

Efficiency of dynamical decoupling sequences in the presence of pulse errors

Zhihao Xiao (肖之昊), Lewei He (何乐为), and Wen-ge Wang (王文阁)*

Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China

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For a generic dynamical decoupling sequence employing a single-axis control, we study its efficiency in the presence of small errors in the direction of the controlling pulses. In the case that the corresponding ideal dynamical decoupling sequence produces sufficiently good results, the impact of the errors is found to scale as ξ^2 , with negligible first-order effect, where ξ is the dispersion of the random errors. This analytical prediction is numerically tested in a model in which the environment is modeled by one qubit coupled to a quantum kicked rotator in chaotic motion. In this model, with periodic pulses applied to the qubit in the environment, it is found numerically that periodic bang-bang control may outperform Uhrig dynamical decoupling.

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I. INTRODUCTION

Dynamical decoupling (DD) has attracted lots of attention in the past years, due to its potential application in quantum-information processes. The basic idea is to insert a sequence of controlling pulses within a time period of interest, such that the system of interest can be effectively decoupled from its environment. As a result, environment-induced decoherence can be effectively suppressed and initial coherence in the system can be preserved.

Several DD schemes have been proposed. For example, the so-called periodic “bang-bang” control [1–3] can suppress decoherence up to the order of $O(T^2)$ for a given period T of coherence preservation, while the Carr-Purcell-Meiboom-Gill sequence has an $O(T^3)$ efficiency [4,5]. A better result can be obtained by an approach called “concatenated dynamical decoupling” [6,7], with an efficiency of the order of $O(T^{N+1})$ achieved by 2^N pulses. Recently, a remarkable progress has been made by Uhrig, showing that decoherence of a single spin, which is induced by a spin-boson bath, can be suppressed up to the order of $O(T^{N+1})$ with only N pulses [5]. Later, the Uhrig dynamical decoupling (UDD) was conjectured [8] and rigorously proved to be model independent for the pure dephasing case [9]. More recently, it has been found that different types of UDD sequences may be integrated to obtain better results [10] and, with appropriate extension, to work for a system with two spins as well [11]. Meanwhile, the efficiencies of UDD and its generalization have been beautifully demonstrated experimentally [12–14].

In practical application of a DD sequence, an important topic is its robustness, i.e., the influence of nonidealness of the controlling pulses in the efficiency of the DD. The nonidealness may come from finite widths of the pulses and/or small deviations of the actual directions of the pulses from their designed directions. In the case that the nonideal parts of the pulses with finite width satisfy certain symmetry requirements, a generalized UDD can be found [9]. However, in a generic circumstance, the accumulation effect of small imperfections in the pulses may have significant consequences

when the number of pulses is not small. There have been several investigations in the impact of systematic pulse errors for some specific DD sequences, both experimentally [15–18] and theoretically [17–20].

In this work, for a generic DD employing a single-axis control, we study accumulation effects of small random errors in the direction of the controlling pulses. Analytical derivations are given in Sec. II. In particular, we show that a DD, which has a sufficiently good performance in the ideal case, is robust in the sense that the pulse errors have negligible first-order effect.

In numerical investigation, we employ a model in which there are periodic pulses in the environment, which are not directly applied to the qubit of interest. Using this model, we can test our analytical predictions and also study another topic of interest, namely, the influence of high-frequency cutoff in the efficiency of UDD. When the spectrum of the environment has a sharp high-frequency cutoff, UDD has been found outperforming all other known DD sequences and is regarded as optimal. However, for environments with soft cutoffs in the spectra, there is no reason to expect that UDD is optimal; in fact, recently it has been shown that protocols with periodic structure, such as the Carr-Purcell-Meiboom-Gill sequence, may have a better performance for this type of environment [21–24].

Specifically, In Sec. III, we study a model in which the environment is simulated by a second qubit coupled to a quantum kicked rotator in chaotic motion. Previous study shows that this model, though a single-particle dynamical system, may simulate a pure-dephasing many-body bath (Caldeira-Leggett model [25]), as well as some non-Markovian environments [26]. Our numerical simulations show that for this type of environment the periodic bang-bang control may outperform UDD. Finally, conclusions and discussions are given in Sec. IV.

II. ANALYTICAL STUDY OF IMPACTS OF SMALL PULSE ERRORS

In this section, we first recall essential properties of DD and then derive expressions for the influence of small pulse errors in the efficiency of a DD that employs a single-axis control.

*wqwang@ustc.edu.cn

A. Dynamical decoupling scheme

Let us first recall some essential properties of a generic DD. Consider a qubit S and its environment \mathcal{E} , under a Hamiltonian,

$$H = H_S + H_I + H_{\mathcal{E}}, \quad (1)$$

where H_S and $H_{\mathcal{E}}$ are the Hamiltonians of S and \mathcal{E} , respectively, and H_I is the interaction Hamiltonian. The self-Hamiltonian H_S is assumed to be a constant, which can be taken as zero, $H_S = 0$. We further assume that the interaction Hamiltonian H_I is commutable with σ_z , $[\sigma_z, H_I] = 0$, where σ_z is the z component of the Pauli operator for the qubit S .

Suppose the qubit S lies initially in a state which is not an eigenstate of the interaction Hamiltonian H_I . Schrödinger evolution under the Hamiltonian H , given by a unitary operator $U(T, 0)$ for a time period T , may induce decoherence to the state of the qubit S . The purpose of a DD is to preserve coherence in the initial state of S within the time period T by inserting a sequence of pulses, e.g., π pulses for the x direction. Using $t_i = \delta_i T$, $i = 1, 2, \dots, n$, to indicate the instants at which pulses are applied, with totally n pulses, the time-evolution operator is now written as

$$R = U(T, t_n) \sigma_x U(t_n, t_{n-1}) \sigma_x \cdots U(t_2, t_1) \sigma_x U(t_1, 0). \quad (2)$$

To measure the preservation of coherence in an initial state in the x direction, one may consider measurement of the observable σ_x , which gives the signal

$$s(T) = \langle \uparrow | D_y^\dagger(-\pi/2) R^\dagger \sigma_x R D_y(-\pi/2) | \uparrow \rangle, \quad (3)$$

where $|\uparrow\rangle$ indicates an eigenstate of σ_z and $D_y(-\pi/2)$ rotates it to an eigenstate of σ_x . After some derivation, one gets

$$s(T) = \text{Re} \langle \downarrow | R^\dagger \sigma_x R | \uparrow \rangle. \quad (4)$$

The coherence is perfectly preserved if $s(T) = 1$.

In UDD, π pulses are applied to the qubit S at times $t_j = \delta_j T$, where

$$\delta_j = \sin^2[\pi j / (2n + 2)] \quad (j = 0, 1, 2, \dots, n, n + 1). \quad (5)$$

For UDD, $s(T) = 1 - O(T^{2n+2})$. In a periodic bang-bang control of DD, pulses are applied at times with $\delta_j = j/n$.

B. $s(T)$ expanded to the second-order term of error

As discussed in the Introduction, controlling pulses in a DD may be subject to small random errors in its direction. Let us consider small random deviation in the y direction for π pulses in the x direction. In this case, σ_x in the ideal time-evolution operator in Eq. (2) should be replaced by

$$\sigma_\varepsilon = \varepsilon_x \sigma_x + \varepsilon_y \sigma_y, \quad (6)$$

where $\varepsilon_x = \sqrt{1 - \varepsilon_y^2}$ and ε_y is a small random number with Gaussian distribution,

$$f(\varepsilon_y) = \frac{1}{\sqrt{2\pi\xi^2}} \exp\left(-\frac{\varepsilon_y^2}{2\xi^2}\right). \quad (7)$$

Here, ξ is the dispersion of the random number, with $\xi \ll 1$. Then, the time-evolution operator, denoted by R_ε , is written as

$$\begin{aligned} R_\varepsilon = & U(T, t_n) \left(\sqrt{1 - \varepsilon_{y,n}^2} \sigma_x + \varepsilon_{y,n} \sigma_y \right) \\ & \times U(t_n, t_{n-1}) \left(\sqrt{1 - \varepsilon_{y,n-1}^2} \sigma_x + \varepsilon_{y,n-1} \sigma_y \right) \\ & \cdots U(t_2, t_1) \left(\sqrt{1 - \varepsilon_{y,1}^2} \sigma_x + \varepsilon_{y,1} \sigma_y \right) U(t_1, 0), \end{aligned} \quad (8)$$

where we use $\varepsilon_{y,i}$ to indicate the value of ε_y for the i th pulse. Now, the signal, denoted by $s_\varepsilon(T)$, has the following expression,

$$s_\varepsilon(T) = \text{Re} \langle \downarrow | R_\varepsilon^\dagger \sigma_x R_\varepsilon | \uparrow \rangle. \quad (9)$$

To evaluate $s_\varepsilon(T)$, we substitute Eq. (8) into Eq. (9) and expand the result in the power of ε_y . Up to the second-order term of ξ , we write the signal in the following form,

$$s_\varepsilon(T) = s_{\varepsilon 0}(T) + s_{\varepsilon 1}(T) + s_{\varepsilon 2}(T) + O(\xi^3). \quad (10)$$

Definitions of the first three terms on the right hand side of Eq. (10) will be given below, when they are treated separately.

The first term on the right-hand side of Eq. (10) is obtained by considering only the contribution of $\varepsilon_x \sigma_x$ in each σ_ε ,

$$s_{\varepsilon 0}(T) = \text{Re} \langle \downarrow | R_{\varepsilon 0}^\dagger \sigma_x R_{\varepsilon 0} | \uparrow \rangle, \quad (11)$$

where

$$\begin{aligned} R_{\varepsilon 0} = & U(T, t_n) \sqrt{1 - \varepsilon_{y,n}^2} \sigma_x U(t_n, t_{n-1}) \\ & \times \sqrt{1 - \varepsilon_{y,n-1}^2} \sigma_x U(t_{n-1}, t_{n-2}) \cdots \sqrt{1 - \varepsilon_{y,1}^2} \sigma_x U(t_1, 0) \\ = & \prod_{k=1}^n (1 - \varepsilon_{y,k}^2)^{\frac{1}{2}} R. \end{aligned} \quad (12)$$

Here, R is the time-evolution operator for the case of ideal pulses in Eq. (2). Substituting Eq. (12) into Eq. (11), we find

$$s_{\varepsilon 0}(T) = \prod_{k=1}^n (1 - \varepsilon_{y,k}^2) s(T). \quad (13)$$

It is seen that, for sufficiently small ξ ,

$$s_{\varepsilon 0}(T) - s(T) \sim -n\xi^2 s(T). \quad (14)$$

Thus, deviation of $s_{\varepsilon 0}(T)$ from the ideal signal $s(T)$ is of the order of $(\xi\sqrt{n})^2$.

Next, we calculate the second term on the right-hand side of Eq. (10), which is the contribution of those multiplication terms that include only one $(\varepsilon_y \sigma_y)$ term in each of them. Noticing that, up to the second-order contribution of ε_y , ε_x in this second term can be taken as 1, we have the following expression for it,

$$s_{\varepsilon 1}(T) = \text{Re} \langle \downarrow | R_{\varepsilon 1}^\dagger \sigma_x R_{\varepsilon 0} + R_{\varepsilon 0}^\dagger \sigma_x R_{\varepsilon 1} | \uparrow \rangle, \quad (15)$$

where

$$\begin{aligned} R_{\varepsilon 1} = & U(T, t_n) \varepsilon_{y,n} \sigma_y U(t_n, t_{n-1}) \sigma_x \cdots U(t_2, t_1) \sigma_x U(t_1, 0) \\ & + U(T, t_n) \sigma_x U(t_n, t_{n-1}) \varepsilon_{y,n-1} \sigma_y \cdots U(t_2, t_1) \sigma_x U(t_1, 0) \\ & + \cdots \\ & + U(T, t_n) \sigma_x U(t_n, t_{n-1}) \sigma_x \cdots U(t_2, t_1) \varepsilon_{y,1} \sigma_y U(t_1, 0). \end{aligned} \quad (16)$$

Making use of the fact that $\sigma_y = i\sigma_x\sigma_z$ and $[\sigma_z, H] = 0$, we find

$$R_{\varepsilon 1} = i \sum_{k=1}^n (-1)^{k+1} \varepsilon_{y,k} R_{\varepsilon 0} \sigma_z. \quad (17)$$

Substituting Eq. (12) into Eq. (17), then into Eq. (15), we obtain

$$s_{\varepsilon 1}(T) = \text{Re} \left[2i \sum_{k=1}^n (-1)^{k+1} \varepsilon_{y,k} \langle \downarrow | R_{\varepsilon 0}^\dagger \sigma_x R_{\varepsilon 0} | \uparrow \rangle \right]. \quad (18)$$

Typically, one has the following estimate,

$$\left| \sum_{k=1}^n (-1)^{k+1} \varepsilon_{y,k} \right| \sim \xi \sqrt{n}, \quad (19)$$

and hence,

$$s_{\varepsilon 1}(T) \sim \pm 2\xi \sqrt{n} \text{Re}[i \langle \downarrow | R_{\varepsilon 0}^\dagger \sigma_x R_{\varepsilon 0} | \uparrow \rangle]. \quad (20)$$

Making use of Eq. (12), after simple algebra, it is found that

$$s_{\varepsilon 1}(T) \sim \pm 2\xi \sqrt{n} q(T), \quad (21)$$

where

$$q(T) = \text{Im} \langle \downarrow | R^\dagger \sigma_x R | \uparrow \rangle. \quad (22)$$

Therefore, $s_{\varepsilon 1}(T)$ gives a first-order correction ($\sim \xi \sqrt{n}$) to the ideal signal.

The first-order correction $s_{\varepsilon 1}(T)$ also depends on the quantity $q(T)$. To give an estimation to $q(T)$, we note that $s(T) + iq(T) = \langle \downarrow | R^\dagger \sigma_x R | \uparrow \rangle$ [see Eqs. (4) and (22)]. Hence,

$$|q(T)| = \sqrt{|\langle \downarrow | R^\dagger \sigma_x R | \uparrow \rangle|^2 - s^2(T)} \leq \sqrt{1 - s^2(T)}, \quad (23)$$

where we have used the fact that $|\langle \downarrow | R^\dagger \sigma_x R | \uparrow \rangle| \leq 1$.

Finally, we discuss the third term on the right-hand side of Eq. (10), which includes all multiplication terms that have only two ($\varepsilon_y \sigma_y$) terms,

$$s_{\varepsilon 2}(T) = \text{Re} \langle \downarrow | R_{\varepsilon 1}^\dagger \sigma_x R_{\varepsilon 1} + R_{\varepsilon 0}^\dagger \sigma_x R_{\varepsilon 2} + R_{\varepsilon 2}^\dagger \sigma_x R_{\varepsilon 0} | \uparrow \rangle, \quad (24)$$

where

$$\begin{aligned} R_{\varepsilon 2} = & U(T, t_n) \varepsilon_{y,n} \sigma_y U(t_n, t_{n-1}) \varepsilon_{y,n-1} \sigma_y \\ & \times U(t_n, t_{n-1}) \sigma_x \cdots U(t_2, t_1) \sigma_x U(t_1, 0) \\ & + U(T, t_n) \varepsilon_{y,n} \sigma_y U(t_n, t_{n-1}) \sigma_x \\ & \times U(t_n, t_{n-1}) \varepsilon_{y,n-2} \sigma_y \cdots U(t_2, t_1) \sigma_x U(t_1, 0) \\ & + \cdots + U(T, t_n) \sigma_x U(t_n, t_{n-1}) \sigma_x \\ & \cdots U(t_3, t_2) \varepsilon_{y,2} \sigma_y U(t_2, t_1) \varepsilon_{y,1} \sigma_y U(t_1, 0). \end{aligned} \quad (25)$$

Following a procedure similar to that for $R_{\varepsilon 1}$, we find

$$R_{\varepsilon 2} = \sum_{k=1}^n \sum_{j(<k)} (-1)^{j+k+1} \varepsilon_{y,j} \varepsilon_{y,k} R_{\varepsilon 0} \sim \pm \frac{1}{2} \xi^2 n R_{\varepsilon 0}. \quad (26)$$

Substituting the above-obtained expressions of $R_{\varepsilon 0}$, $R_{\varepsilon 1}$, and $R_{\varepsilon 2}$ into the expression of $s_{\varepsilon 2}(T)$ in Eq. (24), after some derivation, we obtain an expression for $s_{\varepsilon 2}(T)$. Then, making use of results obtained above in Eqs. (14) and (21), finally, we find

$$s_{\varepsilon}(T) - s(T) \sim \pm 2q(T) \xi \sqrt{n} + C_2 \xi^2 n + O(\xi^3), \quad (27)$$

where $|C_2|$ is of the order of 1.

Of particular interest is the case for a DD with good performance, i.e., with $s(T) \sim 1$. In this case, the inequality (23) shows that $q(T)$ is small. In particular, in the case that $|q(T)| \ll \xi \sqrt{n}$, the first-order term on the right-hand side of Eq. (27) can be neglected and we have

$$s_{\varepsilon}(T) - s(T) \sim C_2 \xi^2 n + O(\xi^3), \quad (28)$$

scaling as $\xi^2 n$. For example, for a UDD with $s(T) = 1 - O(T^{2n+2})$, $q(T)$ is of the order of $O(T^{2n+2})$ or less; hence, for a fixed ξ , $|q(T)| \ll \xi \sqrt{n}$ for a sufficiently large n .

In concluding this section, we remark that it is straightforward to generalize the above discussions to the case of pulses with more generic random errors in their direction. In fact, in this generic case, the time-evolution operator can be obtained by replacing σ_x in Eq. (2) by

$$\sigma_{\varepsilon} = \varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \varepsilon_z \sigma_z, \quad (29)$$

where $\varepsilon_x = \sqrt{1 - \varepsilon_y^2 - \varepsilon_z^2}$ and both ε_y and ε_z are small random numbers with Gaussian distribution. We have found results that are qualitatively the same as those discussed above for the case of σ_{ε} in Eq. (6). In particular, we have found similar estimations as those given in the relations (27) and (28).

III. NUMERICAL SIMULATIONS

In this section, we discuss numerical simulations we have performed for two purposes. One is to check analytical predictions given in the previous section, the other is to study the influence of kicks in the environment in the efficiency of UDD. In fact, since instant kicks in the environment may have non-negligible high-frequency components, the performance of UDD for such an environment may be not as good as that for an environment with a sharp high-frequency cutoff.

A. The model

We consider a model, in which there is a qubit S of interest and an environment \mathcal{E} that is composed of a second qubit A and a quantum kicked rotator denoted by B . The qubit S has interaction with A only, while A interacts with both S and the kicked rotator B . The Hamiltonian is written as

$$H = H_S + H_A + H_B + H_{SA} + H_{AB}, \quad (30)$$

where the self-Hamiltonians are $H_S = 0$, $H_A = \omega_A \sigma_x^A$, and

$$H_B = \frac{p^2}{2} + k \cos \theta \sum_j \delta(t - jT_0), \quad (31)$$

with T_0 being the period of kicking. The interaction Hamiltonians are $H_{SA} = g \sigma_z^S \otimes \sigma_z^A$ and

$$H_{AB} = \lambda \sigma_z^A \cos \theta \sum_j \delta(t - jT_0). \quad (32)$$

Here, for clearness we write explicitly the superscript S in the Pauli operator for the qubit S .

This model has been studied in Ref. [26], showing that the kicked rotator, though a single-particle dynamical system, may simulate a pure-dephasing many-body bath (Caldeira-Leggett model [25]), as well as some non-Markovian environments.

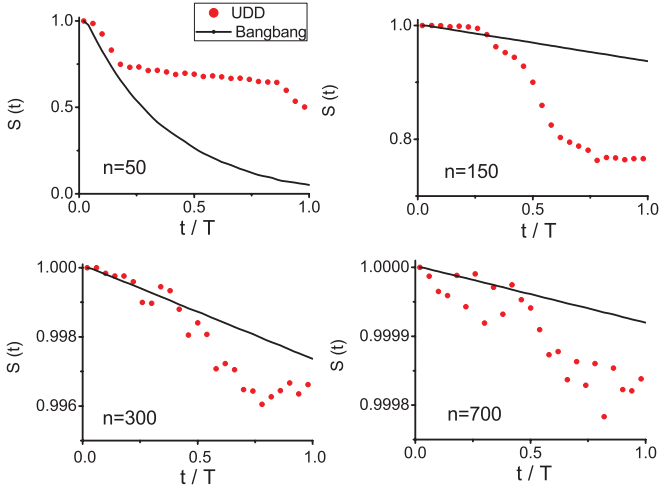


FIG. 1. (Color online) Variation of the signal $s(t)$ in Eq. (34) with time for ideal UDD and ideal periodic bang-bang control. Both UDD and periodic bang-bang control have better performance for larger values of n , the number of controlling pulses. Results of UDD (circles) are better than those of the periodic bang-bang control (solid curve) for $n = 50$. But, for larger values of n , the periodic bang-bang control outperforms UDD. Parameters are $\omega_A = 1.5 \times 10^3$, $\lambda = 10^3$, $g = 100$, $N = 2^{12}$ and $T = 50T_0$, and the kicked rotator is in the chaotic region with $K = 10^3$. The same parameters are also used in the following figures.

Here, we are interested in the chaotic region of the kicked rotator to simulate some random properties of the environment.

The time evolution for one period T_0 is given by the unitary operator

$$\hat{U}_{T_0} = e^{-i(\omega_A T_0 \sigma_x^A + T_0 g \sigma_z^S \otimes \sigma_z^A)} e^{-i T_0 \frac{p^2}{2}} e^{-i(k + \lambda \sigma_z^A) \cos \theta}, \quad (33)$$

where \hbar has been set unit. An effective Planck constant can be introduced, $\hbar_{\text{eff}} = T_0 = 2\pi/N$, where N is the dimension of the Hilbert space of the kicked rotator. The classical limit is obtained by letting $T_0 \rightarrow 0$ and $k \rightarrow \infty$ while keeping $K = kT_0$ fixed, and the classical counterpart is defined on a torus

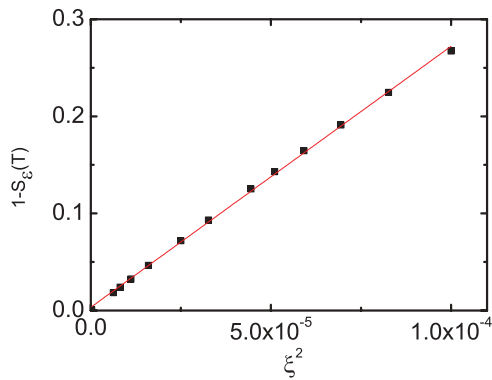


FIG. 2. (Color online) Dependence of $1 - s_\varepsilon(T)$ (solid squares) on ξ^2 for a UDD sequence, with $n = 500$ and $\xi\sqrt{n} \ll 1$. The solid curve is a straight line, indicating a linear dependence, in agreement with the prediction of Eq. (28). For the corresponding ideal UDD, $1 - s(T) \sim 10^{-4}$. Since $1 - s_\varepsilon(T) \gg 1 - s(T)$, the influence of the pulse errors is profound.

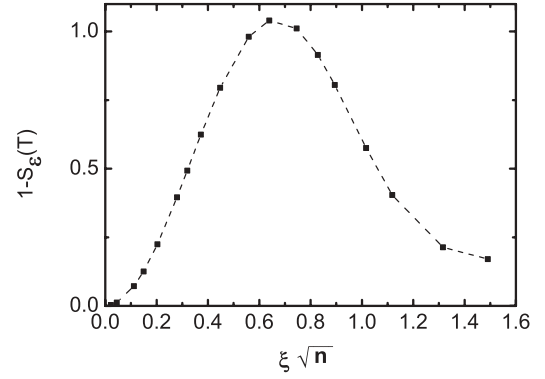


FIG. 3. Variation of $1 - s_\varepsilon(T)$ with $\xi\sqrt{n}$ for a UDD sequence with $n = 500$.

$[0, 2\pi) \otimes [0, 2\pi)$. The kicked rotator has a chaotic motion for K larger than 6 or so.

B. Numerical results

Let us first discuss the performance of ideal UDD and ideal periodic bang-bang control in this model. We are interested in the case that $T \gg T_0$. In this case, due to the time dependency of the Hamiltonian of the environment, Uhrig's strategy for deriving $1 - s(T) \sim O(T^{2n+2})$ is not applicable. It is of interest to know whether UDD is optimal or not in this case.

In a numerical simulation, we calculated $s(t)$ defined by

$$s(t) = \text{Re} \langle \downarrow | R^\dagger(t) \sigma_x R(t) | \uparrow \rangle, \quad (34)$$

where $R(t)$ is obtained by truncation of the time-evolution operator R in Eq. (2) at an intermediate time $t \leq T$. Figure 1 shows that both UDD and periodic bang-bang control have better performance with increasing number n of the controlling pulses. For $n = 50$, UDD outperforms the periodic bang-bang control in the whole time region $(0, T]$. However, with increasing n , results of the periodic bang-bang control become better than those of UDD for $t > T/2$.

Next, we check analytical predictions given above for the impact of random errors in the direction of the controlling pulses, in particular, the behavior of $s_\varepsilon(T)$ in Eq. (28). We use the general form of σ_ε in Eq. (29), with the same dispersion ξ for ε_y and ε_z . An example is given in Fig. 2, showing linear dependence of $1 - s_\varepsilon(T)$ on ξ^2 . To check the details of agreement with analytical predictions, we have numerically computed the corresponding ideal UDD and found that it has a good performance with $1 - s(T) \sim 10^{-4}$. This gives

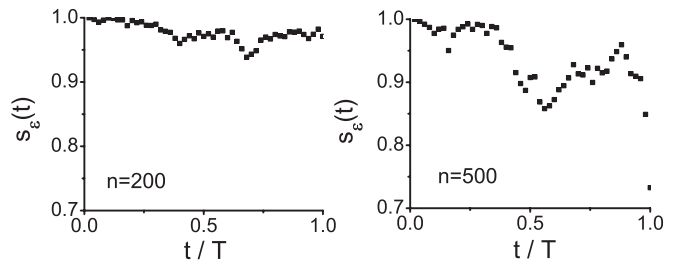


FIG. 4. Variation of the signal $s_\varepsilon(t)$ with time for a UDD with nonideal pulses, $\xi = 10^{-2}$. The UDD with $n = 500$ (right panel) gives results worse than those of the UDD with $n = 200$ (left panel).

$|q(T)| \lesssim 10^{-2}$ [see the estimate in the inequality (23)]. Hence, $|q(T)| \ll \xi\sqrt{n} \sim 10^{-1}$ for the solid squares shown in the figure; as a result, Eq. (28) should hold with $s(T) - s_\varepsilon(T) \simeq 1 - s_\varepsilon(T)$. Thus, the results in Fig. 2 indeed confirm the prediction of Eq. (28).

We have also numerically studied the case of large $\xi\sqrt{n}$. In this case, higher-order terms of ξ in Eq. (27) should be considered. Anyway, this expression of $s_\varepsilon(T)$ suggests that $1 - s_\varepsilon(T)$ may be large for $\xi^2 n \sim 1$. Indeed, numerical results support this expectation, as shown in Fig. 3.

Figure 2 shows that for a fixed n , the influence of pulse errors may become large with increasing ξ , such that $1 - s_\varepsilon(T) \gg 1 - s(T)$. In Fig. 4, we show that for a fixed error dispersion ξ , deviation of $s_\varepsilon(t)$ from 1 enlarges when n is increased, where $s_\varepsilon(t)$ is defined by Eq. (34) with $R(t)$ replaced by the corresponding $R_\varepsilon(t)$.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we have analytically studied the efficiency of a generic DD with a single-axis control when the controlling pulses are subject to small random errors in their direction. We have derived expressions for the influence of the pulse errors up to the second-order term. When the ideal DD has a sufficiently good performance, the influence has a negligible first-order effect; in this sense, good DD is relatively robust.

We have tested the above analytical predictions numerically and shown that accumulation of small pulse errors may have significant influence on the efficiency of DD. For an environment with kicks applied on some part of the environment, it has been found that the periodic bang-bang control may outperform UDD.

A natural question would concern the possibility of having negligible first-order effect of pulse errors in more generic situations, e.g., in the case of more than one layer of controlling pulses with different single-axis control in different layers. For pulse errors appearing only at certain fixed layers with single-axis control, results of this paper may be generalizable. However, the more generic situation with pulse errors in different layers, as well as the case with finite width of the pulses, is much more complex and further investigation is needed before a definite conclusion can be drawn.

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- [1] L. Viola and S. Lloyd, *Phys. Rev. A* **58**, 2733 (1998).
 - [2] L. Viola, E. Knill, and S. Lloyd, *Phys. Rev. Lett.* **82**, 2417 (1999).
 - [3] J. J. L. Morton *et al.*, *Nat. Phys.* **2**, 40 (2006); J. J. L. Morton *et al.*, *Nature (London)* **455**, 1085 (2008).
 - [4] C. P. Slichter, *Principles of Magnetic Resonance* (Springer, Berlin/New York, 1990).
 - [5] G. S. Uhrig, *Phys. Rev. Lett.* **98**, 100504 (2007); G. S. Uhrig, *New J. Phys.* **10**, 083024 (2008).
 - [6] K. Khodjasteh and D. A. Lidar, *Phys. Rev. Lett.* **95**, 180501 (2005).
 - [7] K. Khodjasteh and D. A. Lidar, *Phys. Rev. A* **75**, 062310 (2007).
 - [8] B. Lee, W. M. Witzel, and S. Das Sarma, *Phys. Rev. Lett.* **100**, 160505 (2008).
 - [9] W. Yang and R. B. Liu, *Phys. Rev. Lett.* **101**, 180403 (2008).
 - [10] J. R. West, B. H. Fong, and D. A. Lidar, *Phys. Rev. Lett.* **104**, 130501 (2010).
 - [11] M. Mukhtar, T. B. Saw, W. T. Soh, and J. B. Gong, *Phys. Rev. A* **81**, 012331 (2010).
 - [12] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, *Nature (London)* **458**, 996 (2009).
 - [13] J. F. Du, X. Rong, N. Zhao, Y. Wang, J. H. Yang, and R. B. Liu, *Nature (London)* **461**, 1265 (2009).
 - [14] Y. Wang, X. Rong, P. B. Feng, W. J. Xu, B. Chong, J. H. Su, J. B. Gong, and J. F. Du, *Phys. Rev. Lett.* **106**, 040501 (2011).
 - [15] G. A. Alvarez, A. Ajoy, X. Peng, and D. Suter, *Phys. Rev. A* **82**, 042306 (2010).
 - [16] M. J. Biercuk, H. Uys, A. P. VanDevender, N. Shiga, W. M. Itano, and J. J. Bollinger, *Phys. Rev. A* **79**, 062324 (2009).
 - [17] Z. H. Wang, W. Zhang, A. M. Tyryshkin, S. A. Lyon, J. W. Ager, E. E. Haller, and V. V. Dobrovitski, e-print [arXiv:1011.6417](https://arxiv.org/abs/1011.6417).
 - [18] A. M. Tyryshkin, Z.-H. Wang, W. Zhang, E. E. Haller, J. W. Ager, V. V. Dobrovitski, and S. A. Lyon, e-print [arXiv:1011.1903](https://arxiv.org/abs/1011.1903).
 - [19] Z. H. Wang and V. V. Dobrovitski, e-print [arXiv:1101.0292](https://arxiv.org/abs/1101.0292).
 - [20] J. T. Kao, J. T. Hung, P. Chen, and C. Y. Mou, *Phys. Rev. A* **82**, 062120 (2010).
 - [21] L. Cywinski, R. M. Lutchyn, C. P. Nave, and S. Das Sarma, *Phys. Rev. B* **77**, 174509 (2008).
 - [22] S. Pasini and G. S. Uhrig, *Phys. Rev. A* **81**, 012309 (2010).
 - [23] G. de Lange, Z. H. Wang, D. Riste, V. V. Dobrovitski, and R. Hanson, *Science* **330**, 60 (2010).
 - [24] A. Ajoy, G. A. Alvarez, and D. Suter, *Phys. Rev. A* **83**, 032303 (2011).
 - [25] A. O. Caldeira and A. J. Leggett, *Ann. Phys.* **149**, 374 (1983).
 - [26] D. Rossini, G. Benenti, and G. Casati, *Phys. Rev. E* **74**, 036209 (2006).