Tainted Evidence: Cosmological Model Selection vs. Fitting

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Interpretation of cosmological data to determine the number and values of parameters describing the universe must not rely solely on statistics but involve physical insight. When statistical techniques such as “model selection” or “integrated survey optimization” blindly apply Occam’s Razor, this can lead to painful results. We emphasize that the sensitivity to prior probabilities and to the number of models compared can lead to “prior selection” rather than robust model selection. A concrete example demonstrates that Information Criteria can in fact misinform over a large region of parameter space.

I. INTRODUCTION

Recently several papers (e.g. [1, 2, 3], and references therein) have claimed that fitting cosmological parameters to data may be inadequate, if not misguided, as a means of determining cosmology, in particular the nature of dark energy accelerating the expansion of the universe. They advocate interpreting the evidence for or against a model in a manner intimately tied to the parameter space volume: models with fewer parameters get boosted in preference while models with more parameters – and possibly better fits – should be penalized. This is all in accord with statistical theory. In this approach one should not fit parameters but rather select models; we will refer to the various related techniques along these lines generically as “model selection”.

However, this point of view can be taken too far, and has as well some essential assumptions that prevent model selection from being either a panacea or a substitute for parameter fitting (which admittedly can have pitfalls of its own). In general, statisticians want to model data in an efficient way. That is, they want to go from many data points to just a few parameters in a model that captures the essence of the data accurately and efficiently. Then, one can use that model to predict future outcomes, say (clinical trials, stock market, climate change, etc.). However, physicists do not regard the models as just useful summaries of the data, but as fundamental descriptions of the data based on physical principles. The parameters in our models have (or should have) deep physical meanings, and are not just concise ways of representing the data. Model efficiency takes a back seat to physical fidelity.

We discuss the drawbacks of the model selection paradigm in §II and give several historical examples in §III where model selection would have led physicists astray, especially as a predictive mechanism. In §IV we present a complete numerical example, using simulated cosmological data, that shows how easily model selection can lead us to the wrong physics. Inadequate data, or extrapolation to future, better data, is particularly problematic within a model selection scenario since that rewards mathematical simplicity, which blurry data are more likely to be consistent with. We emphasize in §V that one can too easily misuse this paradigm in attempting survey design, judging data before it is taken. Physics, not just statistics, must retain a central role in interpreting data.

II. BAYESIAN APPROACH

An essential ingredient of the parameter fitting paradigm is that the model described by the parameter set is a reasonable choice. Model selectors ask what if the model is faulty to begin with, and seek to avoid, or at least dilute, this step by considering sets of models, containing parameters, rather than sets of parameters. As far as this goes, it is certainly reasonable – parameter fitters must take care in choosing parameter sets, with respect to validity and lack of bias. However, in parameter fitting to data one already has built in tests to check that the model is reasonable: 1) ideally one starts with a physically motivated model, 2) if the goodness of fit (e.g. chi-squared per degree of freedom) is poor then the model receives careful scrutinization or is discarded, and 3) if subsets of data disagree then this may be a sign that the parameter space should be expanded or abandoned. So parameter fitting can check itself.

Model selection seeks to consider a space of models and find the Bayesian evidence for each. Here the key ingredient is the assigning of prior probabilities to parameter states within each model. That is, in a not-insignificant sense one must guess what the data are going to say before you have them. The final answer from model selection depends on the prior assumptions, assumptions often with relatively little physics guidance. Whether the

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prior is uniform in a parameter itself or in a function of the parameter has substantial impact on the results (for one telling example concerning our very existence, see §4). The range of the prior is another crucial element. Finally, while in parameter fitting one must ensure that the particular parameter values of the fit are physical, in model selection one must extend the probability analysis throughout the entire range of parameter values, some portions of which may be pathological without this being apparent (indeed the “corners” – the extreme ranges of the prior – constitute 99.8% of the volume in a 10 dimensional space []). Only after these assumptions can the model probability be computed.

The final quantity for selecting models is their evidence value, the mean likelihood averaged over the prior. Since probability must sum to unity, large parameter spaces, or spaces where the prior extends widely, are strongly penalized. Doesn’t this make sense, though? Shouldn’t we apply Occam’s Razor to shave the models with more “hair”? We agree that the consistency of data with a model with few parameters can be a useful guide, but there are several points that caution against overly strict application of model selection: shaving blindly with Occam’s Razor is a dangerous activity.

For one thing, there is only one universe accessible to us, so prior probability has a very limited meaning. If only one point in the model parameter space fits the data, nevertheless our universe may be described by that model. Indeed, even the number of models being compared influences the probabilities, as the Bayesian evidence is calculated under the constraint that the total probability must sum to unity.

For another, any model that somewhere fits the data must be allowed to live or else we sacrifice the good with the bad. If there is one “fit” fruit on a tree with bad apples, using model selection criteria everything would be thrown away without exception.

The role of the prior for model selection is crucial, yet the method of selection of the prior is completely undefined. Consider the equation of state phase space \( w-w' \), where \( w \) is the equation of state ratio of the dark energy pressure to energy density, and \( w' = dw/d\ln a \). Should we choose a prior distribution uniform in \( w \) and \( w' \), or perhaps uniform in \( w \) and \( w = Hw' \) – the Bayesian evidence will be different in the two cases and models may be ruled out in one case but not the other. Or should we choose a uniform prior in the field values \( \phi \) and \( \phi' \)? Perhaps \( \phi^2 \) and \( V \), where \( V \) is the potential energy and \( \dot{\phi}^2/2 \) the kinetic energy? Perhaps \( V \) and \( V' \), where \( V' = dV/d\phi' \). What model selection tells us is an acceptable cosmology differs in each case. The foundation, the priors, are essentially undefined by physics but put in by the model selectors by hand. Indeed it is tempting to call this approach “prior selection” rather than model selection. Many of these issues are indeed recognized in the statistics community (see, e.g., [],) if not with full emphasis on physical fidelity.

Mathematical simplicity – the essential weighting for model selection – does not necessarily accord with physical simplicity. We do not always recognize simplicity when we see it. Consider the model spanned by the coefficients of a polynomial function \( f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \). Model selection will deal harshly with it, despite that the apparent complexity – in some cases an infinite number of parameters – sometimes represents simplicity, such as \( f(x) = e^x \), with zero parameters. Conversely, models which appear simple can hide deeper complexity, e.g., where a Gaussian is actually approximating the convolution of multiple functions.

Braneworld models in a spatially flat universe are extremely simple, involving one parameter, the crossover distance. However if we do model selection in terms of the distance-redshift relation \( d(z) \), or the equation of state ratio \( w(z) \), it appears the braneworld models have many more parameters than the cosmological constant model. Model selection will penalize strongly the braneworld model for the apparent number of parameters, despite the underlying physics being quite simple. As another example, a constant equation of state model would appear to have many parameters if discussed in terms of redshift-binned densities \( \rho(z) \) or principal components of the density (also see []). Thus the concept of simplicity is actually quite complicated. We discuss this further in [III] giving historical examples.

Parameter fitting allows us to compare cosmologies equally in the space describing cosmological distances, or densities, or equation of state, or somewhat “non-parametric” forms like principal components. Model selection imposes an unphysical overlay of expectation on such comparisons, based on a perceived simplicity.

Parameter fitting does not require prior knowledge of the probability weighting within the parameter space; it allows the physics to, literally, impose “survival of the fittest”. The most useful parameterizations are moreover motivated physically. Consider the equation of state phase space \( w-w' \). While model selection would weight this as a whole, as two parameters, physics calls out certain subregions of this space as being especially physically relevant []. In particular, thawing dark energy cosmologies, where the dark energy was locked by Hubble expansion drag in a cosmological constant like state in the early universe and more recently released to evolve, follow one-dimensional trajectories in a narrow region, or inverse power law models actually possess only a single equation of state parameter, not two. Model selection deweights these solutions because – although they may be perfectly good fits – they reside in a larger parameter space. Conversely, model selection favors constant equation of state models, with a single parameter, despite their being physically unstable, requiring a precise fine tuning of the ratio between kinetic and potential energies.
III. REALITY CHECK

Prior to 1998, supernova distance data were consistent with a flat, matter dominated universe. Model selection would have claimed strong evidence for the zero parameter SCDM ($\Omega_m = 1$) model, disfavored the general matter dominated universe OCDM ($\Omega_m \in [0,2]$, say), and dismissed a cosmological constant universe $\Lambda$CDM ($\Omega_n + \Lambda$). Simplicity does not automatically bring truth.

Feynman famously pointed out the pitfall in calling on simplicity as a guidepost in physics by rewriting all of the physics as $\vec{U} = 0$, where each element of $\vec{U}$ contained the hidden structure, e.g. Newton’s second law as $U_1 = F - ma$.

Before 1992 the Bayesian evidence was overwhelming against structure in the cosmic microwave background, i.e. comparing the model parametrized by the temperature power spectrum multipoles $C_\ell = 0$ vs. the much larger parameter space spanned by $C_2, C_3$, etc. But no one would have taken such an argument seriously. (Of course using individual $C_\ell$ is a statistical approach not a physical one, but that is part of the point about needing physical input.)

The galaxy two-point spatial correlation function for many years was thought to be a power law, with only two parameters, yet now we know there is little physics in that model; rather the apparent power law is nearly a coincidence from evolution in the true matter clustering and in the bias between mass and light.

The mass power spectrum looks like a smooth curve yet we appreciate the deeper physics only by going to an apparently more complicated halo model involving the addition of multiple terms.

In a particle physics context, the Fermi theory of weak interactions managed to describe all low-energy phenomenology with one single parameter, the Fermi constant $G_F$. However, the underlying renormalizable physical theory, the Glashow-Weinberg-Salam Standard Model of electroweak interactions, has many more free parameters, and has also a realm of application much wider than that of the Fermi theory, which came to be regarded as just a low-energy effective theory.

These historical examples are clearest for information criteria approximations to the full Bayesian evidence, but such simplicity or “predictiveness” arguments apply to the evidence as well. While the full evidence factorizes out unconstrained parameters, these parameters may not be recognized, e.g. when an equation of state parameter is viewed in density or distance (see [3]), and in any case the mere presence of more models affects the evidence.

Purely statistical ideas of simplicity are insufficient in physics. To a large extent, beauty follows truth, not truth beauty. What we consider simple is what we understand. A mess of spectral lines from a hydrogen atom is ugly until we understand the physics behind it, then we realize it is beautiful and simple. The simple cosmological constant hides, within present physics knowledge, all sorts of ugly fine tunings, while a complicated equation of state $w(z)$ may turn into something as beautiful as the hydrogen atom. Model selection forces us to give undue weight to our guess as to which is an ugly duckling and which will be a beautiful swan.

IV. A WORKED-OUT EXAMPLE

As a specific example of the overemphasis that model selection places on models with fewer parameters, despite the physics, we analyze constraints on dark energy from future data. For such data we take a distance-redshift survey over redshifts $z = 0.1 - 1.7$, similar to the supernova half of the proposed SNAP satellite mission [3], supplemented by 300 local Hubble flow ($z = 0.03 - 0.08$) supernovae as will be provided from the Nearby Supernova Factory (SNF) survey [10], and by measurement of the distance to the surface of CMB last scattering provided by the ESA-NASA mission Planck [11].

The supernovae (SNe) numbers and redshift distribution are as given in [12]. We include a linearly rising systematic error floor in each $\Delta z = 0.1$ redshift bin of $\sigma_{\text{Syst}} = 0.02z/1.7$, to be added in quadrature in each redshift bin to the measurement error given by $\sigma_{\text{Int}}/\sqrt{N_{\text{SN}}}$, with $\sigma_{\text{Int}} = 0.12$ and $N_{\text{SN}}$ being the number of Type Ia SNe in that bin. We take a relative precision of 0.7% in the Planck distance measurement.

A standard cosmological fit analysis of actual SNAP + SNF + Planck data will produce a central value in the $(w_0, w_a)$ plane (recall that $w_a = -2w'(z = 1)$) and a contour around it encompassing some chosen confidence level (CL), typically 68, 90, or 95%. The point corresponding to a cosmological constant (-1,0) may or may not lie inside the contour. If it does not, we may say that we exclude $\Lambda$CDM at that CL.

More precisely, in a frequentist analysis in which one aims to prove or disprove $\Lambda$CDM, one would simulate the expected SNAP + SNF + Planck data sample assuming a $\Lambda$CDM universe, and, analyzing this synthetic data sample as if it were the real data, one would draw a, say, 90% CL contour around the (-1,0) point, much like the one depicted as the inner contour in Fig. 1 (Note no assumption of Gaussianity in likelihoods has been made.) That contour tells us that, if $\Lambda$CDM is true, we expect to get a central value inside that contour in 90% of the observations like SNAP + SNF + Planck that we may perform. Therefore, if our one real SNAP + SNF + Planck observation delivers a central value outside that contour, irrespectively of its associated error ellipse, we will be able to say that we have excluded $\Lambda$CDM at greater than 90% CL. For all true models defined by $(w_0, w_a)$ values outside that contour, we expect (in the statistical sense) the measured values to lie, indeed, outside, and so we expect to be able to select them over $\Lambda$CDM at greater than 90% CL.

With the information criteria approach used by model selection advocates, one penalizes models with more parameters. Various information criteria exist (see [3] for
an overview); here we concentrate on the Bayesian Information Criterion (BIC), which is closely related to the Bayesian evidence but simpler to compute. The outer contour in Fig. 1 shows the iso-contour in likelihood $L$ for $-2\Delta \ln L = 9.48$, which, with the penalty from $(w_0, w_a)$ models having two additional parameters beyond $\Lambda$CDM, corresponds to our problem to $\Delta \text{BIC} = -6$.

Values of $\Delta \text{BIC} \leq -6$ indicate strong evidence in favor of the model being probed. That is, for all models defined by $(w_0, w_a)$ inside the outer contour of Fig. 1, BIC would give strong evidence against them and strong evidence supporting, instead, $\Lambda$CDM. In contrast, $-2\Delta \ln L = 9.48$ corresponds to 99.13% CL against $\Lambda$CDM in the fitting analysis. For this combined data sample, one can say that a Bayesian Information Criterion analysis mis-informs us about, i.e. spuriously rules out, all models in-between the contours. In order for BIC to give the same hint against $\Lambda$CDM that the standard fitting analysis gives at 90% CL, one has to go in this case to a contour with $-2\Delta \ln L = 17.48$, or $\Delta \text{BIC} = 2$, which would extend to the margins of the figure. Such a $-2\Delta \ln L = 17.48$ corresponds to 99.98% CL in the fitting analysis. That is, only when the fitting analysis gives $\Lambda$ less than a $2 \times 10^{-4}$ chance of being correct, does model selection start to give a weak hint of preference for the $(w_0, w_a)$ model, turning away from the incorrect model $\Lambda$CDM.

For all models whose $(w_0, w_a)$ lies between the two contours in Fig. 1, the standard fitting analysis would guide us away from $\Lambda$CDM, while the BIC analysis would strongly, and wrongly, discard the true model. Such are the dangers of model selection based on simplicity arguments. Indeed the area of the BIC “misinformation” region is larger than the entire region where $\Lambda$ is viable according to standard fitting. Fig. 1 also shows the physically motivated freezing and thawing regions [8] where models should generically lie. We see that BIC is particularly slanted against the half of the physical space that is thawing models.

V. PREDICTION AND SURVEY DESIGN

As shown from the historical examples in §III model selection is unsuitable for predicting constraints from future data and surveys. Physics input, not statistical probability, is required for assessing the value of future surveys.

Claims for survey optimization that average likelihoods over prior probabilities are basically betting on absence of any physical structure. A similar assumption was addressed in [14] where it was shown that a figure of merit based solely on the area of the confidence region was robust only on abstraction of physics input. Such a “blank map” approach was there called Snarkian statistics, after Lewis Carroll’s poem, “The Hunting of the Snark”. Using model selection to design surveys or plan for the future based on present consistency with the cosmological constant is similarly like saying the snark hunters are well served by a “map representing the sea without the least vestige of land” because the sea so far has been featureless: it may be simple and favored in model selection, but is not to be trusted for navigating if reefs and islands – structure – may exist.

Standard parameter fitting methods should and can test if the model framework is reasonable or not, as mentioned in §III. In particular, the cosmological $w_0$-$w_a$ parameterization was specifically invented from the physical motivation of scalar field behavior [15] and has been carefully shown to be largely safe from bias [14]. Goodness of fit alerts us to problems in the parameterization, informs us if we should enlarge the parameter space (e.g. by comparison of the results of data subsets or multiple probes – a very important aspect of learning about dark energy physics), and allows comparison of disjoint

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1 In frequentist statistics, the contour around $(-1, 0)$ is the proper tool to decide whether a particular $(w_0, w_a)$ model is favored or not with respect to $\Lambda$CDM. For Bayesian statistics, one creates a “halo” of models centered at different $(w_0, w_a)$ whose contours touch $(-1, 0)$ (see, e.g., §III). While the outer contour shown did not come from the halo method, the results are quantitatively quite similar – in fact our contour is conservative: the Bayes contour would be slightly larger.
models that do not share a parameter space. We do not need to know a priori the probabilities at every point in parameter space; the data lead us.

VI. CONCLUSIONS

When coupled with strong physical inputs, tests of robustness against priors, and precise data, model selection techniques can be a useful companion to the standard parameter fitting approach. They are not, however, a panacea or a way to draw conclusions stronger than actual data can support. We cannot get answers about the physics before we get the data. The sensitivity to prior guesses about the probability of the correct physics runs the risk of turning this approach into “prior selection” rather than model selection.

Overenthusiastic application of model selection has led to some claims about the probability of future experiments failing to see characteristics such as dynamics that current data cannot access. Such statistical pronouncements from the shaky foundation of priors are reminiscent of the Dao De Qing, which lessons us that “Without going outside, you may know the whole world, without looking through the window, you may see the ways of heaven. The farther you go, the less you know.” Though a larger parameter space – representing “farther” physics – has many points that fail to fit, only one point needs to be correct, and that large parameter space may arise from a simple, beautiful theory once we understand it. As the eminent statistician C.R. Rao puts it, “The [information] criteria do not take into account the purpose for which the model is estimated and the loss incurred in using the estimated distribution to predict future events” [16].

Parameters should have physical meanings, and statisticians’ model efficiency then takes a back seat to physical fidelity. Through a concrete example we showed how a statistical Information Criterion can decisively rule out a physical model – incorrectly. Furthermore, the misinformation area of model selection exhibited in Fig. II grows even larger with more parameters. Recalling our Standard Model analogy of §III, we should have no expectation that the physics of 75% of the universe’s energy density can be explained by a single number.

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