Separating E and B types of polarization on an incomplete sky

Wen Zhao1,2,3,* and Deepak Baskaran1,2,†

1School of Physics and Astronomy, Cardiff University, Cardiff, CF24 3AA, United Kingdom
2Wales Institute of Mathematical and Computational Sciences, Swansea, SA2 8PP, United Kingdom
3Department of Physics, Zhejiang University of Technology, Hangzhou, 310014, People’s Republic of China

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Detection of magnetic-type (B-type) polarization in CMB radiation plays a crucial role in probing the relic gravitational wave background. In this paper, we propose a new method to deconstruct a polarization map on an incomplete sky in real space into purely electric and magnetic polarization-type maps, $\mathcal{E}(\hat{\gamma})$ and $\mathcal{B}(\hat{\gamma})$, respectively. The main properties of our approach are as follows: First, the fields $\mathcal{E}(\hat{\gamma})$ and $\mathcal{B}(\hat{\gamma})$ are constructed in real space with minimal loss of information. This loss of information arises due to the removal of a narrow edge of the constructed map in order to remove various numerical errors, including those arising from finite pixel size. Second, this method is fast and can be efficiently applied to high resolution maps due to the use of the fast spherical harmonics transformation. Third, the constructed fields, $\mathcal{E}(\hat{\gamma})$ and $\mathcal{B}(\hat{\gamma})$, are scalar fields. For this reason various techniques developed to deal with temperature anisotropy maps can be directly applied to analyze these fields. As a concrete example, we construct and analyze an unbiased estimator for the power spectrum of the $B$ mode of polarization $C_{BB}^l$.

Basing our results on the performance of this estimator, we discuss the relic gravitational wave detection ability of two future ground-based CMB experiments, the Q/U Imaging Experiment and Polarization of Background Microwave Radiation.

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I. INTRODUCTION

The extreme conditions in the very early Universe produce primordial perturbations of two generic types, namely, density perturbations (scalar perturbations) and relic gravitational waves (RGWs, tensor perturbations) [1,2]. In the simplest scenarios, these perturbations are characterized by nearly scale invariant primordial power spectra. The experimental determination of the parameters specifying these power spectra provides an important method to investigate the physics of the very early Universe. The CMB has proved to be a valuable tool in this respect. Scalar and tensor perturbations leave an observable imprint in the temperature and polarization anisotropies of the CMB. Recent experimental effort, including the Wilkinson Microwave Anisotropy Probe (WMAP) satellite [3], QaD [4], BICEP [5], and so on [6–10], has led to a robust determination of the parameters characterizing the primordial density perturbations. On the other hand, the detection of RGWs remains an outstanding experimental challenge, and a key task for current, upcoming, and future planned CMB observations on the ground [4,5,11–17], on balloons [18–20], and in space [21–25].

Both density perturbations and RGWs contribute to the various CMB anisotropy power spectra, namely, $TT$ (temperature), $EE$ (electric-type polarization), and $TE$ (temperature-polarization cross correlation) [26–35]. In addition, RGWs yield the “magnetic”-type ($BB$) polarization that is not produced by density perturbations [36]. In principle, all of these information channels ($TT$, $TE$, $EE$, and $BB$) should be used to infer the RGW signal in the CMB. However, if the contribution of RGWs to the CMB is small ($r \approx 0.05$), the $BB$ channel will be the best venue for detecting RGWs [35].

The separation of the polarization field into electric and magnetic components is a subtle issue. In practice, in a CMB experiment, one directly observes only the Stokes polarization components $Q$ and $U$. Given a full-sky map of the components $Q$ and $U$, one can construct the so-called $E$ mode and $B$ mode of polarization (sometimes referred to as $G$ and $C$ modes, respectively) using spin-spherical harmonics expansion in an unambiguous manner. It is important to keep in mind that, by virtue of construction, $E$ and $B$ modes of polarization are nonlocal quantities, and require the information on Stokes parameters on the complete sky. However, in realistic cases (ground-based, balloon-borne, and satellite experiments), the Stokes parameters are measured only on a fractional portion of the sky. In this situation, the simplest method for constructing electric and magnetic polarization fields, using the spin-spherical harmonics leads to mutual contamination, often referred to as $EB$ mixing. $EB$ mixing can become a dominant hindrance for detecting the RGW signal [37]. In order to overcome this difficulty, numerous methods have been developed to separate $E$ and $B$ types of polarization on an incomplete sky [38–44]. However, these methods suffer from one or more of the following drawbacks—they are slow in practice, they are difficult to realize in pixel space, and/or they lead to partial information loss.

*Wen.Zhao@astro.cf.ac.uk
†Deepak.Baskaran@astro.cf.ac.uk
In the current work, we develop a novel method to separate the electric and magnetic components of polarization on a partial sky. In contrast to previous works [38–40], in this paper we focus mainly on the construction of pure $E$ and $B$ types of polarization in real space, as opposed to constructions in harmonics space. Our method is based on a simple redefinition of electric and magnetic components of polarization, so as to make them only dependent on the differential of the $Q$ and $U$ fields. On a full sky, this definition is equivalent to the standard definition. However, due to the differential nature, our definition is directly extendable to polarization maps given on an incomplete sky. The main advantages of our method are as follows. First, the loss of information is small, and is only caused by the removal of a narrow edge around the observed portion of the sky to reduce numerical errors. Second, this method is easy to realize for pixelized polarization maps, and is sufficiently fast so as to be practical for high resolution full-sky surveys. Third, the method leads to construction of scalar $E$- and $B$-type fields. For this reason, one can directly apply the various techniques developed for temperature anisotropy. In particular, along this route, we discuss the various sources of contamination. We go on to discuss the effect of edge removal, pixel size, and experimental beam size on the resulting contaminations. In Sec. IV, we focus on applying our method to small-sky surveys. Combining our method with the pseudo-$C_\ell$ estimator method, we construct an unbiased estimator for the $B$-mode power spectrum. We show that, in practice, our estimator performs only slightly worse than an estimator in an idealized situation with no loss of information. Based on this estimator, we analyze the ability to detect relic gravitational waves through their signature in the $B$ mode of polarization in two planned ground-based CMB experiments, the Q/U Imaging Experiment (QUIET) and Polarization of Background Microwave Radiation (POLARBEAR). We conclude in Sec. V with a brief summary of our main results.

II. DECOMPOSITION OF THE POLARIZATION FIELD INTO ELECTRIC AND MAGNETIC COMPONENTS ON AN INCOMPLETE SKY

Let us first give a brief recount of the standard procedure to construct the electric $E$ mode and the magnetic $B$ mode of polarization, given a complete sky. For mathematical simplicity, it is convenient to introduce the complex conjugate polarization fields $P_\pm$ as follows:

$$P_\pm(\hat{\gamma}) = Q(\hat{\gamma}) \pm iU(\hat{\gamma}),$$  \hspace{1cm} (1)

where $\hat{\gamma}$ denotes the position on the sky, and $Q$ and $U$ are assumed to be real fields on the sky. The fields $P_\pm$, being $\pm 2$ spin-weighted quantities, can be expanded over appropriate spin-harmonic functions (see [45] for instance):

$$P_\pm(\hat{\gamma}) = \sum_{\ell m} a_{\pm,\ell m} Y_{\ell m}(\hat{\gamma}),$$  \hspace{1cm} (2)

where $Y_{\ell m}(\hat{\gamma})$ are the spin-weighted spherical harmonics. The multipole coefficients $a_{\pm,\ell m}$ can be calculated as

$$a_{\pm,\ell m} = \int d\hat{\gamma} P_\pm(\hat{\gamma}) Y_{\ell m}^*(\hat{\gamma}).$$  \hspace{1cm} (3)

The $E$- and $B$-mode multipoles are defined in terms of the coefficients $a_{\pm,\ell m}$ in the following manner:

$$E_{\ell m} = -\frac{1}{2i} [a_{+,\ell m} + a_{-,\ell m}],$$

$$B_{\ell m} = -\frac{1}{2i} [a_{+,\ell m} - a_{-,\ell m}].$$  \hspace{1cm} (4)

One can now define the electric polarization sky map $E(\hat{\gamma})$ and the magnetic polarization sky map $B(\hat{\gamma})$ as

$$E(\hat{\gamma}) = \sum_{\ell m} E_{\ell m} Y_{\ell m}(\hat{\gamma}),$$

$$B(\hat{\gamma}) = \sum_{\ell m} B_{\ell m} Y_{\ell m}(\hat{\gamma}).$$  \hspace{1cm} (5)

The power spectra of the $E$ and $B$ modes of polarization are defined, in terms of the multipole coefficients $E_{\ell m}$ and $B_{\ell m}$, as

$$C^E_\ell = \frac{1}{2\ell + 1} \sum_m \langle E_{\ell m} E_{\ell m}^* \rangle,$$

$$C^BB_\ell = \frac{1}{2\ell + 1} \sum_m \langle B_{\ell m} B_{\ell m}^* \rangle,$$  \hspace{1cm} (6)

where the brackets denote the average over all realizations.

It is important to note that the polarization sky maps $E(\hat{\gamma})$ and $B(\hat{\gamma})$ are constructed out of underlying $Q(\hat{\gamma})$ and $U(\hat{\gamma})$ maps in a nonlocal manner. This is to say that the value of the $E$ or $B$ field at a given point $\hat{\gamma}$, in virtue of (5), depends on the multipole coefficients $E_{\ell m}$ and $B_{\ell m}$, respectively. These coefficients, in turn, depend on the integral of $P_\pm(\hat{\gamma})$ over the full sky [see (3) and (4)]. Therefore, one requires the knowledge of $Q(\hat{\gamma})$ and $U(\hat{\gamma})$ [or equivalently $P_\pm(\hat{\gamma})$] over the entire sky in order to construct the $E(\hat{\gamma})$ and $B(\hat{\gamma})$ fields.

As was mentioned previously, in realistic scenarios, one does not have information on $Q$ and $U$ fields on the entire sky. For this reason (3)–(5) cannot be applied directly to construct $E$ and $B$ types of polarization maps on an incomplete sky. In order to avoid this problem, in the present paper we adopt a different but related definition for electric
and magnetic polarization maps:
\[ \mathcal{E}(\hat{\gamma}) = -\frac{1}{2} [\delta_1 \delta_2 P_+ (\hat{\gamma}) + \delta_1 \delta_2 P_- (\hat{\gamma})] \]  
(7)
\[ \mathcal{B}(\hat{\gamma}) = -\frac{1}{2i} [\delta_1 \delta_2 P_+ (\hat{\gamma}) - \delta_1 \delta_2 P_- (\hat{\gamma})] \]  
(8)
where \( \delta_1 \) and \( \delta_2 \) (s = 1, 2) are the spin lowering and raising operators, respectively.

The definitions (7) and (8) for \( \mathcal{E} \) and \( \mathcal{B} \) have been previously discussed in the literature [30,41,42,44,46–48]. These have often been denoted as \( \bar{E} \) and \( \bar{B} \), respectively (see, for example, [47,48]). The fields \( \mathcal{E} \) and \( \mathcal{B} \) are equivalent to the fields \( E \) and \( B \) introduced in [30], where these fields were introduced as two independent invariants constructed out of the second covariant derivatives of the polarization tensor [see Eq. (36) in [30]]. In the present paper, to maintain a clear distinction from \( (E, B) \) in (5), we shall use the \( (\mathcal{E}, \mathcal{B}) \) notation.

The constructed electric and magnetic fields are scalar fields on the sphere. Therefore, assuming \( \mathcal{E} \) and \( \mathcal{B} \) given on a full sky, one can determine the spherical harmonics decomposition coefficients

\[ \mathcal{E}_{\ell m} = \int d\hat{\gamma} \mathcal{E}(\hat{\gamma}) Y_{\ell m}^{*}(\hat{\gamma}), \quad \mathcal{B}_{\ell m} = \int d\hat{\gamma} \mathcal{B}(\hat{\gamma}) Y_{\ell m}^{*}(\hat{\gamma}). \]  
(11)
These relations can be inverted to give

\[ \mathcal{E}(\hat{\gamma}) = \sum_{\ell m} \mathcal{E}_{\ell m} Y_{\ell m}(\hat{\gamma}), \quad \mathcal{B}(\hat{\gamma}) = \sum_{\ell m} \mathcal{B}_{\ell m} Y_{\ell m}(\hat{\gamma}). \]  
(12)

The multipole coefficients \( \mathcal{E}_{\ell m} \) and \( \mathcal{B}_{\ell m} \) are related to \( E_{\ell m} \) and \( B_{\ell m} \) defined in (4) by an \( \ell \)-dependent numerical factor \( N_{\ell} = \sqrt{(\ell + 2)!/(\ell - 2)!} \) [30,46,47],

\[ \mathcal{E}_{\ell m} = N_{\ell} E_{\ell m}, \quad \mathcal{B}_{\ell m} = N_{\ell} B_{\ell m}. \]  
(13)
One can also define the power spectra of the \( \mathcal{E} \) and \( \mathcal{B} \) modes of polarization,

\[ C_{\ell}^{EE} = \frac{1}{2\ell + 1} \sum_{m} \langle \mathcal{E}_{\ell m} \mathcal{E}_{\ell m}^{*} \rangle, \quad C_{\ell}^{BB} = \frac{1}{2\ell + 1} \sum_{m} \langle \mathcal{B}_{\ell m} \mathcal{B}_{\ell m}^{*} \rangle. \]  
(14)
These are related with the power spectra \( C_{\ell}^{EE} \) and \( C_{\ell}^{BB} \) through the relations

\[ C_{\ell}^{EE} = N_{\ell}^{2} C_{\ell}^{EE}, \quad C_{\ell}^{BB} = N_{\ell}^{2} C_{\ell}^{BB}. \]  
(15)
Thus, in comparison with \( C_{\ell}^{EE} \) and \( C_{\ell}^{BB} \), the power spectra \( C_{\ell}^{EE} \) and \( C_{\ell}^{BB} \) are “bluer,” due to the factor \( N_{\ell}^{2} \). Note that the relations (13) and (15) assume that the polarization fields are given on a complete sky.

It is important to point out that the quantities \( \mathcal{E} \) and \( \mathcal{B} \) defined in (7) and (8) only depend on the differential of the \( Q \) and \( U \) fields by construction. Therefore, these definitions can be, in principle, applied in the case of \( Q \) and \( U \) given on an incomplete portion of the sky, to construct the \( \mathcal{E} \) and \( \mathcal{B} \) fields on this portion. We now proceed to discuss the relevant steps for this construction on an incomplete sky.

In order to describe the partial-sky observations, we first introduce the mask window function \( W(\hat{\gamma}) \). This mask function is nonzero only in the observational region of the sky. In addition, we shall assume that \( W(\hat{\gamma}) \) is a real function. In particular, the special case with \( W(\hat{\gamma}) = 1 \) in the observational region corresponds to the widely discussed top-hat window function. In the present paper, we denote this special case of a top-hat window function as \( w(\hat{\gamma}) \). With the introduction of the window function \( W(\hat{\gamma}) \), the analysis of the polarization field \( P_{\ell m}(\hat{\gamma}) \) defined on the partial region of the sky becomes equivalent to studying the masked field \( P_{\ell m}(\hat{\gamma}) W(\hat{\gamma}) \) defined on the complete sky.

In the general case of an arbitrary mask, one can define two full-sky maps \( \mathcal{E}(\hat{\gamma}) \) and \( \mathcal{B}(\hat{\gamma}) \) constructed out of observational data,

\[ \mathcal{E}(\hat{\gamma}) = -\frac{1}{2} [\delta_1 \delta_2 (P_{\ell m}(\hat{\gamma}) W)(\hat{\gamma}) + \delta_1 \delta_2 (P_{\ell m}(\hat{\gamma}) W)(\hat{\gamma})], \]  
(16)
\[ \mathcal{B}(\hat{\gamma}) = -\frac{1}{2i} [\delta_1 \delta_2 (P_{\ell m}(\hat{\gamma}) W)(\hat{\gamma}) - \delta_1 \delta_2 (P_{\ell m}(\hat{\gamma}) W)(\hat{\gamma})]. \]  
(17)
Because of the presence of the window function \( W(\hat{\gamma}) \), the two maps \( \mathcal{E}(\hat{\gamma}) \) and \( \mathcal{B}(\hat{\gamma}) \) do not correspond to pure electric and magnetic types of polarization. The main task of this work is to construct pure \( \mathcal{E}(\hat{\gamma}) \) and \( \mathcal{B}(\hat{\gamma}) \) fields out of \( \mathcal{E}(\hat{\gamma}) \) and \( \mathcal{B}(\hat{\gamma}) \).

Moving on, we define the multipole decomposition coefficients \( \tilde{E}_{\ell m} \) and \( \tilde{B}_{\ell m} \) as

\[ \tilde{E}_{\ell m} = \frac{1}{N_{\ell}} \int d\hat{\gamma} \mathcal{E}(\hat{\gamma}) Y_{\ell m}^{*}(\hat{\gamma}), \quad \tilde{B}_{\ell m} = \frac{1}{N_{\ell}} \int d\hat{\gamma} \mathcal{B}(\hat{\gamma}) Y_{\ell m}^{*}(\hat{\gamma}). \]  
(18)

With this definition, the \( \mathcal{E} \) and \( \mathcal{B} \) fields can be expanded in terms of the multipole coefficients \( \tilde{E}_{\ell m} \) and \( \tilde{B}_{\ell m} \) in the following manner:

\[ \mathcal{E}(\hat{\gamma}) = \sum_{\ell m} N_{\ell} \tilde{E}_{\ell m} Y_{\ell m}(\hat{\gamma}), \quad \mathcal{B}(\hat{\gamma}) = \sum_{\ell m} N_{\ell} \tilde{B}_{\ell m} Y_{\ell m}(\hat{\gamma}). \]  
(19)

The multipole decomposition coefficients \( \tilde{E}_{\ell m} \) and \( \tilde{B}_{\ell m} \) can
be calculated in an alternative manner. One can begin by
defining the complex polarization fields \( \tilde{P}_\pm(\hat{\gamma}) = \frac{P_\pm(\hat{\gamma})}{W(\hat{\gamma})} \) and constructing the multiple coefficients using (3) and (4) (with tildes placed on all the relevant quantities). It can be verified that the two definitions are equivalent.

Before proceeding, let us point out some simplifying relations. First, from the definitions of \( \tilde{\delta}_1 \) and \( \tilde{\delta}_2 \), it follows that \( \tilde{\delta}_2 = \tilde{\delta}_2^* \) and \( \tilde{\delta}_1 = (\tilde{\delta}_1)^* \). In light of definitions (7), (8), (16), and (17), it follows that the two sets of fields, \( (E, B) \) and \( (\tilde{E}, \tilde{B}) \), are real. Furthermore, from the definitions one has

\[
\tilde{\delta}_1 \tilde{\delta}_2 P_+ = -E - iB, \quad \tilde{\delta}_1 \tilde{\delta}_2 (P_+ W) = -\tilde{E} - i\tilde{B}. \tag{20}
\]

Thus, in order to determine the relation between the two sets of fields \( (E, B) \) and \( (\tilde{E}, \tilde{B}) \), it suffices to study the relation between \( \tilde{\delta}_1 \tilde{\delta}_2 P_+ \) and \( \tilde{\delta}_1 \tilde{\delta}_2 (P_+ W) \).

In order to derive the relation between the two sets, it is convenient to expand the quantity \( \tilde{\delta}_1 \tilde{\delta}_2 (P_+ W) \), using the definition of the \( \tilde{\delta}_1 \) operator (9), in the following way:

\[
[\tilde{\delta}_1 \tilde{\delta}_2 (P_+ W)]W = ([\tilde{\delta}_1 \tilde{\delta}_2 P_+] W) W + ([\tilde{\delta}_1 \tilde{\delta}_2 P_+] W) + \cot \theta [W(\tilde{\delta}_1 \tilde{\delta}_2 P_+) + P_+(\tilde{\delta}_1 \tilde{\delta}_2 W)] W + (2 + 2\cot^2 \theta) P_+ W^2 + 2\cot \theta W\tilde{\delta}_1 (P_+ W). \tag{21}
\]

Using the following set of relations that follow from (9),

\[
(\tilde{\delta}_1 \tilde{\delta}_2 P_+) W = \tilde{\delta}_2 (P_+ W) - P_+ \tilde{\delta}_2 W - 2\cot \theta P_+ W,
\]

\[
(\tilde{\delta}_1 P_+) W = \tilde{\delta}_1 (P_+ W) - P_+ \tilde{\delta}_1 W - \cot \theta P_+ W,
\]

\[
W(\tilde{\delta}_2 P_+) + P_+(\tilde{\delta}_1 W) = \tilde{\delta}_2 (P_+ W) - 2\cot \theta P_+ W,
\]

along with the expressions in (20), we arrive at the expression

\[
[\mathcal{E} + i\mathcal{B}] W^2 = [\tilde{\mathcal{E}} + i\tilde{\mathcal{B}}] W + (\tilde{\delta}_1 W)[\tilde{\delta}_2 (P_+ W) - P_+ \tilde{\delta}_2 W - 2\cot \theta P_+ W] + (\tilde{\delta}_2 W)[\tilde{\delta}_1 (P_+ W) - P_+ \tilde{\delta}_1 W - \cot \theta P_+ W]
\]

\[
+ \cot \theta \tilde{\delta}_2 (P_+ W) + 2W^2 P_+ + 2W \cot \theta \tilde{\delta}_1 (P_+ W). \tag{22}
\]

This expression can be rewritten in a compact form,

\[
[\mathcal{E} + i\mathcal{B}] W^2 = [\tilde{\mathcal{E}} + i\tilde{\mathcal{B}}] W + \text{ct}, \tag{23}
\]

where “ct” denotes the correction term. This correction term is complex, in general. The real and imaginary parts of the correction term are given as

\[
\text{Re}[\text{ct}] = Q[3\cot \theta WW_x + W(W_{xx} - W_{yy}) - 2(W_x^2 - W_y^2)] + U[2\cot \theta WW_y
\]

\[
+ 2WW_{xy} - 4W_y W_x] + 2W_y[(QW)_x + (UW)_y]
\]

\[
+ 2W_y[(UW)_x - (QW)_y]. \tag{24}
\]

and

\[
\text{Im}[\text{ct}] = U[3\cot \theta WW_x + W(W_{xx} - W_{yy}) - 2(W_x^2 - W_y^2)]
\]

\[
- Q[2\cot \theta WW_y + 2WW_{xy} - 4W_x W_y]
\]

\[
- 2W_y[(QW)_x + (UW)_y]
\]

\[
+ 2W_y[(UW)_x - (QW)_y]. \tag{25}
\]

In the above expressions we have introduced the shorthand notations \( F_x = \frac{\partial F}{\partial \theta}, F_y = \frac{\partial F}{\partial \phi}, F_{xx} = \frac{\partial^2 F}{\partial \theta^2}, F_{yy} = \frac{\partial^2 F}{\partial \phi^2} \), and \( F_{xy} = \frac{\partial^2 F}{\partial \theta \partial \phi} \) for arbitrary function \( F(\hat{\gamma}) \). In Appendix A, we discuss the question of numerically calculating the various terms in the above expression in pixel space.

Finally, one can construct the pure electric and magnetic fields \( \mathcal{E} \) and \( B \) on the observed portion of the sky [i.e. the region of the sky for which \( W(\hat{\gamma}) \neq 0 \)] using the expression

\[
[\mathcal{E} + i\mathcal{B}] = [\tilde{\mathcal{E}} + i\tilde{\mathcal{B}}] W^{-1} + \text{ct} W^{-2}. \tag{26}
\]

The construction of the pure electric and magnetic scalar fields \( \mathcal{E} \) and \( B \) is the main result of this paper. It is worth pointing out that the construction of these fields is independent of the choice of the mask function \( W(\hat{\gamma}) \), as long as the mask is nonzero in the observed portion of the sky. This method for recovering the scalar fields \( \mathcal{E} \) and \( B \) is lossless in real space in the following sense. If one was given the polarization fields \( Q \) and \( U \) on the entire sphere and then constructed the corresponding \( B \) field using (8) [or the \( E \) field using (7)] and compared the resulting scalar field in the observed region with the result of the above procedure (26), one would find the two fields equal. However, due to the ill-behaved nature of \( W^{-1} \) and \( W^{-2} \) at the edge of the observed region, it is difficult to realize the above construction in practice. In order to circumvent this problem, as will be discussed in the following section, one has to remove the edge of the constructed polarization maps.

In conclusion of this section it is instructive to clarify the issues associated with possible leakage from the so-called ambiguous modes. It is known that, on a manifold with a boundary, the decomposition of the polarization field, in addition to pure \( E \) and \( B \) components, contains ambiguous modes that satisfy both \( E \)-mode and \( B \)-mode conditions simultaneously (see [38,39] for details). In particular, when constructing the power spectrum estimators for the \( B \) mode, one has to ensure that there is no leakage from the ambiguous modes. In the current work, the \( B_{\text{rec}}(\hat{\gamma}) \) field does not contain contributions from either \( E \) modes of polarization or ambiguous modes, by virtue of construction.
(analogous to $\chi_B$ in [41]). For this reason, the power spectral estimators constructed from this field will be free from contaminations from both $E$ modes and ambiguous modes.

III. $E/B$ SEPARATION IN PIXEL SPACE

In this section, we shall discuss the issues related to the separation of the electric and magnetic polarizations $E(\hat{r})$ and $B(\hat{r})$ in pixel space using the results of the previous section, in particular, expression (26). We shall discuss this procedure using a toy model. For this toy model, we assume that an experiment will only observe the Stokes parameters $Q$ and $U$ in the northern hemisphere. Following [37,40], we adopt the following axially symmetric form for the mask window function $W(\hat{r})$.

$$W(\hat{r}) = \begin{cases} 1 & \theta < \theta_0 - \theta_1 \\ \frac{1}{2} - \frac{1}{2} \cos \left( \frac{\theta - \theta_0}{\theta_1 - \theta_0} \pi \right) & \theta_0 - \theta_1 < \theta < \theta_0 \\ 0 & \theta > \theta_0. \end{cases}$$ (27)

In the above expression, $\theta_0$ corresponds to the edge of the observational area, and $\theta_1 \geq 0$ is the smoothing scale. The limiting case, $\theta_1 = 0$, corresponds to the top-hat window function $[w(\hat{r}) = 1$ for $\theta < \theta_0$ and $w(\hat{r}) = 0$ for $\theta > \theta_0]$. However, the top-hat function is discontinuous at $\theta = \theta_0$, which makes the quantities $W_{\hat{r}xx}, W_{\hat{r}xy},$ and $W_{\hat{r}yx}$ ill defined at the edge. In order to avoid these difficulties, it is convenient to use a window function with $\theta_1 \neq 0$. Throughout the present section we use the values $\theta_0 = 90^\circ$ and $\theta_1 = 30^\circ$. It is important to point out that the formalism outlined in Sec. II is applicable for arbitrary mask window functions, not necessarily axially symmetric ones. The simple symmetric form (27) for the mask was chosen for simplicity and clarity of presentation. In realistic scenarios one will have to use a more complex mask that will take into account the nonsymmetric form of the observed region and various point-source contaminations.

For an axially symmetric window function $W(\hat{r})$ [i.e. when $W(\hat{r})$ is independent of $\phi$, such as the one considered in the present section, the correction terms in (24) and (25) simplify to

$$\text{Re}[c_r] = Q[3\cot \theta WW_x + WW_{xx} - 2W_x^2] + 2W_x[(QW)_x + (UW)_y],$$ (28)

$$\text{Im}[c_r] = U[3\cot \theta WW_x + WW_{xx} - 2W_x^2] + 2W_x[(UW)_x - (QW)_y].$$ (29)

In order to demonstrate the $E/B$ separation, we shall work with simulated polarization maps. We use the synfast subroutine included in the HEALPix package to generate a full-sky map of the $Q(\hat{r})$ and $U(\hat{r})$ fields. In order to generate this map we use the best-fit WMAP5 values for the cosmological parameters [49]

$$\Omega_m h^2 = 0.02267, \quad \Omega_\Lambda h^2 = 0.1131,$$
$$\Omega_B = 0.076, \quad \tau_{\text{reion}} = 0.084, \quad \alpha_s = 2.446 \times 10^{-9}, \quad \alpha_s = 0.96,$$

and assume no contribution from gravitational waves and cosmic lensing to the $B$ mode of polarization, i.e. $C_{\ell}^{B_B} = 0$. We adopt the pixelization with $N_{\text{side}} = 512$. We set the full width at half maximum (FWHM) for the Gaussian beam to $\theta_f = 30^\circ$. In what follows we shall be solely interested in the determination of the $B$ polarization field, and in studying the possible contaminations to it. In this context, since $C_{\ell}^{B_B} = 0$, a nonzero field $B$ would be wholly attributed to the residual $EB$-mixing contamination.

Before proceeding to construct the pure magnetic field $B$, we shall construct and analyze the auxiliary $\tilde{B}$ field. From the simulated $Q(\hat{r})$ and $U(\hat{r})$ maps we construct the $\tilde{B}(\hat{r})$ in the following manner. First, we construct the multipole coefficients $B_{\ell m}$. This is done by building multipole coefficients $a_{\pm2,\ell m} = \int d\hat{r} P_\pm(\hat{r}) W(\hat{r}) Y_{\pm2,\ell m}(\hat{r})$ using the simulated $(Q(\hat{r}), U(\hat{r}))$ maps along with the window function $W(\hat{r})$ given in (27), and then calculating $B_{\ell m} = \frac{-1}{2\pi} [a_{2,\ell m} - \tilde{a}_{-2,\ell m}]$. These steps were performed numerically using the anafast routine from the HEALPix package. Following this, we use (19) to construct $\tilde{B}$ from the multipole coefficients $\tilde{B}_{\ell m}$. The resulting $\tilde{B}$ map is illustrated in Fig. 1.

One can see that, although the map was generated with $C_{\ell}^{B_B} = 0$, $\tilde{B} \neq 0$ in the region $\theta_0 - \theta_1 < \theta < \theta_0$ (i.e. $60^\circ < \theta < 90^\circ$). This can be viewed as a leakage of the $E$ type of polarization into the $\tilde{B}$ field, due to the presence of the window function $W$. In order to quantify this leakage in the harmonic (multipole) space, we construct the pseudo power spectrum as

$$P_{\ell}^{B_B} = \frac{1}{2\ell + 1} \sum_m B_{\ell m} \tilde{B}_{\ell m},$$

where $\tilde{B}_{\ell m} = \int \tilde{B}(\hat{r}) Y_{\ell m}(\hat{r}) d\hat{r}$. (31)

FIG. 1 (color online). The $\tilde{B}$ map constructed from an input model with no magnetic polarization (in $\mu$K).
The resulting pseudo power spectrum is plotted as a black line in Fig. 3 (see below).

We can now reconstruct the pure magnetic-type field $\mathcal{B}$ using (26) in the following manner:

$$\mathcal{B}_{\text{rec}}(\hat{\gamma}) = \mathcal{B}(\hat{\gamma})W^{-1}(\hat{\gamma}) + \text{Im}(ct(\hat{\gamma}))W^{-2}(\hat{\gamma}).$$  \hspace{1cm} (32)

We use the subscript “rec” to indicate that this field was reconstructed from the $\mathcal{B}$ and the $(Q, U)$ fields. The results of the reconstruction are presented in Fig. 2. Since the input cosmological model assumes no contribution from gravitational waves, one can expect $\mathcal{B}_{\text{rec}}(\hat{\gamma}) = 0$. A visual comparison of Figs. 1 and 2 shows that we have been able to remove much of the leakage that was present in $\mathcal{B}$. The remaining residual contamination in $\mathcal{B}_{\text{rec}}$ is shown (with a magnified scale) in the right panel of Fig. 2. These residuals are a small fraction ($\sim 2\%$) of the total leakage in Fig. 1. In order to quantify these residuals in multipole space, we construct the pseudo spectral estimators, replacing $\mathcal{B}$ with $\mathcal{B}_{\text{rec}}$ in (31). The resulting pseudo power spectrum is plotted as a red (gray) curve in Fig. 3. It can be seen that the resulting leakage power for $\mathcal{B}_{\text{rec}}$ is significantly lower than the corresponding power for $\mathcal{B}$. In particular, in the practically interesting range of multipoles $\ell \in (50, 200)$, the spectrum for the reconstructed $\mathcal{B}_{\text{rec}}$ field is roughly 4 orders of magnitude lower than the spectrum for $\mathcal{B}$. The remaining residuals in $\mathcal{B}_{\text{rec}}$ are attributed to numerical errors.

We believe that the small remaining residuals in $\mathcal{B}_{\text{rec}}$ are a result of two types of numerical errors that cannot be avoided in practice. The first reason for errors is pixelization. In [39], it was argued that pixelization can lead to the mixing of electric and magnetic modes. This point can be intuitively understood in the following way. Imagine a survey that observes polarization on a small square region of the sky. Pixelization introduces a Nyquist wave number $k_N$, such that all modes with wave numbers greater than $k_N$ are aliased to modes with wave numbers less than the Nyquist value. This aliasing completely shuffles the direction of the wave numbers, thus essentially leading to a complete mixing of electric and magnetic modes. Although the complete avoidance of the errors due to pixelization is impossible, these numerical errors can be reduced by using a larger value for $N_{\text{side}}$ (see Sec. III B for details). In the present evaluation, with $N_{\text{side}} = 512$, finite pixelization seems to be the main reason for the residual power spectrum for $\ell > 150$. The main contribution to this residual power spectrum comes from the residual $\mathcal{B}_{\text{rec}}(\hat{\gamma})$ at the pole in real space (see the right panel in Fig. 2).

The second reason for numerical errors is the steep growth of the power spectrum $C_{\ell}^{EE}$ with increasing $\ell$. Because of this, even a small relative numerical error at higher multipoles seeps through to lower multipoles. In other words, these errors occur due to the fact that the various sources of noise and $E$-mode signal are not band limited. We believe that these types of errors mainly account for the residual power spectrum of $\mathcal{B}_{\text{rec}}$ at multipoles $\ell < 150$. These errors are mainly caused by the residual...
$B_{\text{rec}}(\hat{\gamma})$ around the observed edge $\theta = \theta_0$ in real space (see the right panel in Fig. 2).

**A. Dependence of residual leakage on edge removal**

Figure 2 shows that much of the residual leakage occurs around the edge of the observation region. It is therefore instructive to study the edge effects in more detail. The expression (32) for $B_{\text{rec}}$ depends on the $\text{Im}[\psi]$ correction term given by (29). This correction term contains two terms, $(3 \cot \theta W W_x + W W_{xx} - 2 W_x^2)/W^2$ and $2 W_x/W^2$, which have the following asymptotic at the edge of the map as $\theta \to \theta_0$,

$$
\frac{3 \cot \theta W W_x + W W_{xx} - 2 W_x^2}{W^2} \to -\frac{6}{(\theta_0 - \theta)^2},
$$

$$
\frac{2 W_x}{W^2} \to -\frac{16 \theta_0^2}{\pi^2(\theta_0 - \theta)^3}.
$$

Thus, the two functions are divergent for $\theta \to \theta_0$. This implies that the signal-to-noise ratio will tend to zero for data as the boundary of the observed region is approached. Because of this divergence, in numerical calculations, one must remove the edge of the map in order to avoid numerical errors associated with these divergences, thereby introducing a small loss of information.

In order to investigate the dependence of the residual leakage on the edge removal, in Fig. 4 we plot the pseudo power spectrum $D_\ell$ of residual $B_{\text{rec}}$ constructed for the same simulated data but for two different edge removals. The first case (red line) corresponds to a portion of the sky with $W < 0.03$ removed (this corresponds to the removal of data with $\theta_0 - \theta < 0.06$). The second case (green line) corresponds to a portion of the sky with $W < 0.1$ removed (corresponding to the removal of data with $\theta_0 - \theta < 0.1$). The second case corresponds to a larger portion of the sky removed, and thus to a larger loss of information. This loss of information leads to a smaller value for the power spectrum at lower multipoles, $\ell < 150$. This fact can be clearly seen from the right panel of Fig. 2. As one removes more of the data from the equator, the residual spectrum of $B_{\text{rec}}$ becomes smaller. On the other hand, in the region of higher multipoles, where the dominant contribution comes from finite pixelization errors, the two power spectra are comparable.

**B. Dependence of residual leakage on pixelization number $N_{\text{side}}$**

As was pointed out earlier, one of the reasons for residual leakage of power into $B_{\text{rec}}$ is finite pixelization of the sky map. In order to demonstrate the effect of pixelization on the residual power, in Fig. 5 we show the pseudo power spectra $D_\ell$ calculated for two different pixelization numbers, $N_{\text{side}} = 512$ (red line) and $N_{\text{side}} = 1024$ (green line). As one might expect, the increase in the pixelization number reduces the leakage power spectrum. This reduction is most dramatic at higher multipoles, $\ell > 150$, where it is 2 orders of magnitude for this example. At lower multipoles, $\ell < 150$, the reduction is not as dramatic, and is roughly by a factor of 3. These results are consistent with our previous statements about the cause of numerical errors. Indeed, at higher multipoles the main cause of errors seems to be finite pixelization, whereas at lower multipoles the errors are generated by a combination of factors.
C. Dependence of residual leakage on $\theta_F$

The full width at half maximum parameter $\theta_F$ has an important effect on the residual leakage of power into $B_{\text{rec}}$. In order to understand the reason for this, one has to remember that one of the two reasons for residual leakage is the steep growth of the power spectrum $C^{EE}_\ell$ with increasing $\ell$. The parameter $\theta_F$ regulates the exponential damping of this power spectrum at multipoles $\ell \approx \theta_F^{-1}$, and therefore limits the propagation of the power in $C^{EE}_\ell$ into $D_\ell$.

The various contributions to the spectrum $C^{EE}_\ell$ are illustrated in Fig. 6. The main contribution to the power spectrum comes from density perturbations (dashed blue line). For comparison, on this figure, we show the contributions to $C^{EE}_\ell$ (solid blue line) and $C^{BB}_\ell$ (solid red line) from gravitational waves (characterized by the tensor-to-scalar ratio $r = 0.1$), as well as the contribution to $C^{BB}_\ell$ from lensing (red dashed line). The spectrum $C^{EE}_\ell$ from density perturbations at high multipoles acts as the main source for the residual leakage into $D_\ell$.

The contributions to the various spectra at high multipoles are effectively damped by the choice of an appropriate $\theta_F$. This parameter leads to the damping of the power spectrum $C^{EE}_\ell$ proportional to $\exp(-\ell^2 \theta_F^2 / 30)$. In Fig. 7 we show the residual leakage for two choices of the FWHM parameter, $\theta_F = 30'$ (red line) and $\theta_F = 10'$ (green line). For comparison, in this figure we also show the pseudo spectrum $C^{BB}_\ell$ calculated for $\theta_F = 30'$ (black line) and $\theta_F = 10'$ (blue line). As one might expect, the residual power spectrum reduces significantly with an increase in $\theta_F$. For this reason, for the purposes of extracting the magnetic pattern of polarization in experiments with small $\theta_F$ (for example, the POLARBEAR experiment discussed in Sec. IV E), it becomes necessary to artificially increase $\theta_F$ in order to reduce the residual leakage in $B_{\text{rec}}$. In Appendix D, we suggest a “map smoothing” technique to achieve this goal.

IV. $E/B$ SEPARATION AND POWER SPECTRUM ESTIMATION FOR SMALL-SKY SURVEYS

In Secs. II and III we developed a method to construct pure electric $E_{\text{rec}}$ and magnetic $B_{\text{rec}}$ fields out of the original Stokes parameter fields $Q$ and $U$ on a fractional portion of the sky. Ignoring the small numerical errors, it was shown that the resulting fields did not exhibit mixing. A crucial point about the constructed fields is that they are scalar fields. For this reason, one can use all of the robust techniques developed for studying CMB temperature anisotropy to the fields $E_{\text{rec}}$ and $B_{\text{rec}}$. Based on the appropriation of these techniques, in this section, we shall focus on an important practical application, namely, constructing the estimator for the power spectrum of the $B$ mode of polarization $C^{BB}_\ell$. For this reason, as in the previous sections, we shall restrict our analysis to just the magnetic field $B_{\text{rec}}$.

The question of constructing an estimator for the power spectrum $C^{BB}_\ell$ from the field $B_{\text{rec}}$ is analogous to the problem of constructing an estimator for the temperature anisotropy power spectrum $C^{TT}_\ell$ given a temperature map on a partial sky. Fortunately, there are a large number of methods that have been developed for this purpose [50–54]. Amongst these, a popular method is the so-called “pseudo-$C_E$” estimator method [53]. This method can be easily realized in pixel space using fast spherical harmonics transformation, and has been applied to various CMB
observations including WMAP data [55]. However, it is well known that pseudo-$C_\ell$ estimators are suboptimal, particularly for low multipoles. For this reason, many authors have developed alternative estimators that are optimal, in particular, the maximum likelihood estimators in pixel realization [50,52]. The fundamental problem with the maximum likelihood estimators is that these methods are very slow, especially for larger multipoles. For large-sky surveys, such as the Planck satellite, the use of hybrid estimators combines the best sky surveys, such as the Planck satellite, the use of hybrid the maximum likelihood estimators is that these methods are nearly optimal and can be realized on a laptop computer even for large-sky surveys such as Planck.

In the present section we shall focus on small-sky polarization surveys, corresponding to various ground-based CMB experiments [4,5,12–14]. Since these surveys will be primarily sensitive to relatively large multipoles, $\ell \gtrsim 20$, we shall restrict our analysis to pseudo-$C_\ell$ from $B_{\text{rec}}$ in the case of large-sky surveys such as Planck. We leave this exercise for the future.

Below we shall work with a small fraction of the sky characterized by a window function (27) with $\theta_0 = 20^\circ$ and $\theta_1 = 10^\circ$, corresponding to a 3% sky survey. In an ideal case, neglecting numerical errors, the reconstructed field $B_{\text{rec}}(\hat{\gamma})$ would be related to the underlying full-sky field $B(\hat{\gamma})$ through $B_{\text{rec}}(\hat{\gamma}) = B(\hat{\gamma})w(\hat{\gamma})$, where $w(\hat{\gamma})$ is the corresponding top-hat window function. However, as was pointed out in Sec. IIIA, one needs to remove a narrow edge from the observational area in order to avoid excessive numerical errors. For this reason, in practice, we remove the region $\theta_0 - \theta < 0.03$ (corresponding to $\theta > 18^\circ$) from the analysis. Below we shall use the notation $w'(\hat{\gamma})$ to denote the top-hat window function for data with edge removal.

### A. Pseudo estimators

The first step in constructing the pseudo estimator is the definition of the spherical harmonics coefficients $a_{\ell m}$ of the scalar field $B_{\text{rec}}(\hat{\gamma})$ as follows:

$$a_{\ell m} = \int d\hat{\gamma} B_{\text{rec}}(\hat{\gamma}) W(\hat{\gamma}) Y_{\ell m}^*(\hat{\gamma}),$$

(33)

where $W(\hat{\gamma})$ is the weight function. In principle, one can choose an arbitrary form for the weight function. In particular, the choice $W(\hat{\gamma}) = 1$ corresponds to the widely discussed pseudo-$C_\ell$ estimator introduced in [53]. This choice will be the main focus of our attention in the present work. An alternative choice, $W(\hat{\gamma}) = W(\hat{\gamma})$ [where $W(\hat{\gamma})$ is the mask window function in Eq. (27)], corresponds to the analysis in [41], where it was shown that the resulting $a_{\ell m}$ lead to the pure $B$-mode estimators defined in [40].

The comparison of this choice for the weight function with our main choice $W(\hat{\gamma}) = 1$ is discussed in Appendix C. The optimal choice for the weight function in various cases has been discussed in [37,41,56]. In [41] the authors suggest a general method to build the weight function $W(\hat{\gamma})$ for different multipoles $\ell$ in order to optimize the estimator. At this point it is important to emphasize that although $B_{\text{rec}}(\hat{\gamma})$ preserves the available information in real space, a nonoptimal power spectrum estimation will lead to the loss of some of this information. This makes the study of the optimal choice for the weight function particularly important. However, in the current paper we concentrate mainly on the simplistic case, $W(\hat{\gamma}) = 1$, leaving the important but complicated question of the optimal choice of the weight function for future work.

For the choice $W(\hat{\gamma}) = 1$, the spherical harmonics coefficients $a_{\ell m}$ in (33) take the simplified form

$$a_{\ell m} = \int d\hat{\gamma} B_{\text{rec}}(\hat{\gamma}) Y_{\ell m}^*(\hat{\gamma}).$$

(34)

These are related to the coefficients $B_{\ell m}$ [which were defined in (11) in terms of the underlying full-sky map $B(\hat{\gamma})$ through the coupling matrix $K_{\ell m' m''}$ (see, for instance, [54]),

$$a_{\ell m} = \sum_{m'} B_{\ell m} B_{\ell m'} K_{\ell m' m''} = \sum_{m'} B_{\ell m} N_{m'} B_{\ell m'} K_{\ell m' m''},$$

(35)

where $B_{\ell}$ is a window function describing the combined smoothing effects of the beam and the finite pixel size. The coupling matrix $K$ can be expressed in terms of the function $w'(\hat{\gamma})$ as

$$K_{\ell_1 m_1, \ell_2 m_2} = \int d\hat{\gamma} w'(\hat{\gamma}) Y_{\ell_1 m_1}^*(\hat{\gamma}) Y_{\ell_2 m_2}^* (\hat{\gamma}).$$

(36)

The pseudo estimator $D_\ell$ is defined analogous to (31) in terms of the multipole coefficients (34) as

$$D_\ell = \frac{1}{2\ell + 1} \sum_m a_{\ell m} a_{\ell m}^*.$$  

(37)

Using relations (6), (15), and (35), one obtains that the expectation value of this estimator $D_\ell$ is related to the true power spectrum $C^{BB}_\ell$ by the following convolution:

$$\langle D_\ell \rangle = \sum_{\ell'} M_{\ell \ell'} B_{\ell'} C^{BB}_{\ell'} = \sum_{\ell'} M_{\ell \ell'} N_{\ell'} B_{\ell'} C^{BB}_{\ell'}.$$  

(38)

The coupling matrix $M$ in the above expression can be expressed in terms of $3j$ symbols as

$$M_{\ell_1, \ell_2} = (2\ell_2 + 1) \sum_{\ell_3} \frac{(2\ell_3 + 1)}{4\pi} w'_{\ell_3} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right)^2.$$  

(39)

where $w'_{\ell}$ is the power spectrum of the window function $w'(\hat{\gamma})$ defined in an analogous manner to (31).
It can be shown that the covariance matrix for the pseudo estimator $D_\ell$ has the form
\[
\langle \Delta D_\ell \Delta D_{\ell'} \rangle = \frac{2}{(2\ell + 1)(2\ell' + 1)} \sum_{m' m} \sum_{l_1 l_2} \sum_{m' m_1 m_2} B_{l_1}^m N_{l_2}^{m_1} C_{l_1}^{BB} B_{l_2}^{m_2} \times N_{l_2}^{m_2} C_{l_2}^{BB} K_{l m l_1 m_1} K_{\ell' m' l_2 m_2} K_{l' m' l_2 m_2}.
\]

As it stands, this formula is not useful due to the high cost of computation. However, for high multipoles, this formula simplifies to [54]
\[
\langle \Delta D_\ell \Delta D_{\ell'} \rangle = 2 B_{l_1}^m N_{l_2}^{m_1} C_{l_1}^{BB} B_{l_2}^{m_2} N_{l_2}^{m_2} C_{l_2}^{BB} M_{\ell l\ell'} / (2\ell + 1).
\]

In order to implement and verify the above analytical results, we have conducted numerical calculations using simulated data. In the first instance, we generate 1000 random full-sky $(Q, U)$ maps with no contribution from gravitational waves (i.e. $r = 0$) and no lensing. For each realization, we reconstruct the magnetic field $\mathcal{B}_{\text{rec}}(\psi)$ and evaluate the pseudo estimator $D_\ell$. The average over 1000 realizations $\bar{D}_\ell$ is plotted (green line) in Fig. 8. Note that here and below we use the overline to denote averaging over simulated realizations, as opposed to the angle brackets which denote ensemble averaging. The average for the uncleaned spectrum $\tilde{D}_\ell^{BB}$ [defined in (31)] is plotted (black line) for comparison on the same figure. For the next calculations, we simulate 1000 random full-sky maps with contributions from gravitational waves characterized by $r = 0.1$ and contributions to the $B$ mode of polarization from cosmic lensing. The average value of the estimator $\bar{D}_\ell$ is plotted (red line) in Fig. 8. The comparison of curves in Fig. 8 shows that the residual noise contribution to the pseudo estimator due to numerical errors (green line) is negligible in comparison with the contribution to the estimator from the signal (red line). One can therefore conclude that the resulting pseudo estimator $\bar{D}_\ell$ is effectively free from $EB$ mixing.

In Fig. 9, in order to verify (38), we plot the left-hand side (solid blue line) of this equation calculated for a model with $r = 0.1$ and with contribution from cosmic lensing [57]. For comparison, in this figure, we plot the average $\bar{D}_\ell$ over 1000 realizations for the same model (solid red line). As expected, the two lines are close to each other and are practically indistinguishable for multipoles $\ell \geq 20$. For comparison, in Fig. 9 we also plot the individual contributions from gravitational waves (solid magenta line) and cosmic lensing (solid green line). Finally, we plot the square root of the average over 1000 realizations of the diagonal terms in the covariance matrix $(\Delta D_\ell \Delta D_\ell)^{1/2}$ (dashed red line). In order to check the analytical approximation (41), we also plot the diagonal term $(\Delta D_\ell \Delta D_\ell)^{1/2}$ evaluated using the right side of expression (41) (dashed blue line). As expected, the two curves for the covariance matrix practically coincide for large multipoles, $\ell \geq 80$, which corresponds to the region of applicability of the approximation (41).
B. Unbiased estimators for $C_{\ell}^{BB}$

Having constructed the pseudo estimator $D_{\ell}$, we are one step away from constructing an unbiased estimator for the $B$-mode power spectrum $C_{\ell}^{BB}$. In this subsection, we shall discuss this construction. Let us for the moment assume that the coupling matrix $M_{\ell \ell'}$ in (38) is invertible. In this case it is possible to bin the pseudo estimator data into multipole bins, and construct an unbiased estimator for the binned power spectrum. Following the analysis for temperature anisotropy [58], we build the so-called full-sky CMB band powers $P_b^{BB}$ as

$$P_b^{BB} = \sum_{\ell'} M_{bb'}^{-1} \sum_{\ell} p_{\ell b} D_{\ell},$$  

where the subscript $b$ denotes the multipole bands, $p_{\ell b}$ is a binning operator in $\ell$-space defined as

$$p_{\ell b} = \begin{cases} \frac{\ell + 1}{2\pi N_2^{c}(\ell_{low}^{(b+1)} - \ell_{low}^{(b)})} & \text{if } \ell_{low}^{(b+1)} \leq \ell < \ell_{low}^{(b)} \\ 0 & \text{otherwise.} \end{cases}$$

The nonsingular binned coupling matrix $M_{bb'}$ participating in (43) is constructed from the coupling matrix $M_{\ell \ell'}$,

$$M_{bb'} = \sum_{\ell} p_{\ell b} \sum_{\ell'} M_{\ell \ell'} B_{\ell}^{2} q_{\ell b}. \quad (45)$$

The function $B_{\ell}^{2}$ takes into account the effects arising due to finite beam size and finite pixelization. In the above expression, $q_{\ell b}$ is the reciprocal operator of $p_{\ell b}$,

$$q_{\ell b} = \begin{cases} \frac{2\pi N_2^{c}}{\pi (\ell + 1)} & \text{if } \ell_{low}^{(b+1)} \leq \ell < \ell_{low}^{(b)} \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

It is straightforward to verify that $P_b^{BB}$ is an unbiased estimator of the $B$ mode of the polarization power spectrum $\ell(\ell + 1)C_{\ell}^{BB}/2\pi$, i.e.,

$$\langle P_b^{BB} \rangle = \frac{\ell(\ell + 1)}{2\pi} C_{\ell}^{BB}. \quad (47)$$

The covariance matrix of the band powers is related to the covariance matrix $\langle \Delta D_{\ell} \Delta D_{\ell'} \rangle$ in (40) by [59]

$$\langle \Delta P_b^{BB} \Delta P_{b'}^{BB} \rangle = M_{bb'}^{-1} p_{\ell b} \langle \Delta D_{\ell} \Delta D_{\ell'} \rangle (p_{b'e})^T (M_{bb'}^{-1})^T. \quad (47)$$

In Fig. 10 we plot the value of the band power $P_b^{BB}$ (red dots) averaged over 1000 realizations. The realizations were generated for a model including contributions from gravitational waves with $r = 0.1$ and cosmic lensing. The multipole bins were chosen with $\Delta \ell = 10$ for each bin. The analysis shows that (up to discrepancies that can be attributed to a finite number of realizations) the average of the power spectrum estimators coincides with the theoretical (input) spectrum. The error bars $\langle \Delta P_b^{BB} \Delta P_{b'}^{BB} \rangle^{1/2}$ (red error bars) were calculated using (47), with the ensemble average replaced by an average over realizations. As can be expected, the error bars are large for the first three data points, due to the small sky coverage. In addition to evaluating error bars, it was verified that the correlation between various multipole bins is quite weak (all of the correlation coefficients are smaller than 0.3). Note that the correlation matrices and corresponding error bars calculated here do not include contributions from instrumental and astrophysical foreground noises. The left panel in Fig. 10 shows the power spectrum estimation with both gravitational wave and cosmic lensing contributions in-

![FIG. 10 (color online). The averaged values (larger red dots) and error bars (larger red error bars) following from simulations of the unbiased estimators of the power spectra $\ell(\ell + 1)C_{\ell}^{BB}$ (total)/2$\pi$ (left panel) and $\ell(\ell + 1)C_{\ell}^{BB}$ (gw)/2$\pi$ (right panel). In both panels, the black solid line denotes the theoretical values of the underlying power spectra. For comparison, in both panels we plot the averaged values (smaller blue dots) and simulated error bars (smaller blue error bars) of the unbiased estimators for an ideal case without information loss (see text for details). In both panels, we have considered a case with no instrumental noise.](023001-11)
constructed directly from the pseudo estimators \( C'_B \) and becomes one of the main contaminations for the magnitude of the covariance matrix for the unbiased estimator and detection of gravitational waves. In [37], the authors found the mixing contamination to the covariance matrix of the tensor-to-scalar ratio that can be probed to increase the error bars of the spectral estimator, is sufficiently small, \( \leq 15\% \).

In order to quantify the detectability of the gravitational wave signal, it is convenient to introduce the total signal-to-noise ratio as follows:

\[
S/N = \sqrt{\sum_{b \ell} (P_{b}^{BB}(gw))(Cov^{-1})_{bb}(P_{b}^{BB}(gw))},
\] (48)

where \( Cov_{bb'} = \langle \Delta P_{b}^{BB} \Delta P_{b'}^{BB} \rangle \) is the covariance matrix of the band-power estimator [47]. For the example considered above with \( r = 0.1 \), we find \( S/N = 8.26 \).

It is important to emphasize that our pseudo-\( C'_{B} \) estimator is quite different from the pseudo-\( C_{T} \) polarization estimators suggested in [37,61], or an equivalent estimator suggested in [62]. In [37,61], the unbiased estimators are constructed directly from the pseudo estimators \( C_{EE}^{\text{RE}} \) and \( C_{BB}^{\text{RE}} \), both of which are a mixture of electric and magnetic types of polarization. The resultant mixing increases the magnitude of the covariance matrix for the unbiased estimator and becomes one of the main contaminations for the detection of gravitational waves. In [37], the authors found that, for small sky surveys covering 1\% or 2\% of the sky, the mixing contamination to the covariance matrix of the \( B \)-mode power spectrum estimator typically limits the tensor-to-scalar ratio that can be probed to \( r \approx 0.05 \).

On the other hand, the pseudo-\( C_{T} \) method suggested in the present work explicitly separates the electric and magnetic types of polarization up to very small numerical errors. For this reason, the effects of mixing of the electric and magnetic modes, which are completely removed (reduced to negligible levels) in our case, do not put a limit on the ability to detect gravitational waves. This is the main advantage of our method, and it is the main motivation for this paper.

At the end of the subsection, we would like to point out that, if one proceeds to construct an unbiased estimator for \( C_{BB}^{\text{RE}} \) using the \( B_{\text{rec}}(\hat{\chi})W(\hat{\chi}) \) field [instead of the \( B_{\text{rec}}(\hat{\chi}) \) field used above], the resulting unbiased estimator will be equivalent to the “pure \( B \)-mode” estimator defined in [40]. This has been discussed in Appendix C. On the other hand, if one constructs the unbiased estimator for \( C_{BB}^{\text{RE}} \) using \( \tilde{B}(\hat{\chi}) \) [instead of \( B_{\text{rec}}(\hat{\chi}) \)] adopting a top-hat window function [instead of \( W(\hat{\chi}) \)], one would return to the \( B \)-mode estimator defined in [44]. As was pointed out in [44], the resulting estimator suffers from large \( EB \) mixing at the edge of the observed field.

### C. Information loss due to edge removal

As was emphasized in Sec. III, in practical calculations, the edge of the partial sky map has to be removed in order to reduce numerical errors. The edge removal leads to the partial loss of information. In this subsection we study the impact of this information loss on the performance of the \( B \)-mode of the polarization power spectrum estimator \( P_{bb}^{BB} \).

In order to study the performance of the estimator \( P_{bb}^{BB} \) in an ideal case with no edge removal, we perform the following steps:

1. We generate 1000 full-sky \( (Q,U) \) maps, for a cosmological model with \( r = 0.1 \) and contributions from cosmic lensing. For each of these maps we calculate the multipole coefficients \( B_{\ell m} \) using (2)–(4).

2. With the multipole coefficients \( B_{\ell m} \) we construct the full-sky map \( \mathcal{B}(\hat{\chi}) \) using (12).

3. We construct the top-hat mask window function \( W(\hat{\chi}) \) equal to unity for \( \theta < \theta_{0} = 20^\circ \) and zero otherwise. We now construct the masked magnetic field \( \mathcal{B}_{\text{rec}}(\hat{\chi}) = \mathcal{B}(\hat{\chi}) W(\hat{\chi}) \). The masked field \( \mathcal{B}_{\text{rec}}(\hat{\chi}) \) constructed in this manner corresponds to a reconstructed magnetic field map in an idealized case with no edge removal in the absence of numerical errors.

4. Working with the field \( \mathcal{B}_{\text{rec}}(\hat{\chi}) \), following the steps outlined in Secs. IV A and IV B, we build the unbiased estimator \( P_{bb}^{BB} \) and calculate its covariance matrix. The resulting estimator is equivalent to one that could be constructed in an ideal situation, without numerical errors, in which we could have worked without edge removal.

The resulting averaged value for the estimator (blue dots) and the corresponding error bars (blue error bars) are plotted in Fig. 10. Once again, we find that the average values of the estimators are practically coincident with the theoretical (input) values. In both panels, the blue error bars are slightly smaller than the red ones for all multipole bins. The difference reflects the loss of information due to edge removal. In Fig. 11, we plot the ratio of the two error bars as a function of the multipole bin. This ratio is almost everywhere less than 1.2. The signal-to-noise ratio (48) calculated for the ideal case is \( S/N = 9.59 \), which is less than 15\% higher than the practically relevant example considered in the previous subsection. The results of this section demonstrate that the loss of information, gauged by the increase in the error bars of the spectral estimator, is sufficiently small, \( \approx 15\% \).

### D. Power spectrum estimators in the presence of instrumental noise

In the previous subsections we have considered a situation in which the magnetic type of polarization was generated solely by gravitational waves and cosmic lensing. In realistic observations, in addition to these two
where $\mathcal{N}_{\ell}^{BB}$ is the pseudo estimator for the full-sky noise power spectrum $N_{\ell}^{BB}$. The expectation value of this noise estimator is

$$\langle \mathcal{N}_{\ell}^{BB} \rangle = \sum_{\ell'} M_{\ell\ell'}^2 N_{\ell'}^{BB}.$$  \hfill (50)

The presence of noise leads to a redefinition of the unbiased estimator $P_{bb}^{BB}$,

$$P_{bb}^{BB} = \sum_{\ell'} M_{\ell\ell'}^{-1} \sum_{\ell} p_{\ell'}(D_{\ell} - \langle \mathcal{N}_{\ell}^{BB} \rangle),$$  \hfill (51)

with matrices $p_{\ell'}$ and $M_{bb'}$ given in (44) and (45). The covariance matrix for this estimator has the form given by (47), where $\langle \Delta D_{\ell} \Delta D_{\ell'} \rangle$ are calculated from the right side of (40), with $B_{\ell\ell'}^{2C_{BB}}$ terms replaced by $(B_{\ell\ell'}^{2C_{BB}} + N_{\ell\ell'})$.

The estimator $P_{bb}^{BB}$ defined in (51) is an unbiased estimator for the $B$ mode of the polarization power spectrum $\ell(\ell + 1)C_{\ell}^{BB}/2\pi$, where $C_{\ell}^{BB}$ contains contributions from both gravitational waves (gw) and cosmic lensing (lens),

$$C_{\ell}^{BB} = C_{\ell}^{BB}(\text{gw}) + C_{\ell}^{BB}(\text{lens}).$$

However, if we are primarily interested in detection of gravitational waves, we can treat the cosmic lensing contribution as effective noise, and define an unbiased estimator for the $B$-mode power spectrum due to gravitational waves $\ell(\ell + 1)C_{\ell}^{BB}(\text{gw})/2\pi$ as

$$P_{bb}^{BB}(\text{gw}) = \sum_{\ell'} M_{\ell\ell'}^{-1} \sum_{\ell} p_{\ell'}(D_{\ell} - \langle \mathcal{N}_{\ell}^{BB} \rangle),$$  \hfill (52)

where the effective noise term $\langle \mathcal{N}_{\ell}^{BB} \rangle$ contains contributions from instrumental noises and cosmic lensing,

$$\langle \mathcal{N}_{\ell}^{BB} \rangle = \sum_{\ell'} M_{\ell\ell'}^2 (B_{\ell\ell'}^{2C_{BB}(\text{lens})} + N_{\ell\ell'}).$$  \hfill (53)

The covariance matrix for this estimator is the same as that calculated for the estimator in (51).

FIG. 11. The ratio of practically achievable error bars (red error bars in Fig. 10) to the corresponding error bars in an information lossless case (blue error bars in Fig. 10), as a function of the multipole $\ell$.

contributions, there are contaminating contributions from various other sources like instrumental noise and astrophysical foregrounds. This said, it is reasonable to assume that, for an appropriate choice of observed sky region, the astrophysical foregrounds are typically expected to be small in comparison with instrumental noises [5]. For this reason, we shall ignore the foreground contaminations and restrict our analysis to the study of power spectrum estimators in the presence of only instrumental noises.

The pseudo estimator $D_{\ell}$ (37) has the following expectation in the presence of noise [compared with the no noise case (38)],

$$\langle D_{\ell} \rangle = \sum_{\ell'} M_{\ell\ell'} N_{\ell'}^{2} B_{\ell\ell'}^{2C_{BB}} + \langle \mathcal{N}_{\ell}^{BB} \rangle.$$  \hfill (49)

FIG. 12 (color online). The averaged values and simulated error bars of the unbiased estimators for the power spectra $\ell(\ell + 1)C_{\ell}^{BB}/2\pi$ (green dots and larger green error bars) and $\ell(\ell + 1)C_{\ell}^{BB}(\text{gw})/2\pi$ (red dots and larger red error bars). In both panels, the solid lines denote the theoretical values for these power spectra. In this figure, we have considered the instrumental noise for the QUIET experiment. The left panel shows the results for an input cosmological model with $r = 0.1$, while the right panel shows the results for an $r = 0.01$ model. In both panels, the smaller error bars calculated using the analytical approximation (54) are plotted in grey.
E. Expected performance of ground-based CMB experiments

In this subsection we shall investigate the prospects of detecting the \( B \)-mode signature from relic gravitational waves by two future ground-based experiments, QUIET [14] and POLARBEAR [13].

Let us first consider the QUIET experiment. We shall restrict our analysis to the 40 GHz frequency channel. The FWHM for the Gaussian beam at this channel is \( \theta_f = 23' \), and the expected instrumental noise is \( N^\text{BB}_f = 2.72 \times 10^{-7} \mu \text{K}^2 \) [14] (see also [60]). We shall assume that the experiment will observe \( f_{\text{sky}} = 3\% \), corresponding to \( \theta_0 = 20' \). Following the steps outlined in Secs. IVA and IVB, using the experimental characteristics for the QUIET experiment, we construct the unbiased estimators \( p_b^{BB} \) and \( b_b^{BB}(\text{gw}) \) and their covariance matrices for 1000 realizations with \( r = 0.1 \) and \( r = 0.01 \). The average values for the estimators and their corresponding error bars are plotted in Fig. 12, for \( r = 0.1 \) (left panel) and \( r = 0.01 \) (right panel). The error bars in this case are larger than the error bars in Fig. 10 due to larger instrumental noise in comparison with QUIET. The error bars for the POLARBEAR experiment are considerably larger than those in Fig. 12 (and Fig. 10) due to larger instrumental noise in comparison with QUIET. Finally, we calculate the signal-to-noise ratio for the POLARBEAR experiment to be \( S/N = 4.31 \) for a model with \( r = 0.1 \).

We now turn to the POLARBEAR experiment. Once again, we restrict our study to the performance of the best frequency channel at 150 GHz. The FWHM for the Gaussian beam is \( \theta_f = 4' \), and the expected instrumental noise is \( N^\text{BB}_f = 4.22 \times 10^{-6} \mu \text{K}^2 \) [13]. As above, we assume \( f_{\text{sky}} = 3\% \). In order to study the performance of POLARBEAR, we simulate 1000 realizations of \((Q, U)\) maps with \( r = 0.1 \). Before proceeding to construct the power spectrum estimators, one should notice that, in comparison with QUIET, the value of \( \theta_f = 4' \) for POLARBEAR is substantially smaller. Thus, in order to avoid leakage from higher multipole electric-type polarization, we first apply the “map smoothing” procedure outline in Appendix D. Following this, we construct the estimators \( p_b^{BB} \) and \( b_b^{BB}(\text{gw}) \) and their covariance matrices following the steps explained in Secs. IVA and IVB. In Fig. 13 we plot the average values of the estimators and their error bars. The error bars for the POLARBEAR experiment are considerably larger than those in Fig. 12 (and Fig. 10) due to larger instrumental noise in comparison with QUIET. Finally, we calculate the signal-to-noise ratio for the POLARBEAR experiment to be \( S/N = 4.31 \) for a model with \( r = 0.1 \).

It is worth pointing out that, although in our estimations above we relied on the performances of a single best frequency channel for QUIET and POLARBEAR, these experiments will be observed in several frequency channels. Combining data from several frequency channels will have the effect of reducing the total effective instrumental noise. In addition, these experiments could potentially observe larger portions of the sky. Both these points could potentially increase the detection ability of these experiments. On the other hand, one should remember that various foregrounds [63] and systematic errors [64] would increase the effective noise, thereby reducing the detection ability. One should remember these caveats when looking at various signal-to-noise estimates, including the ones presented above.

At the end of this subsection we shall briefly discuss a widely used analytical approximation for the signal-to-noise ratio. In this approximation \( S/N \propto \sqrt{f_{\text{sky}}} \), where \( f_{\text{sky}} \) is the sky-cut factor. This approximation follows from the following considerations. In the case of full-sky coverage, one can construct an unbiased estimator \( D^{XX}_\ell \) (where \( X = T, E, \) or \( B \)) for the various power spectra \( C^{XX}_\ell \) in a straightforward manner (see, for example, [35,65] for details). In this case the covariance matrix is diagonal with

\[
\sqrt{\Delta D^{XX}_\ell \Delta D^{XX}_\ell} = \sqrt{\frac{2}{2\ell + 1} (C^{XX}_\ell + N^{XX}_\ell B^{-1}_\ell)},
\]

where \( 2\ell + 1 \) plays the role of the number of degrees of freedom for a given multipole \( \ell \). The above expression was elegantly extrapolated for the temperature anisotropy power spectrum estimator \( D^{TT}_\ell \) to partial sky surveys in [66]. The author proposed to replace \( (2\ell + 1) \) with the effective number of degrees of freedom, \( (2\ell + 1)f_{\text{sky}} \), in the above expression, to account for the loss of information that arises due to partial sky coverage. This simple consideration was extended to the \( B \)-mode power spectrum esti-
mator in [67], and was further extended to account for multipole binning [68]. These approximations lead to

\[ \langle \Delta P_{bb}^{gw}(\ell) \rangle = \frac{2}{(2\ell + 1)\Delta f_{\text{sky}}(\ell(\ell + 1)/2\pi)} \times (C_{bb}^{B} + N_{bb}^{B}\beta_{\ell}^{-2}), \quad (54) \]

with \( \ell \) being the central multipole in each bin. In this approximation, the total signal-to-noise ratio for the gravitational wave signal in the \( B \) mode of polarization takes the form

\[ S/N = \sqrt{\sum_{\ell} \left( \frac{\langle P_{bb}^{gw}(\ell) \rangle}{\langle \Delta P_{bb}^{gw}(\ell) \rangle} \right)^2}. \quad (55) \]

In order to gauge the performance of this approximation, in Figs. 11 and 12, we plot the error bars calculated using (54) (grey error bars). For this calculation we have set \( f_{\text{sky}} = 0.024 \), corresponding to an effective top-hat window function with \( \theta_0 = 18^\circ \). One can see that, in both figures, the analytical approximation leads to smaller error bars than those obtained from numerical simulations. We use (55) to calculate the analytical signal-to-noise ratio for the two considered experiments. The results for the signal-to-noise ratio are summarized in Table I. It can be seen that the analytical approximation for the signal-to-noise ratio (55) considerably overestimates the detection ability, particularly for smaller values of actual \( S/N \). Several works [39, 69] have pointed out that the analytical approximation (55) exaggerates the detection ability. However, these papers argued that the primary reason for overestimation is due to the omission of possible contaminations from \( EB \) mixing. However, our approach shows that the analytical approximation (55) with an effective sky-cut factor also overvalues the \( S/N \) in comparison to the case with no \( EB \) mixing. One should therefore use this approximation with caution [70]. At the same time, it is very important to point out that this conclusion about overestimation is based on the analysis of small sky coverage and the use of pseudo-\( C_{\ell} \) estimators with the uniform weight function \( W(\hat{\gamma}) \). In contrast, for large scale surveys [54] or small scale surveys using the pseudo-\( C_{\ell} \) estimators with the optimal choice of the weight function \( W(\hat{\gamma}) \) [41], the conclusion might change. In particular, for the large scale surveys and maximum likelihood estimators, the discussed analytical approximation may underestimate the true \( S/N \), as was shown for temperature anisotropy in [54].

### V. CONCLUSION

In this paper we have proposed a new method to construct pure electric- and magnetic-type fields \( E(\hat{\gamma}) \) and \( B(\hat{\gamma}) \) from the polarization field given on an incomplete sky. Because of the differential definitions of these fields, we avoid the so-called \( EB \)-mixing problem. In practice, when working with pixelized maps, residual leakages from various numerical errors require the removal of data from a narrow edge on the boundary of the observed sky. This leads to a minor loss of information in comparison with the idealized lossless case considered in Sec. IV C.

A major advantage of our approach is that the constructed fields \( E(\hat{\gamma}) \) and \( B(\hat{\gamma}) \) are scalar fields. For this reason, the various techniques developed for the analysis of temperature anisotropy maps can be directly applied to these fields. As an important and motivating application, we discuss the construction of an unbiased estimator for the \( B \)-mode power spectrum \( C_{bb}^{E} \), using the pseudo-\( C_{\ell} \) estimator approach. We find that our method is computationally feasible even in the case of high resolution maps. In particular, it takes 2.5 min on a laptop (2.4 GHz processor and 2 GB memory) to perform all of the calculations, including the calculation of \( \tilde{B}(\hat{\gamma}) \) in pixel space with \( N_{\text{side}} = 512 \) and the construction of unbiased estimators for \( C_{bb}^{E} \).

With the help of the constructed unbiased estimator, we have investigated the ability to detect gravitational waves through the \( B \) mode of polarization in CMB experiments covering 3% of the sky. In the absence of instrumental noise, we find \( S/N = 8.26 \) for a model with \( r = 0.1 \). This value is 14% smaller than an idealized situation with no information loss. In the case of realistic experiments, the signal-to-noise ratio reduces to \( S/N = 7.05 \) for QUIET and \( S/N = 4.31 \) for POLARBEAR.

In conclusion, we would like to point out that a similar analysis can be applied to large sky surveys. In particular, for the Planck satellite and the planned CMBPol experiment, one can construct unbiased estimators for the polarization power spectra \( C_{\ell}^{EE} \) and \( C_{\ell}^{BB} \), by synthesizing the approach outlined in this paper together with the hybrid estimator method suggested in [54]. We leave this task for future work.
ollowing way. One first defines the multiple expansion coefficients $W_{\ell m}$ in the standard way,

$$W_{\ell m} = \int W(\hat{\gamma}) Y_{\ell m}(\hat{\gamma}) d\hat{\gamma}.$$ 

Following this, one calculates

$$W_\ell(\hat{\gamma}) = \frac{\partial W}{\partial \theta} = \sum_{\ell m} W_{\ell m} \left( \frac{\partial}{\partial \theta} Y_{\ell m}(\hat{\gamma}) \right)$$

$$= \sum_{\ell m} W_{\ell m} \left( \ell \frac{\partial Y_{\ell m}(\hat{\gamma})}{\partial \theta} - \frac{1}{\sin \theta} \right) \times \sqrt{\frac{2\ell + 1}{2\ell - 1}} (\ell^2 - m^2) Y_{\ell - 1 m}(\hat{\gamma}).$$

The other quantities can be calculated in an analogous manner. It is important to point out that the steps mentioned above can be realized in a straightforward manner using the anafast and synfast routines in the HEALPix package.

APPENDIX B: CONSTRUCTION OF THE MAGNETIC MAP $B_{\text{rec}}(\hat{\gamma})$ FROM SIMULATED POLARIZATION MAPS

In this appendix, we outline the steps which were used to simulate the polarization maps and construct the pure magnetic map $B_{\text{rec}}(\hat{\gamma})$.

1. We generate the mask window function $W(\hat{\gamma})$ using (27) in pixel space using the standard pixelization scheme used in HEALPix with $N_{\text{side}} = 512$ (or $N_{\text{side}} = 1024$ in the example in Sec. III B).

2. Using the synfast HEALPix routine, we generate full-sky $(Q(\hat{\gamma}), U(\hat{\gamma}))$ maps with $N_{\text{side}} = 512$ or 1024, using cosmological parameters (30) and an appropriate value of the tensor-to-scalar ratio $r$ as input. Using the window function $W(\hat{\gamma})$, we build the masked $(\hat{Q}(\hat{\gamma}), \hat{U}(\hat{\gamma}))$ maps (where $\hat{Q} = QW$ and $\hat{U} = UW$).

3. Using the anafast HEALPix routine, we calculate the coefficients $(\hat{E}_{\ell m}, \hat{B}_{\ell m})$. The field $\hat{B}(\hat{\gamma})$ is calculated from $\hat{B}_{\ell m}$ according to (19) using the synfast routine.

4. With the coefficients $(\hat{E}_{\ell m}, \hat{B}_{\ell m})$ we construct the fields $QW(\hat{\gamma}), UW(\hat{\gamma}), (UW)(\hat{\gamma})$, and $(QW)(\hat{\gamma})$, using the 5th option in the synfast routine.

5. Using the fields $UW(\hat{\gamma}), (UW)(\hat{\gamma})$, and $(QW)(\hat{\gamma})$ constructed in the previous step and the analytical expressions for $W, W_x$, and $W_{xx}$ [derived by differentiating (27)], we calculate Im[ct] in (29).

6. The pure magnetic field $B_{\text{rec}}(\hat{\gamma})$ is now constructed from $\hat{B}, W$ and Im[ct] using (32). The pure magnetic field $B_{\text{rec}}(\hat{\gamma})$ is truncated at the edges in order to remove residual leakages associated with numerical errors.
APPENDIX C: PSEUDO ESTIMATORS FOR A SPECIAL CHOICE OF WEIGHT FUNCTION 
\[ W(\hat{\gamma}) = W(\gamma) \]

In Sec. IVA it was pointed out that, in principle, one can construct pseudo estimators of the power spectrum by adopting an arbitrary weight function \( W(\gamma) \) in (33). Above, in the main text, we have focused on a specific case corresponding to a uniform weight function \( W(\gamma) = 1 \). This choice is nearly optimal for high multipoles. However, this choice becomes suboptimal at lower multipoles [41]. In this appendix we study another possible choice for the weight function, namely, \( W(\gamma) = W(\gamma) \), where \( W(\gamma) \) is the mask window function in (27) with \( \theta_0 = 20^\circ \) and \( \theta_1 = 10^\circ \). Note that the function \( W(\gamma) \) is the same window function that was used for constructing \( B_{\text{rec}}(\gamma) \). With this choice, the resulting pseudo estimator is equivalent to the pure \( B \)-mode estimator studied in [40].

The construction of the pseudo estimators and the corresponding unbiased estimators closely follows the discussion in Sec. IV. The only difference is that the definition of the coefficients \( a_{\ell m} \) in (34) are modified to

\[ a_{\ell m} = \int d\gamma B_{\text{rec}}(\gamma) W(\gamma) Y_{\ell m}^*(\gamma), \]  

(C1)

and the quantities \( w' \) and \( w'' \) in (36) and (39) would now be replaced by \( W \) and its power spectrum, respectively.

In Fig. 14, we plot the unbiased estimators for the power spectra \( \ell(\ell + 1)C_{\ell}^{BB}(\text{total})/2\pi \) (left panel) and \( \ell(\ell + 1)C_{\ell}^{BB}(\text{gw})/2\pi \) (right panel) together with the corresponding error bars (thin blue error bars). It can be seen that, in comparison with the estimators in the case of a uniform weight function, the error bars of the new estimators are larger at high multipoles, but are smaller at low multipoles.

This result is consistent with findings in [41] that the optimal weight functions for high multipoles tend to the top-hat function. On the other hand, for low multipoles, the optimal weight function tends to smooth out (see the right panel of Fig. 2 in [41] for a concrete example).

APPENDIX D: SMOOTHING THE POLARIZATION MAPS

The high value of the power spectrum \( C_{\ell}^{EE} \) of the electric component at large values of multipoles (due to the presence of the \( N^2 \) factor) leads to substantial leakage of power into the reconstructed pure magnetic field \( B_{\text{rec}} \). This leakage seeps through to low multipoles, playing the role of residual effective noise. In order to reduce this contamination, below we introduce a map smoothing procedure for polarization maps. The idea behind this method is similar to the "prewhitening" method suggested in [73].

The smoothing procedure is simple and straightforward in the case of full-sky coverage. Given the \( (Q, U) \) polarization maps on a full sky, one can calculate the multipole coefficients \( E_{\ell m} \) and \( B_{\ell m} \) using (1)–(4). In order to smooth the polarization maps, we first use a damping function to modify the multipole coefficients,

\[ E'_{\ell m} = E_{\ell m} e^{-(\ell^2\theta_0^2)/(8\ln23)}, \]

\[ B'_{\ell m} = B_{\ell m} e^{-(\ell^2\theta_0^2)/(8\ln23)}. \]  

(D1)

Following this, we reconstruct the smoothed polarization fields \( Q' \) and \( U' \) using the modified coefficients \( E'_{\ell m} \) and \( B'_{\ell m} \) in the standard way. The reconstructed maps can be thought of as the result of observing the original \( (Q, U) \) polarization field in an experiment with the FWHM of the Gaussian beam equal to \( \theta_0 \). Overall, the smoothing has the effect of exponentially damping the power in high multipoles.

We can extend the smoothing procedure to the case of partial sky coverage. Given the \( (Q, U) \) polarization maps
on a partial sky, we calculate the coefficients $\tilde{E}_{\ell m}$ and $\tilde{B}_{\ell m}$. These coefficients are smoothed in analogy with the full-sky case,
\begin{equation}
\tilde{E}_{\ell m} = E_{\ell m} e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)},
\end{equation}
\begin{equation}
\tilde{B}_{\ell m} = B_{\ell m} e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)}.
\end{equation}

The smoothed polarization maps $(Q', U')$ are reconstructed from the modified multipole coefficients $E_{\ell m}$ and $B_{\ell m}$.

It is important to point out that, in the case of partial sky coverage, the smoothing procedure outlined above introduces a mixture of electric and magnetic polarizations. In particular, even if the original $(Q, U)$ did not contain a magnetic type of polarization, the smoothed map $(Q', U')$ would contain it. We have verified numerically that, in practically interesting cases, the resulting mixture is very small, and would not significantly affect the ability to detect gravitational waves.

In order to verify the smallness of the resulting mixing, we have performed the following calculation. Using an input model with no $B$ mode of polarization (i.e. $C_\ell^BB = 0$) and $\theta_F = 30'$, we generated a full-sky $(Q, U)$ map. Following this, we truncate the map to keep the data from only the northern hemisphere. Using the procedure outlined in Secs. II and III, we construct the field $B_{\text{rec}}$, which is expected to be zero equal except for the residual leakage. The pseudo power spectrum $D_\ell$ for this field (red line) is plotted in Fig. 15. Following this, for the same input model, we generate the $(Q, U)$ map with $\theta_F = 10'$, once again restricting the data to just the northern hemisphere. We now smooth this map with $\theta_F = 30'$ using the anafast, alteralrm, and synfast HEALPix routines. We construct $B_{\text{rec}}$ from the smoothed $(Q', U')$ map and plot the corresponding pseudo power spectrum $D_\ell'$ (green line) in Fig. 15. The difference between the two spectra $D_\ell$ and $D_\ell'$ can be interpreted as the result of mixing introduced by map smoothing (blue line in Fig. 15). It can be seen that the mixing due to smoothing is quite small, comparable to leakage due to numerical errors, at all the relevant multipoles. The power spectrum of the leakage due to smoothing peaks at $\ell \sim 400$. It can be completely neglected at low multipoles ($\ell \leq 50$).

Below we give a heuristic argument to understand these results. The two sets of multipole coefficients, $(\tilde{E}_{\ell m}, \tilde{B}_{\ell m})$ and $(E_{\ell m}, B_{\ell m})$, are related by the following expression (see, for instance, [37]),
\begin{equation}
\tilde{E}_{\ell m} + i\tilde{B}_{\ell m} = \sum_{\ell' m'} (2 I_{(\ell m)(\ell' m')}) [E_{\ell' m'} + iB_{\ell' m'}].
\end{equation}

where $2 I_{(\ell m)(\ell' m')}$ is the coupling matrix, which depends only on the mask window function. From this relation, it formally follows that
\begin{equation}
E_{\ell m} + iB_{\ell m} = \sum_{\ell' m'} (2 I^{-1})_{(\ell m)(\ell' m')} [\tilde{E}_{\ell' m'} + i\tilde{B}_{\ell' m'}].
\end{equation}

For the smoothed multipole coefficients one has
\begin{equation}
E_{\ell m} + iB_{\ell m} = \sum_{\ell' m'} (2 I^{-1})_{(\ell m)(\ell' m')} [\tilde{E}_{\ell' m'} + i\tilde{B}_{\ell' m'}] = \sum_{\ell' m'} (2 I^{-1})_{(\ell m)(\ell' m')} \tilde{E}_{\ell' m'} + i\tilde{B}_{\ell' m'} e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)}.
\end{equation}

Since the coupling matrix $(2 I_{(\ell m)(\ell' m')})$ is sharply peaked at $\ell = \ell'$ [37], the above expression can be approximated by
\begin{equation}
E_{\ell m} + iB_{\ell m} = \sum_{\ell' m'} (2 I^{-1})_{(\ell m)(\ell' m')} \tilde{E}_{\ell' m'} + i\tilde{B}_{\ell' m'} \times e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)} = (E_{\ell m} + iB_{\ell m}) e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)}.
\end{equation}

It therefore follows that
\begin{equation}
E_{\ell m} = E_{\ell m} e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)},
\end{equation}
\begin{equation}
B_{\ell m} = B_{\ell m} e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)}.
\end{equation}

Thus, in this approximation, there is no mixing introduced by smoothing. Moreover, $e^{-\left(1/2\right)\left(|\ell^2|\ell(\theta_F^2)/(8 \ln 2)\right)} = 1$ for $\ell' \ll 1/\theta_F$, leading to $E_{\ell m} = E_{\ell m}$, $B_{\ell m} = B_{\ell m}$ for low multipoles. In reality the coupling matrix $(2 I_{(\ell m)(\ell' m')})$ is often noninvertible, and therefore (D4) might not be valid. However, the resulting leakage is still small, and completely negligible for $\ell \ll 1/\theta_F.$


[70] In [58], the authors found that formulas similar to (54) and (55) can over-value the detection of the temperature anisotropy power spectrum. However, the authors argued that this over-valuation could be corrected by using a filter function when building the unbiased estimators. We expect that a similar analysis can be applied in our method.

[71] E. Hivon (private communication).


