Stability of quantum motion in regular systems: A uniform semiclassical approach

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We study the stability of quantum motion of classically regular systems in the presence of small perturbations. On the basis of a uniform semiclassical theory we derive the fidelity decay which displays a quite complex behavior, from Gaussian to power law decay $t^{-\alpha}$, with $1 \leq \alpha \leq 2$. Semiclassical estimates are given for the time scales separating the different decaying regions, and numerical results are presented which confirm our theoretical predictions.

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Stable manipulation of quantum states is of importance in many research fields such as in quantum information processing and in Bose-Einstein condensation. A measure of the stability of quantum motion is the so-called fidelity or quantum Loschmidt echo [1], which characterizes the stability of quantum dynamics under small perturbations of the Hamiltonian that may derive from static imperfections or from interaction with an external environment. From a more general point of view it is interesting to understand the behavior of fidelity in relation to the dynamical properties of the system.

While fidelity decay in classically chaotic systems has been extensively studied [2–11], the situation in regular systems is much less clear [5,10–19] and only for the particular case of vanishing time-average perturbations has a clear understanding been achieved [12]. Moreover, for a general perturbation and for a single initial Gaussian wave packet, a Gaussian decay has been predicted in some time interval which is not exactly specified [5]. On the other hand, numerical investigations show a much richer behavior of the fidelity decay, ranging from power law to exponential up to a Gaussian decay, depending on initial conditions, perturbation strength, and time interval [15]. In addition, a somehow unexpected regime in which the fidelity decay in regular systems is faster than in classically chaotic systems has been found in [5]. All of the above call for a theory which can account for these diverse analytical and numerical findings.

We would like to draw the reader's attention to the fact that in the general theory of dynamical systems, integrability is the exception rather than the rule and it is therefore extremely rare. However, the theory we present in this paper applies also to integrable regimes of systems with divided phase space in which both chaotic and integrable components are present and this is indeed the typical situation. Moreover, as shown in [20], for a proper operability of a quantum computer it is desirable to remain below the border for transition to quantum chaos, a situation which is likely to correspond to quasi-integrable behavior. Finally, some quantum algorithm can have a phase space representation and, in particular, the Grover algorithm can be interpreted as

a simple quantum map which turns out to be a regular map [21].

The above considerations have motivated our interest in the stability of integrable motion. In this paper, we develop a uniform semiclassical approach to the fidelity decay in regular systems and we provide a unified description together with the corresponding time scales. Numerical computations confirm our analytical estimates.

Quantitatively, the fidelity for an initial state $|\Psi_0\rangle$ is defined as $M(t) = |m(t)|^2$, where

$$m(t) = \langle \Psi_0 | \exp(iHt/\hbar) \exp(-iH_0 t/\hbar) | \Psi_0 \rangle. \quad (1)$$

Here H_0 and $H = H_0 + \epsilon V$ are the unperturbed and perturbed Hamiltonians, with ϵ a small quantity and V a generic perturbing potential.

Consider an initial Gaussian wave packet in a $2d$-dimensional phase space, centered at $(\tilde{\mathbf{r}}_0, \tilde{\mathbf{p}}_0)$,

$$\psi_0(\mathbf{r}_0) = (\pi\xi^2)^{-d/4} \exp[i\tilde{\mathbf{p}}_0 \cdot \mathbf{r}_0/\hbar - (\mathbf{r}_0 - \tilde{\mathbf{r}}_0)^2/(2\xi^2)]. \quad (2)$$

For a sufficiently narrow, initial Gaussian packet, the semiclassical approximation to the fidelity amplitude is

$$m_{\text{sc}}(t) \simeq (\pi w_p^2)^{-d/2} \int d\mathbf{p}_0 \exp\left[\frac{i}{\hbar} \Delta S - \frac{(\mathbf{p}_0 - \tilde{\mathbf{p}}_0)^2}{w_p^2} \right], \quad (3)$$

where $w_p = \hbar/\xi$ and ΔS is the action difference between the two nearby trajectories of the two systems H and H_0 starting at $(\mathbf{p}_0, \tilde{\mathbf{r}}_0)$ [7]. We mention that, for not very narrow initial Gaussian packets, Eq. (3) may still hold with a redefinition of w_p [9]. For more general initial states, the fidelity amplitude can be expressed in terms of the Wigner function of the initial state in a uniform semiclassical approach [10].

The action difference can be calculated in the first-order classical perturbation theory:

$$\Delta S \simeq \epsilon \int_0^t dt' V[\mathbf{r}(t'), \mathbf{p}(t')], \qquad (4)$$

with V evaluated along one of the two trajectories. Equations (3) and (4) give quite accurate predictions even for relatively long times, much more accurate than that usually expected for a first-order perturbation treatment [7,8]. The reason for the unexpected accuracy is explained in [10] by making use of the shadowing theorem, the two trajectories for ΔS being in fact the so-called shadowing trajectories of the two systems with slightly different initial conditions.

Equation (3) shows that the behavior of fidelity is mainly determined by two factors: (i) ΔS as a function of $\mathbf{p}_0$ and t and (ii) the Gaussian term, which specifies an effective window in the $\mathbf{p}_0$ space with size w_p. For simplicity, in what follows, we consider kicked systems with $d=1$ and set the domains of r and p to be $[0, 2\pi)$.

For an integrable system or a regular region of a mixed system, at least locally there exist action-angle variables (θ, I), connected to the variables (r, p) by a canonical transformation. The integrand in Eq. (4) can then be written as

$$V_t \equiv V[I(t), \theta(t)], \quad \text{with } \theta(t) = \theta_0 + \nu t, \qquad (5)$$

$I(t) \simeq I_0$, and $\nu \equiv \partial H(I)/\partial I$.

The main features of ΔS as a function of t and p_0 can be seen by substituting Eq. (5) into Eq. (4), replacing νt by $\phi = \nu t$, and noting the periodicity of the angle θ. This gives

$$\Delta S \simeq \epsilon(U_I t + S_f), \quad \text{where } U_I \equiv \frac{1}{2\pi} \int_0^{2\pi} V(I, \theta) d\theta, \qquad (6)$$

$$S_f \equiv \frac{1}{\nu} \left[\int_0^b V(I, \theta_0 + \phi) d\phi - b U_I \right]. \qquad (7)$$

Here $b \equiv \nu t - 2\pi n_t$, where n_t is the integer part of $\nu t/2\pi$. Therefore, for a fixed p_0, b is a sawtooth-type function of time oscillating between 0 and 2π with a frequency $\nu/2\pi$. As a result S_f oscillates correspondingly; hence, ΔS oscillates around its linearly increasing part $\epsilon U_I t$. On the other hand, for a fixed t, U_I as a function of p_0 changes almost linearly in the neighborhood of $\tilde{p}_0$ while S_f oscillates with a frequency $\tilde{\nu}' t/2\pi$, where $\nu' \equiv \partial \nu/\partial p_0$. Therefore the average slope of ΔS with respect to p_0 is $\epsilon U_I' t$, where $U_I' \equiv \partial U_I/\partial p_0$. [We use a tilde above a quantity to indicate its value taken at the center $(\tilde{r}_0, \tilde{p}_0)$.]

We first study the fidelity decay for times $t < \tau_1$, where τ_1 is the time scale before which the right-hand side of Eq. (3) can be calculated by a linear approximation to ΔS with respect to $p_0 - \tilde{p}_0$. This gives

$$M_{sc1}(t) \simeq \exp\left[-\frac{1}{2}(\sigma w_p \tilde{k}_p)^2 \right], \quad \text{with } k_p \equiv \frac{1}{\epsilon} \frac{\partial \Delta S}{\partial p_0}, \qquad (8)$$

where $\sigma = \epsilon/\hbar$ is the strength of perturbation. The explicit dependence of k_p on time t can be calculated by using Eq. (6). The leading terms give
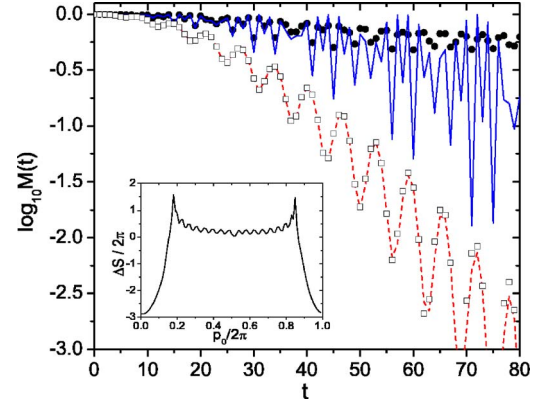

FIG. 1. (Color online) Comparison between the numerically computed fidelity decay (black circles and open squares) and the semiclassical prediction $M_{sc1}(t)$ in Eq. (8), in the kicked rotator model with $k=0.3, \sigma=1.5, N=2^{17}$, and $\xi^2 = \hbar/20$. Centers of the initial Gaussian packets are $(\tilde{r}_0, \tilde{p}_0) = (1.2\pi, 0.6\pi)$ for the circles and $(1.2\pi, 0.2\pi)$ for the squares. The solid and dashed curves represent the semiclassical predictions, respectively, with k_p evaluated numerically in the classical systems. Inset: $\Delta S/2\pi$ vs $p_0/2\pi$, for $r_0 = 1.2\pi$ at $t=20$. The average slope of ΔS in the monotonically increasing part is much larger than that in the central part; therefore, the fidelity decay at $\tilde{p}_0 = 0.2\pi$ is faster than that at $\tilde{p}_0 = 0.6\pi$. The oscillations of ΔS in the central region imply larger second- and higher-order terms in the expansion of ΔS around $\tilde{p}_0 = 0.6\pi$, hence larger deviations of the fidelity from $M_{sc1}(t)$.

$$k_p \simeq (U_I' + U_\theta)t, \quad \text{where } U_\theta \equiv \frac{1}{t}\frac{\partial S_f}{\partial p_0} \simeq (V_t - U_I)\frac{\nu'}{\nu}. \qquad (9)$$

Since U_θ oscillates in time around zero, the fidelity has, on average, an initial Gaussian decay with a rate depending on initial conditions, with larger values of $|U_I'|$ implying faster decay.

To test the above predictions, we consider the kicked rotator model $H = \frac{1}{2}p^2 + k_0 \cos r \sum_{n=0}^{\infty} \delta(t - nT)$. The quantized system has a finite Hilbert space with dimension N. We take $k_0 = kN/2\pi$ and $T = 2\pi/N$, with $k = k_0 T$ independent of N. The classical limit corresponds to $N \to \infty$. The one-period quantum evolution is given by the Floquet operator $U = \exp[-i\hat{p}^2 T/2]\exp[-ik_0 \cos(\hat{r})]$ and is numerically computed by the method of fast Fourier transform. Here $2\pi/N$ serves as an effective Planck constant. The inset of Fig. 1 shows an example of ΔS vs p_0, in which the values of $|U_I'|$ are large at the borders while quite small in the central, oscillating region of p_0. In agreement with our theory, under the perturbation $k \to k + \epsilon$, the fidelity for initial states lying in the two regions of p_0 has a quite different decaying rate, as seen in Fig. 1. The agreement with the theory is particularly good in the case where the oscillations of ΔS are not too strong.

The time scale τ_1 can be estimated by the time at which the second-order term in the Taylor expansion of $\Delta S/\hbar$ is order of 1 at the point $p_0 = \tilde{p}_0 + w_p$—i.e., $\sigma w_p^2 |\tilde{k}_{pp}(t)| \sim 1$ for $t = \tau_1$, where
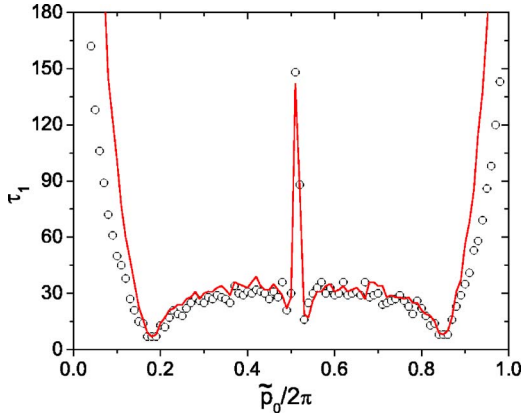
FIG. 2. (Color online) The time scale τ_1 versus $\tilde{p}_0$, for the same parameters as in Fig. 1. The circles give the values of τ_1 calculated by the first time at which $|[M(t)-M_{sc1}(t)]/M(t)|>0.1$ [see Eq. (8)]. The solid curve is the semiclassical estimate given by $\sigma w_p^2|\tilde{k}_{pp}(\tau_1)|=1$, with $\tilde{k}_{pp}(t)$ numerically computed in the corresponding classical system.

$$k_{pp}(t) \equiv \frac{\partial k_p}{\partial p_0} = \frac{(\nu')^2}{\nu}\frac{\partial V}{\partial \theta}t^2 + O(t). \tag{10}$$

A numerical check of this prediction is presented in Fig. 2. When the t^2 term dominates, $\tau_1 \propto \sqrt{1/\sigma w_p^2}$.

Beyond τ_1, a direct analytical computation of the fidelity is difficult since higher- and higher-order terms in the Taylor expansion of ΔS with respect to $p_0 - \tilde{p}_0$ need to be considered. We take therefore the following approach: we divide the domain of p_0 into segments separated by points p_{0j} (with $p_{00}=\tilde{p}_0$), in such a way that S_f completes one oscillation period within each segment. (We will use the subscript j to indicate quantities taken at the point p_{0j}.) It is easy to see that $(p_{0j+1}-p_{0j})\nu'_j t \simeq 2\pi$ and therefore the number of segments increases linearly with time t. The quantity $m_{sc}(t)$ in Eq. (3) can now be written as a sum of contributions $m_j(t)$ of different segments: namely, $m_{sc}(t)=\Sigma_j m_j(t)$.

Let us introduce the time scale τ_s, at which S_f completes one oscillation within the window $W_p=(\tilde{p}_0-w_p,\tilde{p}_0+w_p)$. From Eq. (7), $\tau_s \approx \pi/(\tilde{\nu}' w_p)$. For $t \gg \tau_s$, there are many segments within W_p; hence, within each segment, the variation of the Gaussian term on the right-hand side of Eq. (3) is negligible. Therefore one may write

$$m_j(t) \simeq \frac{e^{-(p_{0j}-\tilde{p}_0)^2/w_p^2}}{\sqrt{\pi}w_p \nu'_j t}e^{i\Delta S_j/\hbar}F(t), \tag{11}$$

$$F(t) = \int_0^{2\pi} d\phi \exp\left\{\frac{i\sigma}{\tilde{\nu}}\int_{\tilde{\theta}(t)}^{\tilde{\theta}(t)+\phi} V(\tilde{I},\theta')d\theta'\right\}. \tag{12}$$

Indeed for p_{0j} not far from $\tilde{p}_0$, $F(t)$ is independent of j, and since $|F(t)|$ does not decay with time, we will not consider it further.

First we notice that Eq. (11) predicts a plateau in the fidelity decay when the change of $\Delta S_j/\hbar$ within the window W_p is negligible. The plateau disappears when $\sigma w_p \tilde{U}'_I t \sim \pi$.
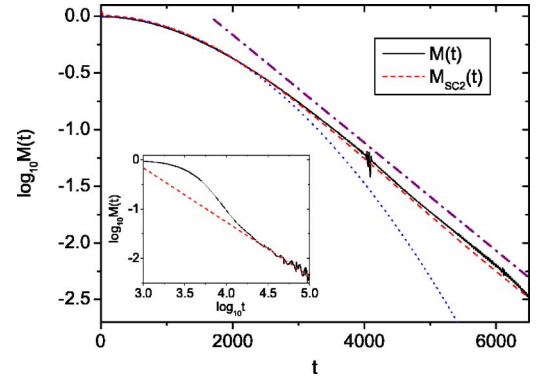

FIG. 3. (Color online) Solid curve: fidelity decay for $\sigma=0.2$, $N=2^{12}$, $k=0.3$, $\xi^2=\hbar/20$, and $(\tilde{r}_0,\tilde{p}_0)=(1.2\pi,0.8\pi)$. Dashed curve: the prediction of Eq. (14) with $c=1$ and $\tilde{U}'_I \approx -0.019$, $\tilde{U}''_I \approx 0.042$ numerically computed from the *classical* system. Dotted curve: the Gaussian decay $e^{-(w_p\sigma\tilde{U}'_I t)^2/2}$. [For small values of σ, the difference between this decay and M_{sc1} in Eq. (8) is small before τ_1.] The dash-dotted line shows the approximate, intermediate, exponential decay. Inset: long time decay for the same case with $\tilde{r}_0=0.6\pi$ (every 500 steps shown). The dashed line gives the $1/t^{1.1}$ decay.

If, for example, $U_I=0$ for the system H_0, then $U_I \propto \epsilon$ for the system H, and the plateau will end at a time proportional to $\hbar^{1/2}\epsilon^{-2}$ for $\xi \propto \hbar^{1/2}$ in agreement with the result of [12].

Due to the Gaussian term on the right-hand side of Eq. (11), the main contribution to $m_{sc}(t)$ comes from p_{0j} not far from $\tilde{p}_0$. For these p_{0j}, ν'_j is almost a constant, as well as the value of S_f; hence, apart from a common phase, ΔS_j in Eq. (11) can be approximated by $\epsilon U_{Ij}t$. For a sufficiently large number of segments, the sum $\Sigma_j m_j(t)$ can be replaced by an integral over p_0. Then, setting $q=p_0-\tilde{p}_0$, we have

$$m_{sc}(t) \simeq \frac{F(t)}{2\pi\sqrt{\pi}w_p}\int_{-\infty}^{\infty}dq e^{-q^2/w_p^2}e^{i\sigma t U_I}. \tag{13}$$

Expanding U_I to the second-order terms in q, we have

$$M_{sc2}(t) \simeq \frac{2c}{\sqrt{4+(w_p^2\sigma\tilde{U}''_I t)^2}}\exp\left[\frac{-2(w_p\sigma\tilde{U}'_I t)^2}{4+(w_p^2\sigma\tilde{U}''_I t)^2}\right], \tag{14}$$

where $U''_I \equiv \partial U'_I/\partial p_0$ and c is a constant with $c \approx 1$ for sufficiently small σ.

From Eq. (14) it is seen that for $(w_p^2\sigma\tilde{U}''_I t)^2 \ll 4$, the fidelity has a Gaussian decay $e^{-(w_p\sigma\tilde{U}'_I t)^2/2}$, which agrees with the one given in [5] for weak perturbations. On the other hand, if $(w_p^2\sigma\tilde{U}''_I t)^2 \gg 4$—i.e., for larger time t—$M_{sc2}(t)$ in Eq. (14) has a power law decay $1/t$. In the transition region between the two decays, the fidelity may have an approximate exponential decay (see Fig. 3) which can explain the exponential-like decay found numerically in [15].

With further increasing time t, higher-order terms in the Taylor expansion of U_I will become important and will modify the $1/t$ decay. In order to evaluate the effect of higher-order terms, we divide the interval W_p into subintervals labeled by l, in such a way that inside each of them the linear approximation of U_I can be used. Thus their width

must be of the order $\delta q \simeq 1/\sqrt{|U_I''|\sigma t}$ and their number in the region W_p is $l_m \propto \sqrt{\sigma t}$. For sufficiently long time t, $l_m \gg 1$ and the width of each subinterval is much smaller than W_p. As a result, the Gaussian term in Eq. (13) can be regarded as constant within each subinterval. Then, the integral (13) reduces to $m_{sc}(t) \simeq \Sigma_l m_l(t) = \Sigma_l c_l e^{i\phi_l(t)}/\sigma t$, with some coefficients c_l and phases $\phi_l(t)$. The detailed behavior of this sum depends on the properties of the function U_I. However, one can give estimates in some limiting cases: (i) Since $|\Sigma_l c_l e^{i\phi_l(t)}| \leq \Sigma_l |c_l| \propto (\sigma t)^{1/2}$, the slowest decay is $1/\sqrt{\sigma t}$. (ii) The fastest decay is obtained when $|\Sigma_l c_l e^{i\phi_l(t)}|$ is a constant (which may happen due to mutual cancellation of phases ϕ_l). In this case, $m_{sc} \sim 1/\sigma t$. (iii) In the case of random phases ϕ_l, one has $|\Sigma_l c_l e^{i\phi_l(t)}| \propto (\sigma t)^{1/4}$, then $m_{sc} \sim 1/(\sigma t)^{3/4}$, which coincides with the result given in Ref. [13] for averaged fidelity. (For an analysis of Gaussian and power law decay of fidelity, which is based on statistics of action difference, see Ref. [10(c)]).

Therefore, in general, the decay of $M(t)$ has the power law dependence $(\epsilon t/\hbar)^{-\alpha}$ with $1 \leq \alpha \leq 2$. This is in agreement with numerical results in [15], as well as with our extensive numerical simulations (see, e.g., Fig. 4 and the inset of Fig. 3). The cases of $\alpha \simeq 1$ or 2 have been found to be quite rare in our simulations.

It is important to remark that, contrary to chaotic systems for which the decay rate depends on the strength but not on the shape of the perturbation, integrable systems lack of such generic behavior. This is typical in the general theory of dynamical systems, and it is due to the peculiarity of integrability. In the present case the numerical value of α depends on the particular shape of U_I.

Our approach can also explain an interesting feature of fidelity decay in regular systems observed numerically in [15], that is the fact that, in some cases, the decay rate may decrease with increasing the perturbation strength σ. First we note that Eq. (14) is valid only when $|\Delta S_{j+1} - \Delta S_j|/\hbar$ is small compared with π, since it is obtained by replacing the sum $\Sigma_j m_j(t)$ by an integral over p_0. Since U_I' is given by $(\Delta S_{j+1} - \Delta S_j)/(p_{0j+1} - p_{0j})\epsilon t$, then $|\Delta S_j - \Delta S_j|/\hbar \simeq 2\pi\sigma|U_{Ij}'/\nu_j'|$, which may exceed π for sufficiently large σ. In this case, we can replace $\Delta S_j/\hbar$ by $\psi_j \equiv \Delta S_j/\hbar - 2\pi m j$, where m is an integer such that $|\psi_{j+1} - \psi_j| \leq \pi$. Then using the fact that, for p_{0j} close to $\tilde{p}_0$, $(p_{0j} - \tilde{p}_0)\tilde{\nu}' t \simeq 2\pi j$, we find that the term $\sigma\tilde{U}_I'$ in Eq. (14) can be replaced by $\beta \equiv |\sigma\tilde{U}_I' - m\tilde{\nu}'|$. With increasing σ, the value of β, which gives the decay rate, oscillates between 0 and $|\tilde{\nu}'|/2$, with a period $|\tilde{\nu}'/\tilde{U}_I'|$. The results of Fig. 4 nicely confirm the above analysis.
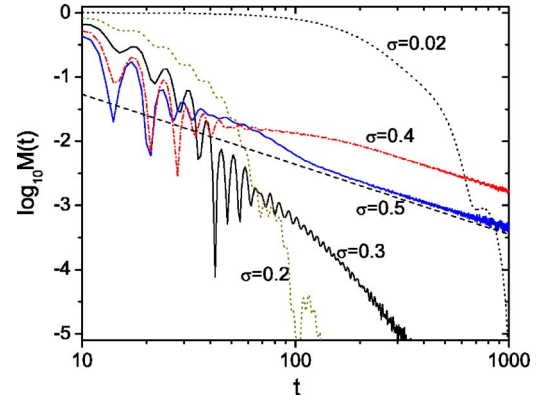
Finally a word of comment on the comparison of fidelity

FIG. 4. (Color online) Fidelity decay for several values of σ with $k=0.3$, $N=2^{12}$, $\xi^2 = \hbar/20$, and $(\tilde{r}_0, \tilde{p}_0) = (1.2\pi, 0.28\pi)$. For t around 100, the rate of fidelity decay increases for $0 < \sigma < 0.2$, then decreases for $0.2 < \sigma < 0.4$, and then increases again. This is in agreement with the semiclassical prediction that the decay rate β oscillates with a period $|\tilde{\nu}'/\tilde{U}_I'| \approx 0.4$ (see text). The values $\tilde{U}_I' \approx 1.85$ and $\tilde{\nu}' \approx 0.76$ are obtained from the classical motion. Notice that the transition time from Gaussian to power law decay of fidelity is proportional to $1/\sigma$ (see text). Here the power law decay is visible for $\sigma=0.4$ and 0.5 (the dashed straight line gives the $1/t^{1.09}$ decay).

decay in classically regular and chaotic systems. As shown in [5], in a sufficiently weak perturbation regime $\sigma < \sigma_c \propto \sqrt{\hbar}$, the Gaussian decay $e^{-(w_p\sigma\tilde{U}_I' t)^2/2}$ in the regular case can be faster than the Fermi-golden-rule decay in the chaotic case. This is not surprising as it may look at first sight, since in this regime, $w_p\sigma_c \propto \hbar$ can be quite small and the fidelity may remain close to 1 for a time comparable to the Heisenberg time $\sim 1/\hbar$, which can be quite long. Moreover, the above Gaussian decay in the regular case is followed by a power law decay, which is slower than the exponential decay in the chaotic case.

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