almost every possible initial states is equal to the average over (energy surface in) phase space.

$$\langle f \rangle_t = \langle f \rangle_{ps}$$

$$\langle f \rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(q,p)dt \qquad \langle f \rangle_{ps} = \int_{\mathcal{M}} f(q,p)d\mu$$

Where $d\mu$ is the invariant measure in the phase space.

The meaning of ergodicity is that almost every trajectory explores all the possible regions (on the energy surface) in phase space, with a weight proportional to $d\mu$.

Mixing: One may use the following picture to illustrate the concept of mixing

(a) Take a shaker that consists of 20% rum and 80% cola, with the part of rum representing the initial distribution of the considered initial states as “incompressible fluid” in phase space.

(b) Shaking the shaker for a time long enough, then, every part of the shaker (of macroscopic scale), however small, will contain “approximately” 20% rum, representing that every part of the phase space contains 20% of the trajectories at time t.

Or one may imagine the dispersion of a drop of ink in a glass of water.

Mathematical expression

An area preserving map $\mathbf{M}$ of a compact region S is mixing on S, if given any two subsets σ and σ' of S, where σ and σ' have positive Lebesgue measure ($\mu_L(\sigma) > 0, \mu_L(\sigma') > 0$), then,

$$\frac{\mu_L(\sigma)}{\mu_L(S)} = \lim_{m \to \infty} \frac{\mu_L[\sigma' \cap \mathbf{M}^m(\sigma)]}{\mu_L(\sigma')}$$

Mixing implies ergodicity, but, the converse is not true.

K-systems have invariant sets with positive KS (Krylov, Kolmogorov, Sinai) entropy.

It has been proved that KS entropy is the summation of positive Lyapunov exponents.

$$h_k = \sum_{\sigma_i > 0} \sigma_i$$

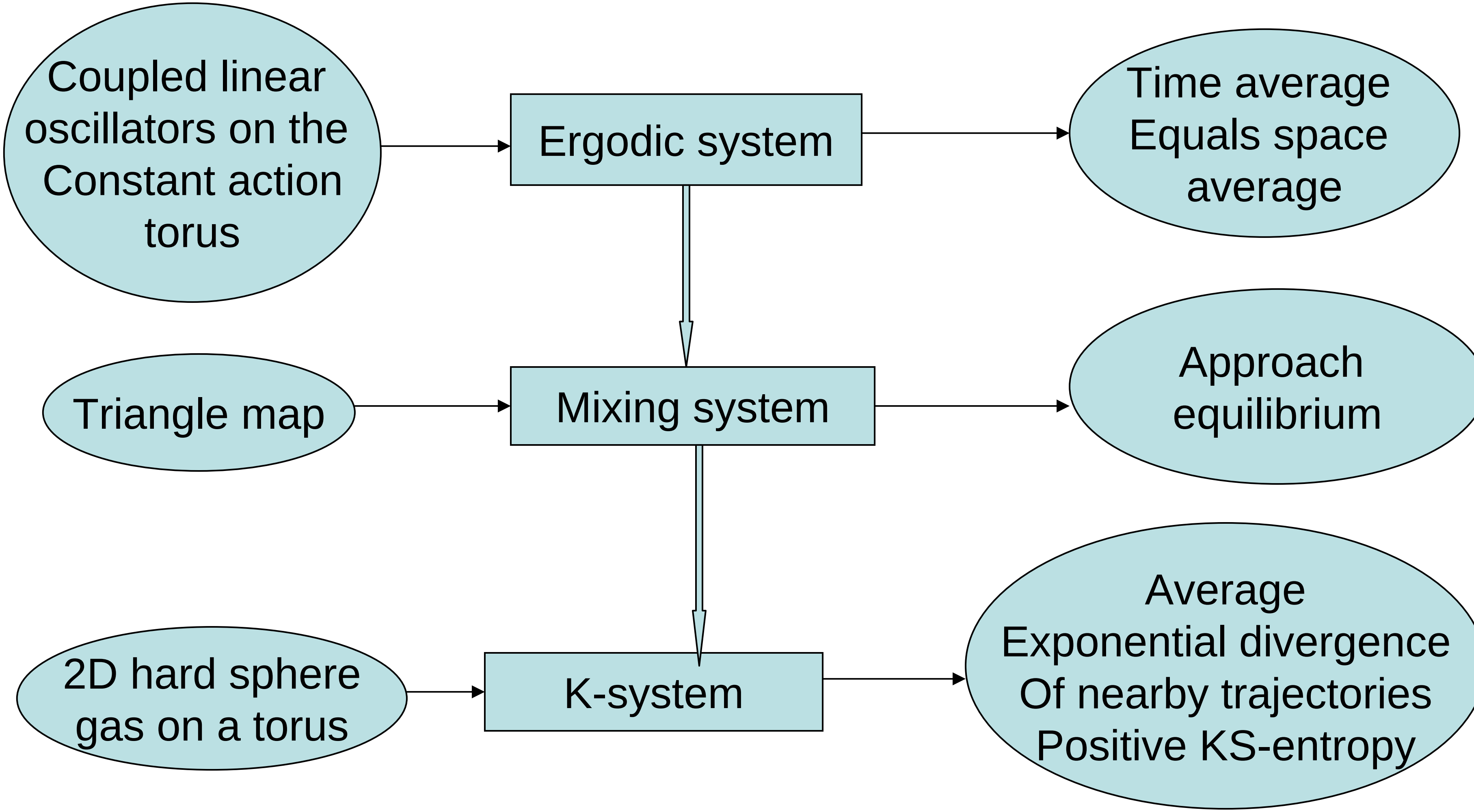
Therefore, a K-system has positive Lyapunov exponent.

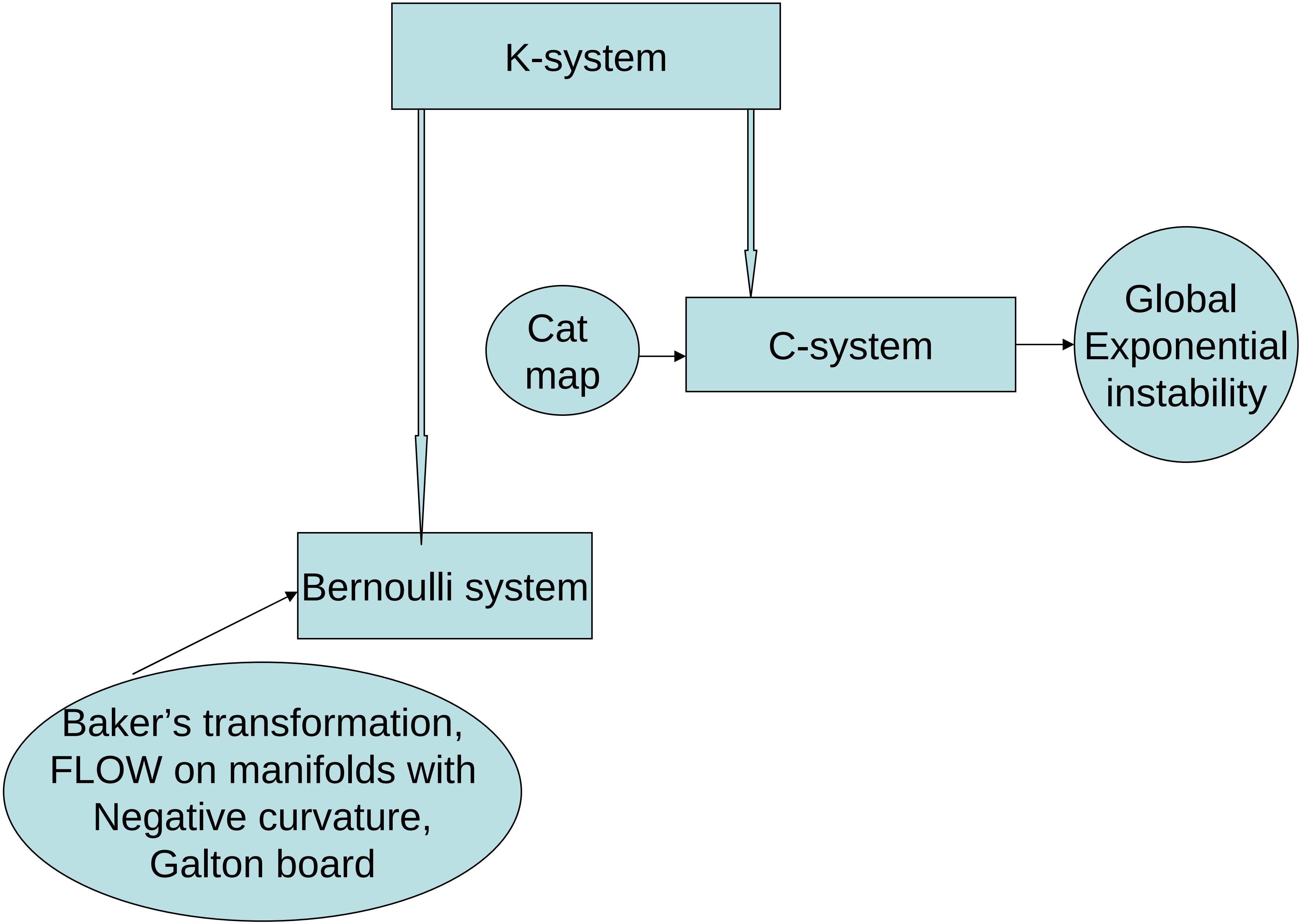
When we speak of a chaotic system, we usually mean a K-system.

We mention that σ_1 is not usually the same constant for all stochastic regions; distinct, isolated regions of stochasticity generally have different values of σ_1.

A C-system, also called Anosov-system, is one which is chaotic and is hyperbolic at every point in the phase space (not just on the invariant set).

A Bernoulli system is a system which can be represented as a symbolic dynamics consisting of a full shift on a finite number of symbols.


Coupled linear oscillators on the Constant action torus
Ergodic system
Time average Equals space average
Triangle map
Mixing system
Approach equilibrium
2D hard sphere gas on a torus
K-system
Average Exponential divergence Of nearby trajectories Positive KS-entropy


K-system
Cat map
C-system
Global Exponential instability
Bernoulli system
Baker's transformation, FLOW on manifolds with Negative curvature, Galton board

II. Chaos in dissipative systems

Attractor
Area contraction
Bifurcation
Fractal dimension

1 Attractor and bifurcation

In a dissipative system with a N-dimensional phase space, phase space volume may shrink to some stable, steady motion on a surface with a dimension smaller than N.

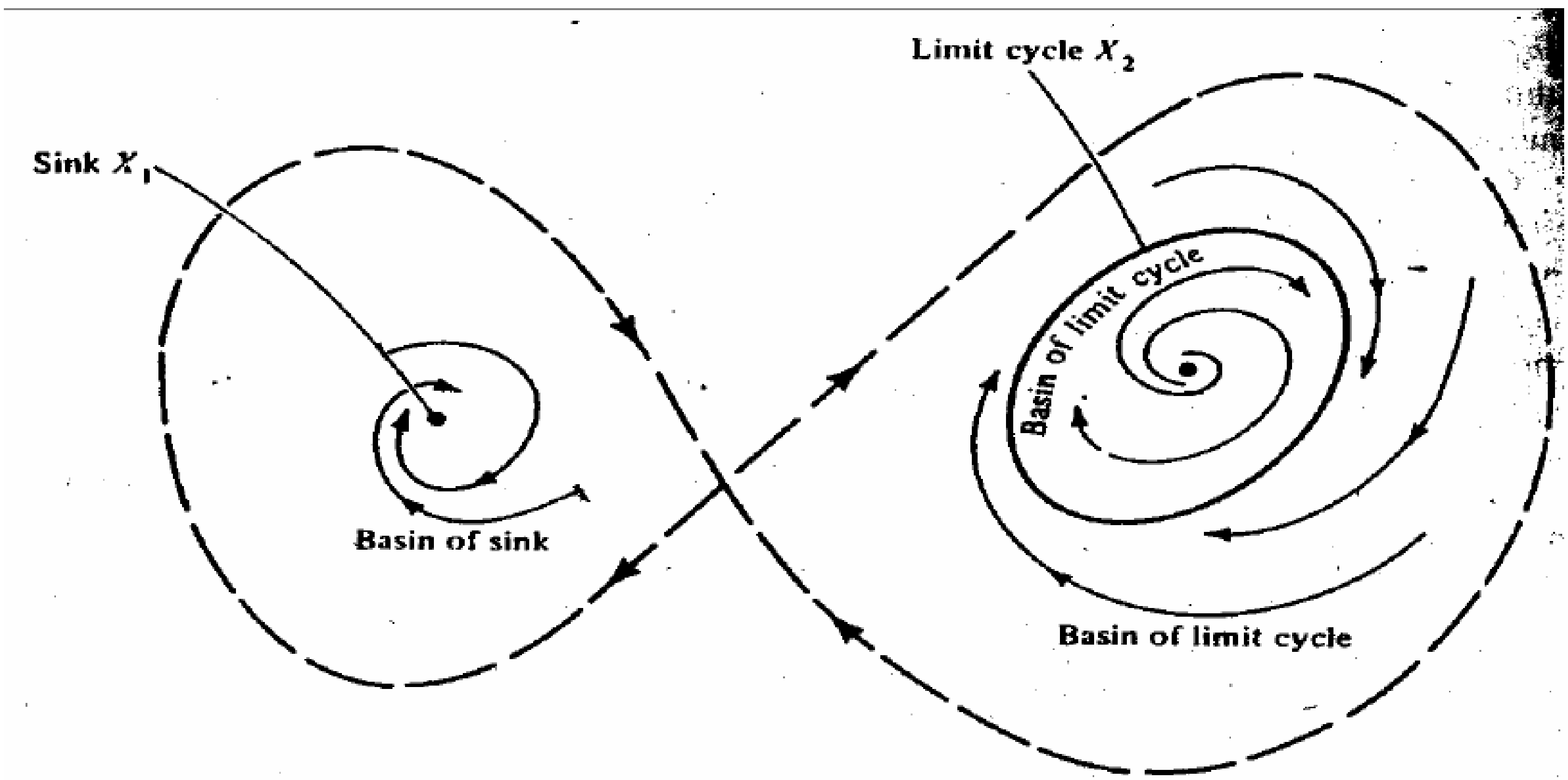
Such a surface is usually called “attractor”.

We call a subset X of a phase space an **attractor**, if

- X is invariant under the flow
- There is an (open) neighborhood around X that shrinks down to X uder the flow.
- no part of X is transient.
- X cannot be decomposed into two nonoverlapping invariant pieces.

Basin of attraction

The "*basin of attraction*" of X is the set of states in phase space that approach X as $t \to \infty$. Often there are a finite number of attractors $X_1, \cdots, X_M$ for an N-dimensional flow, although cases are known that have infinitely many attractors. Except for a set of measure zero, all initial states lie in the basin of one of the M attractors. (See Fig.).



Bifurcation

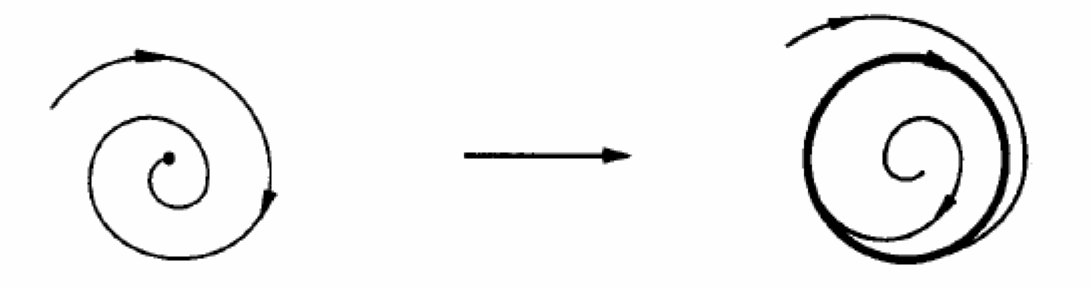
The real part of one or more of the eigenvalues may pass through zero at a critical paramter value, leading to *bifurcation phenomena.*

We can shift the
origin of μ so that the bifurcation occurs at $\mu = 0$. If a single real eigenvalue passes through zero, then the bifurcation is essentially one dimensional; i.e. the bifurcation can be found in one-dimensional flows or one one-dimensional manifolds embedded in higher dimensional flows.

Some types of bifurcation

- Pitchfork Bifurcation
  - A period one fixed point of the map becomes unstable and a pair of stable fixed points of period two appear.
- Tangent Bifurcation
  - A stable and unstable fixed point approach each other and disappear into the complex plane. This leads to so-called "intermittency"
- Hopf Bifurcation

The transition from a point attractor to a limit cycle



Strange attractor