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Relatively-long time decay of Loschmidt echo of a Bose-Einstein condensate in a double-well potential

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Abstract. We study the long-time decay of quantum Loschmidt echo (LE) of a Bose-Einstein condensate (BEC) in a double-well potential. In the tunneling and self-trapping phases of the BEC, the LE has exponential and Gaussian decays, respectively, for relatively-long times. In the crossover region, the LE behaves differently from both the tunneling and the self-trapping phases. These results indicate that relatively-long time decay of the LE is suitable for characterizing the dynamical phase transition of the BEC.

1 Introduction

Since Bose-Einstein condensate (BEC) was realized in experiments [1–3], a new epoch for studying its dynamical properties has also been launched. One of the exciting fields is BEC in optical lattice [4]. Optical lattice has the advantage of being almost completely controllable. This feature is certainly not available in any solid-state experiment, and it makes BEC in optical lattice an ideal quantum simulator to verify predictions of condensed matter theories [5]. For example, quantum phase transition [6] between superfluid and Mott insulator phases in the Bose-Hubbard model [7] has been observed experimentally in a BEC system [8]. In the study of quantum phase transition, in addition to standard tools such as order parameter, correlation function, and scaling law, the concepts of geometric phase [9,10] and entanglement [11,12] have also been made use of.

Recently, aroused by the work of Zanardi and Paunkovic [13] and Quan et al. [14], there have been investigations, which show that fidelity and (quantum) Loschmidt echo (LE) decay much faster in the neighborhood of critical points of quantum phase transitions than in regimes far from the critical points [15–17]. A merit of this approach of characterizing quantum phase transition is that it does not require any a priori knowledge in correlation functions or in order parameters, since the fidelity is a purely geometrical quantity in the Hilbert space.

The LE is defined by the overlap of the time evolution of the same initial state under two slightly different Hamiltonians,

$$M(t) = |\langle\Psi_0|\exp(iHt/\hbar)\exp(-iH_0t/\hbar)|\Psi_0\rangle|^2, \quad (1)$$

where H_0 and H are the unperturbed and perturbed Hamiltonians, respectively, $H = H_0 + \epsilon H_1$ with a small parameter ϵ. This quantity was first introduced by Peres more than twenty years ago in the study of the stability of quantum motion [18], and has been extensively studied in recent years [19–26]. It is also of relevance in the study of decoherence [27] and concurrence in two-qubit systems coupled to spin chains [28].

It is known that a BEC system in a double-well potential may undergo a dynamical phase transition [29,30], between a tunneling phase and a self-trapping phase. This phenomenon was first predicted theoretically [31,32], then, observed experimentally [33] and studied in detail analytically [34–38]. Recently, it has been found that the decay of the LE for not long times has a Gaussian form in the self-trapping phase, while has a stretched exponential form in the tunneling phase [39,40].

The LE may have more than one types of decay in different time regimes. For example, Weinstein and Hellberg noticed that, in the nearly-integrable regime of the quantum kicked top (QKT) model, the LE has exponential or power-law decay beyond an initial Gaussian decay [41]. Using a semiclassical method, these rich decaying behaviors of the LE had been analytically derived [42]. These interesting results on the two-stages LE decays inspire us to study relatively-long time decay of LE, to see whether it may characterize the dynamical phase transition in a BEC system in a double-well potential. Our numerical simulations indeed confirmed this expectation. Specifically, we found that in the tunneling and self-trapping phases of the BEC, the LE has exponential and Gaussian decays, respectively, for relatively-long times; while in the crossover region, the LE behaves differently. These results indicate

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that the LE is suitable for characterizing the dynamical phase transition of the BEC.

This paper is structured as following: in Section 2, the double-well potential BEC model is introduced and the dynamical phase transition is briefly discussed. Section 3 is devoted to a study of the relatively-long time decay of the LE in the neighborhoods of the dynamical phase transition. Section 4 gives conclusions.

2 BEC in double-well potential

2.1 Model

We consider a N-atom BEC in a double-well potential, with the wells indicated by A and B, respectively. The Hamiltonian in the second quantization form is written as [43],

$$H = \frac{\gamma}{2}(a^\dagger a - b^\dagger b) + \frac{c}{2N}(a^\dagger a^\dagger aa + b^\dagger b^\dagger bb) - \frac{v}{2}(a^\dagger b + b^\dagger a). \tag{2}$$

Here $\gamma = E_a - E_b$ is level separation between the two wells, $E_i = \int[(1/2m|\nabla\varphi_i|^2 + V(r)|\varphi_i|^2|]dr$, $c = (4\pi aN/m)\int|\varphi_i|^4 dr$ is the effective interaction constant of atoms, determined by the s-wave scattering length a between atoms, $v = \int[(1/2m)\nabla\varphi_a\nabla\varphi_b + V(r)\varphi_a\varphi_b]dr$ is the coupling strength between the two wells, $\varphi_i(i = a, b)$ is the wave function for each well. And $a^\dagger$ and $b^\dagger$ (a, b) are boson creation (annihilation) operators for the two wells, respectively.

In this paper, we consider the case of a symmetric double-well potential with $\gamma = 0$ and $c > 0$ corresponding to a repulsive interaction between atoms. A scaling can make the Hamiltonian depend on only one of the two parameters v and c. Hence, we set unit the coupling strength v and consider the variation of c. Experimentally, the atom-atom interaction may be adjusted via Feshbach resonances [44].

Time evolution of the state is given by Schrödinger equation

$$i\frac{d|\varphi(t)\rangle}{dt} = H|\varphi(t)\rangle, \tag{3}$$

where the Planck constant is set unit, $\hbar = 1$. In the Fock space, the state of the BEC in the double-well potential is

$$|\varphi(t)\rangle = \sum_{n=0}^{N} a_n(t)|n\rangle. \tag{4}$$

Here we employ the Fock states $|n\rangle$ as the basis states, where n is the number of the atoms in the well A, with $(N - n)$ atoms in the well B. This BEC model is solvable by the algebraic Bethe ansatz [45,46]. But, in the study of LE decay, it is convenient to adopt the direct numerical diagonalization method as used in reference [47]. In the numerical calculations given below, we used a fixed particle number $N = 1000$.

2.2 Two phases

The BEC system have two phases: a tunneling phase and a self-trapping phase. For relatively weak nonlinear term, i.e., small value of c, if all the atoms are initially in one of the two wells, they can travel to the other well by tunneling. On the other hand, for sufficiently large c, the tunneling effect can be suppressed, resulting in a self-trapping phase.

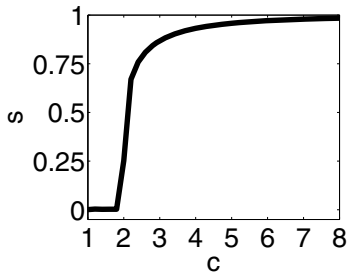

Fig. 1. Variation of mean-atom-number difference s with respect to the coupling strength c for the initial state $|0, N\rangle$.

The transition between the two phases of the BEC in a double-well potential is not a typically static QPT in the ground state, but a dynamical phase transition. When the initial state is $|0\rangle$, the critical value $c_{cr} = 2$ [48]. For displaying the two phases of this model, we consider the difference in the mean atom numbers, defined as

$$s = \{N_A - N_B\}/N, \tag{5}$$

here $N_A = \langle\varphi(t)|a^\dagger a|\varphi(t)\rangle$, $N_B = \langle\varphi(t)|b^\dagger b|\varphi(t)\rangle$, and $\{\ \}$ denotes the average over time. Figure 1 shows variation of s with respect to coupling strength c. $s \sim 0$ corresponds to the tunneling phase and $s \sim 1$ corresponds to the self-trapping phase. Note that when s is not large enough, e.g., $s < 0.5$, the atom number difference between two wells is small and only a small portion of the atoms are self-trapped. The region $s \sim 0.5$ corresponds to a crossover between the two phases. Due to competition of the two phases, behaviors of the system in the crossover region are more complex. It is reasonable to expect that the LE decay may have some intriguing behavior in the crossover region.

3 Decay of Loschmidt echo

3.1 Loschmidt echo

In this section, the decay of LE around the transition point $c_{cr} = 2$ is investigated. Let us rewrite the wave function in equation (4) as $|\varphi(t, c)\rangle$, with the dependence on the parameter c indicated explicitly. Similarly, the wave function of the perturbed system is written as

$$|\phi(t, c')\rangle = \sum_{n=0}^{N} a'_n(t)|n\rangle, \tag{6}$$

where $c' = c+\delta c$. With the dependence of H on c indicated explicitly, the two states can also been written as

$$|\varphi(t, c)\rangle = \exp[-iH(c)t]|\varphi_0\rangle,\ |\phi(t, c')\rangle = \exp[-iH(c')t]|\varphi_0\rangle. \tag{7}$$

Then, the LE can be written as

$$M(t) = |\langle\varphi_0| \exp[iH(c',t)t] \exp[-iH_0(c,t)t]|\varphi_0\rangle|^2 = |\sum_{n=0}^{N} a'_n(t)^* a_n(t)|^2, \quad (8)$$

where * denotes complex conjugate.

3.2 Relatively-long time decay of LE

The most useful analytical tool in the study of the LE decay in regular systems seems the semiclassical approach. Before presenting our numerical simulations, let us first briefly recall the main results of this approach. We present it in a general d-dimensional configuration space and consider the general expression of the LE given in equation (1) [19]. For an initial wave function $\psi_0(\mathbf{r}_0)$, its time evolution propagated by the semiclassical Van Vleck-Gutzwiller propagator is written as

$$\psi_{\rm sc}(\mathbf{r};t) = \int d\mathbf{r}_0 K_{\rm sc}(\mathbf{r},\mathbf{r}_0;t)\psi_0(\mathbf{r}_0), \quad (9)$$

where $K_{\rm sc}(\mathbf{r},\mathbf{r}_0;t) = \sum_c K_c(\mathbf{r},\mathbf{r}_0;t)$, with

$$K_c(\mathbf{r},\mathbf{r}_0;t) = \frac{C_c^{1/2}}{(2\pi i\hbar)^{d/2}} \exp\left[\frac{i}{\hbar}S_c(\mathbf{r},\mathbf{r}_0;t) - \frac{i\pi}{2}\mu_c\right]. \quad (10)$$

Here, the label c (more exactly $c(\mathbf{r},\mathbf{r}_0;t)$) indicates classical trajectories starting from $\mathbf{r}_0$ and ending at $\mathbf{r}$ within the time t, $S_c(\mathbf{r},\mathbf{r}_0;t)$ is the action (the time integral of the Lagrangian $\mathcal{L}$) along the trajectory c, $S_c(\mathbf{r},\mathbf{r}_0;t) = \int_0^t dt'\mathcal{L}$, and $C_c = |\det(\partial^2 S_c/\partial r_{0i}\partial r_j)|$. μ_c is the Maslov index counting the conjugate points. (Here, we write the Planck constant explicitly in the semiclassical expressions.)

Consider an initial Gaussian wave packet centered at $\widetilde{\mathbf{r}}_0$, with dispersion ξ and mean momentum $\widetilde{\mathbf{p}}_0$,

$$\psi_0(\mathbf{r}_0) = \left(\frac{1}{\pi\xi^2}\right)^{d/4} \exp\left[\frac{i}{\hbar}\widetilde{\mathbf{p}}_0\cdot\mathbf{r}_0 - \frac{(\mathbf{r}_0-\widetilde{\mathbf{r}}_0)^2}{2\xi^2}\right]. \quad (11)$$

When ξ is small enough, the amplitude $m(t)$ of the LE is written as [19,21]

$$m_{\rm sc}(t) \simeq \left(\pi w_p^2\right)^{-d/2} \int d\mathbf{p}_0 \exp\left[\frac{i}{\hbar}\Delta S_{\mathbf{p}_0} - \frac{(\mathbf{p}_0-\widetilde{\mathbf{p}}_0)^2}{w_p^2}\right], \quad (12)$$

where $w_p = \hbar/\xi$ and $\Delta S_{\mathbf{p}_0}$ is the action difference between two nearby trajectories of the two systems H and H_0 starting at $(\mathbf{p}_0,\widetilde{\mathbf{r}}_0)$. The semiclassical expression of the LE can then be calculated, $M_{\rm sc}(t) = |m_{\rm sc}(t)|^2$. The action difference can be calculated in the first order classical perturbation theory:

$$\Delta S_{\mathbf{p}_0} \simeq \epsilon \int_0^t dt' V[\mathbf{r}(t'),\mathbf{p}(t')] \quad (13)$$

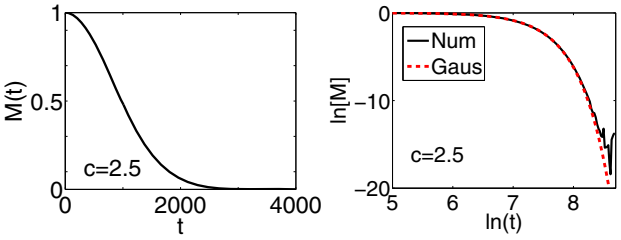

Fig. 2. (Color online) In the self-trapping phase with $c = 2.5$, the LE has a Gaussian decay in almost the whole decay region. Left panel: plotted in the natural units. Right panel: plotted in logarithm scale. The solid curve represents numerical results and the dashed curve indicates a fitting Gaussian decay. The strength of perturbation $\delta c = 10^{-4}$.

with V being a perturbation Hamiltonian evaluated along one of the two trajectories.

In a regular system with $d = 1$, the above semiclassical approximation gives the following prediction for the LE of an initial narrow Gaussian wave packet,

$$M(t) \simeq c_0(1+\xi^2 t^2)^{-1/2} e^{-\Gamma t^2/(1+\xi^2 t^2)}, \quad (14)$$

where $c_0 \sim 1$, Γ and ξ are determined by the Planck constant and some quantities in the corresponding classical system, with $\Gamma \sim \epsilon^2$ and $\xi \sim \epsilon$ [42]. Here $\epsilon = \delta c$. It is seen that the LE has a Gaussian decay $e^{-\Gamma t^2}$ for initial times with $t \ll \tau = 1/\xi$ and has a power-law decay $1/\xi t$ for relatively-long times with $t \gg \tau$. In reference [42], it is shown that between the Gaussian and power-law decays, there is a crossover region in which the LE has a transient exponential decay.

We have made extensive numerical simulations in the LE decay. In the self-trapping regime with $c \geq 2.3$, we found that the LE has a Gaussian decay in the whole decay region, as shown in Figure 2. The Gaussian decay can be explained as follows. The initial state $|0\rangle$ is a coherent state of SU(2). It is known that the LE in an integrable system may have a Gaussian decay for initial Gaussian wave packets or initial coherent states [20,42] (see also Eq. (14)). In this regime, the wave packet of the system was found localized in the time evolution, hence, the Gaussian decay of the LE may persist in a relatively-long time period.

In the crossover region, $2.0 < c < 2.3$, the LE has a complex behavior. Numerically, we found that the LE has an initial Gaussian decay, in consistence with the semiclassical prediction in equation (14). For relatively long times, the LE has a decay faster than exponential but slower than Gaussian, as shown in Figure 3. We found that the long time decay of the LE can not be fitted well by stretched exponential decay, either. In the transition region between the initial Gaussian decay and the long-time decay in Figure 3, we found that the LE has a decay somewhat close to a power-law decay with a power between –3 and –4. The complex behavior of the LE decay is due to competition of the two phases. In fact, in the crossover region, wave functions have been found having some multi-fractal structure with strong fluctuations in all the scales [48].

In the tunneling phase with $c < 2$, it is shown in reference [40] that the LE has a stretched exponential for

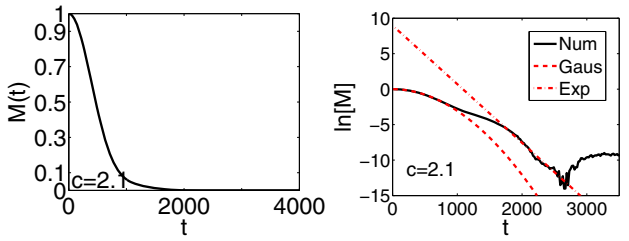

Fig. 3. (Color online) The LE decay in the crossover region with $c = 2.1$. Right panel: the initial decay of the LE is Gaussian (dashed curve), and the long time decay is slower than Gaussian and faster than exponential (dashed-dotted line). The strength of perturbation $\delta c = 10^{-4}$.

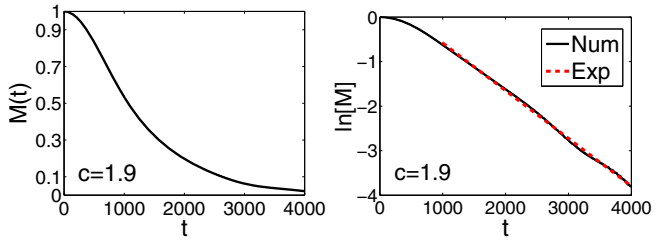

Fig. 4. (Color online) In the tunneling phase with $c = 1.9$, for long times, the LE has an exponential decay, $M \propto \exp(-kt)$. The two panels give the same curve in different vertical scales. The solid curve is the numerical result and the dashed line is a fitting straight line. The strength of perturbation $\delta c = 10^{-4}$.

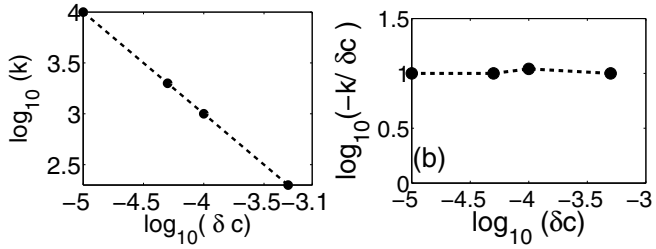

Fig. 5. (a) Variation of the exponential decay rate k in the tunneling phase ($c = 1.9$) with the perturbation strength δc. (b) Variation of $-k/\delta c$ with δc.

times not long, which can be explained by making use of a semiclassical approach. We further found that the stretched exponential decay is followed by an exponential decay for longer times, as shown in Figure 4. One may note that an exponential decay of the LE in a regular system has been observed numerically [41] and explained as a crossover from Gaussian decay to power-law decay in the semiclassical expression equation (14) [42]. A difference in our case here is that the exponential decay follows a stretched exponential decay, not a Gaussian decay. Interestingly, as shown below, the rate k of the exponential decay here has a dependence on $\delta c = \epsilon$ similar to that predicted in equation (14). We would mention that we found qualitatively the same decaying behaviors of the LE for $\delta c \in [10^{-6},\ 2 \times 10^{-3}]$.

In Figure 5, we show the dependence of k on the perturbation strength δc, where k is the rate of the exponential decay $M(t) = e^{-kt}$ in the tunneling phase as shown in Figure 4. It is seen that $k \sim \delta c$. It is of interest to note a property of the prediction given in equation (14), i.e., the approximate exponential decay of the $M(t)$ in the crossover region from Gaussian decay to power law decay has a rate proportional $\epsilon = \delta c$. (We have also checked this property of Eq. (14) numerically.) Thus, the decay rate in Figure 5 has a dependence on δc similar to the corresponding one predicted in equation (14).

4 Conclusion

In conclusion, the long-time decay of the LE of a BEC in a symmetric double-well potential has been studied by numerical simulations. The BEC has two phases, a self-trapping phase and a tunneling phase, with a crossover between the two phases. We have found that in the tunneling and self-trapping phases of the BEC the LE has exponential and Gaussian decays, respectively, for relatively-long times. In the crossover region, the LE has a complex behavior as a result of competition of the two phases. These results indicate that relatively-long time decay of the LE is suitable for characterizing the dynamical phase transition of the BEC.

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