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Improved calculation of relic gravitational waves

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In this paper, we have improved the calculation of the relic gravitational waves (RGW) in two aspects. First, we investigate the transfer function by taking into consideration the redshift-suppression effect, the accelerating expansion effect, the damping effect of free-streaming relativistic particles, and the damping effect of cosmic phase transition, and give a simple approximate analytic expression, which clearly illustrates the dependence on the cosmological parameters. Second, we develop a numerical method to calculate the primordial power spectrum of RGW in a very wide frequency range, where the observed constraints on $n_{\rm s}$ (the scalar spectral index) and $P_{\rm S}(k_0)$ (the amplitude of primordial scalar spectrum) and the Hamilton–Jacobi equation are used. This method is applied to two kinds of inflationary models, which satisfy the current constraints on $n_{\rm s}$, α (the running of $n_{\rm s}$) and r (the tensor–scalar ratio). We plot them in the $r-\Omega_{\rm g}$ diagram, where $\Omega_{\rm g}$ is the strength of RGW, and study their measurements from the cosmic microwave background (CMB) experiments and laser interferometers.

Keywords: relic gravitational waves, inflation
PACC: 0430, 9880

1. Introduction

Recently, many observations on the cosmic microwave background (CMB) power spectra[1−9] and the large scale structure (LSS)[10−12] have supported inflation as a good phenomenological model of describing the evolution of the universe at very early stage, which naturally answers the origin of the primordial fluctuations with a nearly scale-invariant and gaussian spectrum. In addition to the density perturbations, the inflationary models also predict a stochastic background of relic gravitational waves (RGW), which is also called the tensor perturbation. The amplitude of RGW is directly related to the energy scale of inflation. Although this background has not yet been observed, and only some loose constraints have been achieved,[3−5,13−15] its detection can provide incontrovertible evidence that the inflation has actually occurred and can set strong constraints on the dynamic of inflation.[16−20] So it is always regarded as the ‘smoking-gun’ evidence for the inflation.

There are two main kinds of experiments that are underway to detect the RGW at different frequencies. The first kind of experiment is the CMB experiment, which can find the RGW by observing the CMB B-polarization power spectrum.[21−25] This method is sensitive to the waves with very low frequencies, $\nu \in (10^{-17},\ 10^{-15})$Hz. Now, the first-three-year results of Wilkinson microwave anisotropy probe (WMAP)[3] have not revealed evidence of the gravitational waves, but have only given a constraint $r < 0.28$ (95% confidence limit), where r is the so-called tensor–scalar ratio. The next experiment, the Planck satellite[26], has a higher sensitivity to polarization, is scheduled for launch in 2007, and is expected to be able to observe the RGW if $r > 0.1$. The ground-based experiment, Clover (Cl-Observer), which is a bolometric CMB polarization imaging experiment, is also under development[27], and is expected to be able to observe the RGW if $r > 0.005$. The second kind of experiment is laser interferometry, including the BBO (Big Bang Observer)[28,29] and DECIGO (Deci-hertz Interferometer Gravitational Wave Observatory),[30] which can detect the gravitational waves with very high frequencies $\nu \sim 0.1$Hz. The former can detect the RGW when $\Omega_{\rm g} > 2\times10^{-17}$ holds true, where $\Omega_{\rm g}$ is the strength of RGW at 0.1Hz, and the latter is expected to be able to observe the RGW if $\Omega_{\rm g}h^2 > 10^{-20}$. It should be noticed that the waves with very high frequencies can be observed by an electromagnetic resonant system.[31−35] This is also an important method to detect the relic gravitational waves.

A lot of studies of the RGW detection by these experiments have been carried out.[36−38] In the previous work,[39] we have discussed the predicted values of RGW (r and $\Omega_{\rm g}$) for some kinds of inflationary mod-

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els, where we have used a simple power-law function to describe the primordial power spectrum of RGW, which is a very good approximation for the waves with very low frequencies, but for the waves with high frequencies, this may generate a large error. In that work, we have not considered the damping effect of cosmic phase transition on the RGW, such as the quark confined into hadrons (QCD transition).[40−44] e^+e^- annihilation and so on. In this paper, we discuss this topic more precisely. First, we consider the damping effect of a general cosmic phase transformation, which can be described by a simple damping factor. And then we give a simple form of the total transfer function, which applies to the waves with $\nu \gg 10^{-16}$ Hz. This function is dependent on the values of Ω_Λ and $\Omega_{\rm m}$ (the present energy densities of vacuum and matter, respectively), the value of τ_0 (the age of the universe), the value of H_0 (the present Hubble constant), the values of g_* and $g_{*{\rm s}}$ (the effective number of relativistic degrees of freedom when the waves exactly crossed the horizon), and the fraction f of the background (critical) energy density of the free-streaming relativistic particles in the universe when the waves exactly crossed the horizon. So this function includes abundant cosmic information. Second, we use a numerical method to calculate the primordial power spectrum of RGW, where the Hamilton–Jacobi formula is used. Compared with the result from the simple power-law form, this numerical result has little change in the value of $\Omega_{\rm g}$ when r is smaller, say, $r < 0.02$. But when the value of r is larger, the numerical result is obviously smaller than that from the simple power-law approximation.

The rest of this paper is organized as follows: in Section 2, we simply review the evolutive equation of the RGW. In Section 3, we mainly discuss the damping effects. In Section 4, we introduce the numerical method by discussing two kinds of inflationary models. Finally, we give a conclusion and discussion in Section 5.

2. The relic gravitational waves

Incorporating the perturbation into the spatially flat Robertson–Walker (FRW) spacetime, the metric is

$$ {\rm d}s^2 = a(\tau)^2[{\rm d}\tau^2 - (\delta_{ij} + h_{ij}){\rm d}x^i{\rm d}x^j] \ , \qquad (1) $$

where a is the scale factor of the universe, τ is the conformal time, which is related to the cosmic time by $a{\rm d}\tau \equiv {\rm d}t$. The perturbation of spacetime h_{ij} is a 3×3 symmetric matrix. The gravitational wave field is the tensorial portion of h_{ij}, which is transverse-traceless $\partial_i h^{ij} = 0$, and $\delta^{ij}h_{ij} = 0$. Since the gravitational waves are very weak, $|h_{ij}| \ll 1$, one needs to study just the following linearized evolutive equation:

$$ \partial_\mu(\sqrt{-g}\partial^\mu h_{ij}) = 16\pi G a^2(\tau)\Pi_{ij} \ , \qquad (2) $$

where Π_{ij} is the tensor part of the anisotropy stress, which satisfies $\Pi_{ii} = 0$ and $\partial_i \Pi_{ij} = 0$, and couples with h_{ij} like an external source in this equation, which is always generated by the free-streaming relativistic particles,[45−47] the cosmic magnetic,[48,49] and so on. It is convenient to perform the Fourier transform as follows:

$$ h_{ij}(\tau,\boldsymbol{x}) = \sum_\lambda \sqrt{16\pi G}\int \frac{{\rm d}\,\boldsymbol{k}}{(2\pi)^{3/2}}\epsilon_{ij}^{(\lambda)}(\boldsymbol{k})h_{\boldsymbol{k}}^{\lambda}(\tau){\rm e}^{{\rm i}\boldsymbol{kx}}, \quad (3) $$

$$ \Pi_{ij}(\tau,\boldsymbol{x}) = \sum_\lambda \sqrt{16\pi G}\int \frac{{\rm d}\,\boldsymbol{k}}{(2\pi)^{3/2}}\epsilon_{ij}^{(\lambda)}(\boldsymbol{k})\Pi_{\boldsymbol{k}}^{\lambda}(\tau){\rm e}^{{\rm i}\boldsymbol{kx}}, \quad (4) $$

where λ = '$+$' and '$\times$' denote the two polarization states of the gravitational waves. The polarization tensors are symmetries, transverse-traceless $k^i\epsilon_{ij}^{(\lambda)}(\boldsymbol{k}) = 0$, and $\delta^{ij}\epsilon_{ij}^{(\lambda)}(\boldsymbol{k}) = 0$, and satisfy the conditions $\epsilon^{(\lambda)ij}(\boldsymbol{k})\epsilon_{ij}^{(\lambda')}(\boldsymbol{k}) = 2\delta_{\lambda\lambda'}$ and $\epsilon_{ij}^{(\lambda)}(-\boldsymbol{k}) = \epsilon_{ij}^{(\lambda)}(\boldsymbol{k})$. Since the RGW under consideration is isotropic, and polarization states each are the same, $h_{\boldsymbol{k}}^{(\lambda)}(\tau)$ can be denoted by $h_k(\tau)$, and $\Pi_{\boldsymbol{k}}^{(\lambda)}(\tau)$ by $\Pi_k(\tau)$, where $k = |\boldsymbol{k}|$ is the wavenumber of the gravitational wave, which is related to the frequency by $\nu \equiv k/2\pi$ (the present scale factor is set to be $a_0 = 1$). So Eq.(2) can be rewritten as

$$ \ddot{h}_k + 2\frac{\dot{a}}{a}\dot{h}_k + k^2 h_k = 16\pi G a^2(\tau)\Pi_k(\tau) \ , \qquad (5) $$

where the overdot indicates a conformal time derivative ${\rm d}/{\rm d}\tau$.

The RGW was generated during the early inflation stage. The inflation is an extremely attractive idea of describing the very early universe, which has received strong supports from the observations of CMB anisotropies and from the studies of the large-scale distribution of galaxy. In this paper, we consider only the simplest single field model. This kind of model is enough to account for the current observations on $n_{\rm s}$, α, and r. In the context of slow-roll inflationary model, the most observables depend on three slow-roll parameters:[50,51]

$$ \epsilon_{_V} \equiv \frac{M_{\rm Pl}^2}{2}\Big(\frac{V'}{V}\Big)^2 \ , \quad \eta_{_V} \equiv M_{\rm Pl}^2\Big(\frac{V''}{V}\Big) \ , $$
$$ \text{and} \ \ \xi_{_V} \equiv M_{\rm Pl}^4\Big(\frac{V'V'''}{V^2}\Big) \ , \qquad (6) $$

where $M_{\mathrm{Pl}} \equiv (8\pi G)^{-1/2} = m_{\mathrm{Pl}}/\sqrt{8\pi}$ is the reduced Planck energy. In the following discussion, we use the units of $M_{\mathrm{Pl}} \equiv 1$ and $m_{\mathrm{Pl}} = \sqrt{8\pi}$. $V(\phi)$ is the inflationary potential, and prime denotes derivative with respect to the field ϕ. Here, $\epsilon_{_V}$, $\eta_{_V}$ and $\xi_{_V}$ quantitate the 'steepness' of the slope of the potential, the 'curvature' of the potential, and the 'jerk' respectively. All the parameters must be smaller than one for the inflation to occur. The most important prediction of the inflationary model is the primordial scalar perturbation power spectrum, which is almost gaussian and nearly scale-invariant. This spectrum is always written in the form

$$P_{\mathrm{S}}(k) = P_{\mathrm{S}}(k_0)\Big(\frac{k}{k_0}\Big)^{n_{\mathrm{s}}(k_0)-1+\frac{1}{2}\alpha\ln(k/k_0)} , \quad (7)$$

where $\alpha \equiv \mathrm{d}n_{\mathrm{s}}/\mathrm{d}\ln k$, and k_0 is some pivot wavenumber. In this paper, $k_0 = 0.002\ \mathrm{Mpc}^{-1}$ is used. The observations of WMAP indicate $P_{\mathrm{S}}(k_0) \simeq 2.95 \times 10^{-9} A(k_0)$ and $A(k_0) = 0.813^{+0.042}_{-0.052}$.[3] Another key prediction of inflationary model is that the existence of the RGW. The primordial power spectrum of RGW is defined as

$$P_{\mathrm{T}}(k) \equiv \frac{32Gk^3}{\pi} h_k^+ h_k \ . \quad (8)$$

The strength of the gravitational wave is characterized by their energy spectrum

$$\Omega_{\mathrm{g}}(k) = \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{\mathrm{g}}}{\mathrm{d}\ln k} \ , \quad (9)$$

where $\rho_c = 3H_0^2/8\pi G$ is the critical density and $H_0 = 100h$ km s^{-1} Mpc^{-1} is the present Hubble constant. One can relate Ω_{g} to the primordial power spectrum by the following formula:[36,47]

$$\Omega_{\mathrm{g}}(k) = \frac{1}{12H_0^2} k^2 P_{\mathrm{T}}(k) T_{\mathrm{f}}^2(k) \ , \quad (10)$$

where the transfer function $T_{\mathrm{f}}(k)$ reflects the damping effect of the gravitational wave when evolving in the expansion universe. It is convenient to define a function $\mathcal{T}(k) \equiv k^2 T_{\mathrm{f}}^2/12H_0^2$, so the strength of RGW becomes $\Omega_{\mathrm{g}}(k) = \mathcal{T}(k) P_{\mathrm{T}}(k)$. In the following sections, we discuss $\mathcal{T}(k)$ and $P_{\mathrm{T}}(k)$, separately.

3. The transfer function

In this section, we discuss three kinds of damping effects. First we ignore the anisotropies stress in Eq.(5), and consider only the redshift-suppression effect. So Eq.(5) becomes

$$\ddot{h}_k + 2\frac{\dot{a}}{a}\dot{h}_k + k^2 h_k = 0 \, . \quad (11)$$

This is the evolutive equation of RGW in vacuum, which only depends on the evolution of the scale factor $a(\tau)$. It is clear that the mode function of the gravitational waves behaves simply in two regimes when evolving in the universe: far outside the horizon ($k \ll aH$), and far inside the horizon ($k \gg aH$). When waves are far outside the horizon, the amplitude of h_k keeps constant, and when inside the horizon, they damp with the expansion of the universe

$$h_k \propto \frac{1}{a} \, . \quad (12)$$

In the simple cosmic model, the evolution of the universe may be separated into three stages: the radiation-dominant stage, the matter-dominant stage, and the vacuum-dominant stage. In this model, by numerically integrating Eq.(11), one has found that the transfer function can be approximately described with a damping function (for the waves with $k \gg 10^{-18}$ Hz)[39,52−55]

$$t_1(k) = \frac{3}{(k\tau_0)^2}\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\sqrt{1.0+1.36\Big(\frac{k}{k_{\mathrm{eq}}}\Big)+2.50\Big(\frac{k}{k_{\mathrm{eq}}}\Big)^2} , \quad (13)$$

where $k_{\mathrm{eq}} = 0.073\Omega_{\mathrm{m}}h^2$ Mpc^{-1} is the wavenumber corresponding to the Hubble radius at the time when matter and radiation have equal energy densities. And $\tau_0 = 1.41 \times 10^4$ Mpc is the present conformal time. Ω_{m} and Ω_Λ are the present energy densities of matter and vacuum, respectively. It is obvious that, when $k \ll k_{\mathrm{eq}}$, under which the waves have entered the horizon in the matter-dominant or vacuum-dominant stage, $t_1(k) \propto k^{-2}$, but when $k \gg k_{\mathrm{eq}}$, under which the waves have entered the horizon in the radiation-dominant stage, $t_1(k) \propto k^{-1}$, which is for the different evolutions of scale factor in different stages. The factor $\Omega_{\mathrm{m}}/\Omega_\Lambda$ is the effect of accelerating expansion, which has been discussed in the previous work.[39,53,56,57]

The second is the damping effect of the free-streaming relativistic particles,[45] especially the neutrinos, which can generate the anisotropic stress Π_k on the right-hand side of Eq.(5), when they are the free-streaming relativistic particles. This effect was first considered by Weinberg, and Eq.(5) can be rewritten as a fairly simple integro-differential equation. The solution shows that anisotropy stress can reduce the amplitude for the waves that have re-entered the horizon during the radiation-dominated stage, and the damping factor is only dependent on the fraction f of the background (critical) energy density of the free-streaming relativistic particles in the universe. The effect is less for the waves that enter the horizon at later time. A lot of work has been done on simplifying this effect, and in the work[47] the authors found that it could be approximately described by a transfer function t_2 for the waves with $\nu > 10^{-16}$ Hz (which re-enter the horizon at the radiation-dominant stage),

$$t_2 = \frac{15(14406f^4 - 55770f^3 + 3152975f^2 - 48118000f + 324135000)}{343(15+4f)(50+4f)(105+4f)(180+4f)} \ . \tag{14}$$

When the waves with frequencies in a range of 10^{-16} Hz $< \nu < 10^{-10}$ Hz re-enter the horizon, the temperature in the universe is relatively low (<1 MeV), we are fairly confident that the neutrinos are the only free-streaming relativistic particles. So we choose $f = 0.4052$, corresponding to three standard neutrino species, and the damping factor to be 0.80313. But for the waves with very high frequencies ($\nu > 10^{-10}$ Hz), the temperature of the universe is very high when they re-enter the horizon, and the value of f is much uncertain. Thus, the detection of RGW at this frequency offers the probability of learning about the free-streaming fraction f in the very early universe.

The third is the effect due to the successive changes in the relativistic degrees of freedom during the radiation-dominant stage, here we also call it the effect due to the cosmic phase transition, which includes the QCD transition, the e^+e^- annihilation, the electroweak phase transition and so on. In an adiabatic system, the entropy per unit comoving volume must be conserved,[40,58]

$$S(T) = s(T)a^3(T) = \text{constant},$$
$$\text{and}\ \ s(T) = \frac{2\pi^2}{45}g_{*\mathrm{s}}(T)T^3, \tag{15}$$

where the entropy density, $s(T)$, is given by the energy density and pressure: $s = (\rho + p)/T$. Combining it with the expressions of energy density and pressure in the radiation-dominant universe,

$$\rho(T) = \frac{\pi^2}{30}g_*(T)T^4, \quad p(T) = \frac{1}{3}\rho(T)\ . \tag{16}$$

one can immediately obtain the relation

$$\rho \propto g_* g_{*\mathrm{s}}^{-4/3} a^{-4}, \tag{17}$$

where we have defined the 'effective numbers of relativistic degrees of freedom', g_* and $g_{*\mathrm{s}}$, following the Refs.[40,58]. These quantities, g_* and $g_{*\mathrm{s}}$, count the effective numbers of relativistic species contributing to the radiation energy density and entropy, respectively. From this relation, one can find that, if the phase transitions are not considered, g_* and $g_{*\mathrm{s}}$ are both constant, and this relation turns into the general expression of $\rho \propto a^{-4}$. However, it does not always hold, as some particles become non-relativistic before the others stop contributing to the radiation energy density. In other words, the evolution of ρ during the radiation era is sensitive to how many relativistic species the universe has at a given epoch. As the equation of gravitational waves constraints $(\dot{a}/a)\dot{h}_k$, the solution of h_k can be affected by g_* and $g_{*\mathrm{s}}$ via the Friedmann equation:

$$\left(\frac{H(\tau)}{H_0}\right)^2 = \left(\frac{g_*}{g_{*0}}\right)\left(\frac{g_{*\mathrm{s}}}{g_{*\mathrm{s}0}}\right)^{-4/3}\Omega_r\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\mathrm{m}}\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda, \tag{18}$$

where the subscript 0 denotes the quantity with the present value. Here we have considered the Friedmann equation in a Lambda cold dark matter (ΛCDM) universe, which is supported by a number of observations.[1–9,59–62] Inserting this into Eq.(11), one can numerically calculate the value of h_k,[40] which can take a very long computer time, since one must integrate that equation from the end of the inflation to the present time, and calculate the waves from $\nu = 10^{-16}$Hz to 0.1 Hz which we are interested in. Here we give an approximate method, which can describe this effect by a simple factor t_3. We consider the wave h_k with the wavenumber k, which crosses the horizon at $a = a_k$, and the corresponding Hubble parameter H_k. So one has $k = a_k H_k / a_0$. It is known that when the waves are in side the horizon, $h_k(\tau) \propto 1/a(\tau)$, damping with the expansion of universe, and when the waves are out side the horizon, the h_k = constant, keeping its initial value. So one can define a factor as follows:

$$F_k \equiv \frac{h_k(\tau_0)}{h_k(\tau_i)} = \frac{a_k}{a_0} \ , \tag{19}$$

where τ_i is the conformal time at the beginning of the radiation era. During the radiation era, one has

$$H = B\Big(\frac{g_*}{g_{*0}}\Big)^{1/2}\Big(\frac{g_{*\rm s}}{g_{*\rm s0}}\Big)^{-2/3}\Big(\frac{a}{a_0}\Big)^{-2}, \tag{20}$$

where $B = H_0\Omega_r^{1/2}$, which is a constant. Using expressions (19) and (20) and the relation of $k = a_kH_k/a_0$, one obtains

$$F_k = \frac{B}{k}\Big(\frac{g_*(T_k)}{g_{*0}}\Big)^{1/2}\Big(\frac{g_{*\rm s}(T_k)}{g_{*\rm s0}}\Big)^{-2/3}, \tag{21}$$

where T_k is the temperature when the wave h_k exactly crosses the horizon. First we can assume that $g_* = g_{*0}$ and $g_{*\rm s} = g_{*\rm s0}$ are always satisfied, which is the condition without changes in the relativistic degrees of freedom during the radiation era, which follows that $\tilde{F}_k = B/k$. Inserting this into expression (10), one finds that the value of $\Omega_{\rm g}(k)$ is independent of the wavenumber k. However, here we are interested in the condition where g_* and $g_{*\rm s}$ are variable, and the factor t_3 exactly denotes the difference between the two conditions, i.e.

$$t_3 = \frac{F_k}{\tilde{F}_k} = \Big(\frac{g_*(T_k)}{g_{*0}}\Big)^{1/2}\Big(\frac{g_{*\rm s}(T_k)}{g_{*\rm s0}}\Big)^{-2/3}, \tag{22}$$

where $g_{*0} = 3.3626$ and $g_{*\rm s0} = 3.9091$. This factor depends on the values of g_* and $g_{*\rm s}$ at the early universe. Figure 1 presents the evolutions of the values of g_* and $g_{*\rm s}$, showing that the value of g_* has an obvious accretion when $T > 0.1$MeV. The difference between g_* and $g_{*\rm s}$ exists only when $T < 0.1$MeV. In the expression of $\Omega_{\rm g}(k)$, this effect is described by a factor ${t_3}^2$. Compared with the accurate numerical calculations, this approximation has an error smaller than 10%. The total transfer function is the combination of these three effects

$$T_{\rm f}(k) = t_1 \times t_2 \times t_3 \ , \tag{23}$$

where t_1 is most important, and it approximately shows the evolution of RGW in the expanding universe. The function of t_2 is most uncertain in this discussion. In one extreme condition where f and t_2 are fixed at 0 and 1 respectively, i.e. no damping, and in another extreme condition where f and t_2 are fixed at 1 and 0.59 respectively, this function arrives at its smallest value. The case of $f = 0.4052$ and $t_2 = 0.80321$ only contributes a damping factor of 0.645 to the strength of the RGW. The value of t_3 is fairly small. For the extreme condition with $T_k > 10^6$MeV ($k > 2\times10^{-4}$Hz), one has $g_* = g_{*\rm s} = 106.75$ in the Standard Model ($g_* = g_{*\rm s} = 228.75$ in the Minimal Supersymmetric Standard Model (MSSM)), the case of $t_3 = 0.62$ ($t_3 = 0.55$ in MSSM) only contributes a damping factor of 0.38 (0.30 in MSSM) to the strength of the RGW.

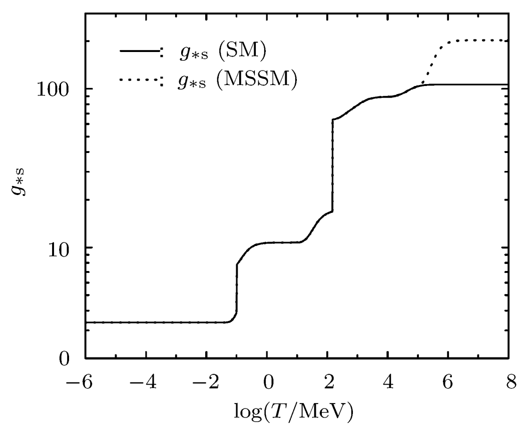

Fig.1. Evolution of g_* with temperature. The solid and dot lines represent g_* in the Standard Model (SM) and in the Minimal Supersymmetric Standard Model (MSSM), respectively. At the energy scales below $\sim$ 0.1 MeV, $g_* = 3.3626$ and $g_{*\rm s} = 3.9091$; $g_* = g_{*\rm s}$ otherwise. This figure is cited from Ref.[40].

The experiments which can directly detect the RGW are all sensitive to waves with $k \gg k_{\rm eq}$, which have re-entered the horizon during the radiation era. From the previous discussion, one can obtain a simple expression of all these damping effects

$$\mathcal{T}(k) = \Big(\frac{15}{8k_{\rm eq}^2H_0^2\tau_0^4}\Big)\Big(\frac{\Omega_{\rm m}}{\Omega_\Lambda}\Big)^2\Big(\frac{g_*(T_k)}{g_{*0}}\Big)\Big(\frac{g_{*\rm s}(T_k)}{g_{*\rm s0}}\Big)^{-4/3} \times\Big(\frac{15(14406f_k^4-55770f_k^3+3152975f_k^2-48118000f_k+324135000)}{343(15+4f_k)(50+4f_k)(105+4f_k)(180+4f_k)}\Big)^2 , \tag{24}$$

where f_k is the value of the function f when wave h_k exactly crosses the horizon. This function is dependent on the values of Ω_Λ and $\Omega_{\rm m}$ (the present energy densities of vacuum and matter, respectively), the value of τ_0 (the age of the universe), the value of H_0 (the present Hubble constant), the values of $g_*(T_k)$ and $g_{*\rm s}(T_k)$ (the effective numbers of relativistic degrees of freedom), and f_k (the fraction of the background (critical) energy density of the free-streaming relativistic particles in the universe). So this function includes abundant cosmic information. Using this, the strength of RGW becomes

$$\Omega_{\rm g}(k) = P_{\rm T}(k)\mathcal{T}(k) \ . \tag{25}$$

Here we are interested in the wave with $\nu = 0.1$ Hz, which is the sensitive frequency of laser interferometers, BBO and DECIGO. Choosing the cosmic parameters $h = 0.72$, $\Omega_{\rm m} = 0.27$, $\Omega_\Lambda = 0.73$, $g_* = g_{*\rm s} = 106.75$ and $f_k = 0$, one acquires

$$\mathcal{T}(k) = 4.15 \times 10^{-7}, \quad \text{and } \Omega_{\rm g}(k) = 4.15 \times 10^{-7} P_{\rm T}(k). \tag{26}$$

4. The primordial power spectrum of RGW

The primordial spectrum of RGW is always described in a simple form

$$P_{\rm T}(k) = P_{\rm T}(k_0)\Big(\frac{k}{k_0}\Big)^{n_t(k_0)+\frac{1}{2}\alpha_t \ln(k/k_0)} \ , \tag{27}$$

where $n_t(k)$ is the tensor spectral index, and $\alpha_t \equiv {\rm d}n_t/{\rm d}\ln k$ is its running. In the single-field inflationary model, a standard slow-roll analysis gives the following relations between observable quantities and slow-roll parameters:

$$n_t = -\frac{r}{8} \ , \quad \alpha_t = \frac{r}{8}\Big[\Big(n_{\rm s} - 1\Big) + \frac{r}{8}\Big] \ , \quad \text{and} \quad r = \frac{8}{3}(1 - n_{\rm s}) + \frac{16}{3}\eta_V \ , \tag{28}$$

where $r(k) \equiv P_{\rm T}(k)/P_{\rm S}(k)$, is the so-called tensor–scalar ratio. These formulae relate n_t and α_t to the other two functions $n_{\rm s}$ and r, which are easy to observe. But the relation between r and $n_{\rm s}$ is dependent on η_V, which depends on the specific inflationary potential. Inserting these into expression (27), one obtain

$$P_{\rm T}(k) = P_{\rm S}(k_0) \times r \times \Big(\frac{k}{k_0}\Big)^{(-r/8)+(r/16)[(n_{\rm s}-1)+r/8]\ln(k/k_0)} \ , \tag{29}$$

where r denotes the tensor–scalar ratio at $k = k_0$, i.e. $r \equiv r(k_0)$, which also holds true in the following sections. So the primordial spectrum of RGW only depends on $n_{\rm s}$ and r. The recent constraints come from the observations of three-year WMAP,[3] which are

$$n_{\rm s} = 0.951^{+0.015}_{-0.019} \ (68\% \ \text{C.L.}) \ , \quad \text{and} \quad r < 0.28 \ (95\% \ \text{C.L.}) \ . \tag{30}$$

Using expression (26), one obtains

$$\Omega_{\rm g}(k) = 9.98 \times 10^{-16} r \Big(\frac{k}{k_0}\Big)^{(-r/8)+(r/16)[(n_{\rm s}-1)+r/8]\ln(k/k_0)} \ , \tag{31}$$

where we have chosen $A(k_0) = 0.813$. We have plotted the function $\Omega_{\rm g}$ (where $\Omega_{\rm g} \equiv \Omega_{\rm g}(k_1)$, and $k_1 = 0.1$Hz) versus r in Fig.(2), where $n_{\rm s} = 0.951$ is used. This result is consistent with that in our previous work, a larger r leads to a larger $\Omega_{\rm g}$. It is well known that the formula in expression (27) is a very good approximation when the wavenumber k is not much larger (or smaller) than k_0. But it may be not a good approximation at k_1, which is 16 order more than k_0. So it is necessary to numerically calculate $P_{\rm T}(k)$. It is not easy to perform the exact numerical calculation, since one must calculate the spectrum in a very wide range in wavenumber (larger than 16 order), and for each k one must integrate it from the initial condition to the end of the inflation. In this section, we use a semi-numerical method to calculate the primordial power spectrum of RGW. We introduce this method by discussing two kinds of inflationary models, which satisfy the current constraints of n_s, α and r.

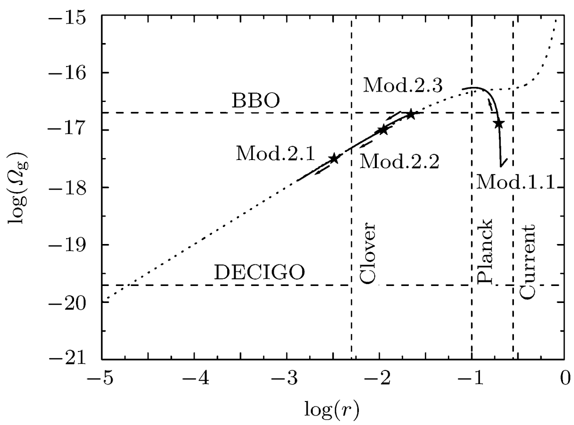

Fig.2. The inflationary models in the $r-\Omega_\mathrm{g}$ diagram. r is the tensor–scalar ratio at $k=0.002\ \mathrm{Mpc}^{-1}$, and Ω_g is the strength of RGW at $k=2\pi\times0.1$Hz. The dot line represents the curve of the approximate formula in expression (31) and $n_\mathrm{s}=0.951$. The vertical (dot) lines from right to left are the sensitive limit curves of current observations, Planck and Clover, respectively. The horizontal (dot) lines from up to down are the sensitive limit curves of BBO and ultimate DECIGO, respectively. The solid lines are the predicted curves of the inflationary models, where $n_\mathrm{s}\in[0.94,\ 0.98]$, and the arrows denote the direction of increasing n_s. The stars denote the models with $n_\mathrm{s}=0.951$.

First we consider the model with potential (Mod.1.1) $V(\phi)=\Lambda^4(\phi/\mu)^2$, which belongs to the large-field model, and predicts a fairly larger r.[63−65] From expression (6), one obtains

$$\epsilon_{_V}=\frac{2}{\phi^2},\quad \eta_{_V}=\frac{2}{\phi^2},\quad \text{and}\quad \xi_{_V}=0, \tag{32}$$

where we have used $M_\mathrm{Pl}\equiv1$. So the slow-roll condition requires that $\phi\gg\sqrt{2}$, which is the so-called large-field model. At the end of inflation, $\epsilon_{_V}=1$ is satisfied, which leads to $\phi_\mathrm{end}=\sqrt{2}$. In the initial condition, one has[50,51]

$$\begin{aligned}&n_\mathrm{s}-1=-6\epsilon_{_V}+2\eta_{_V},\quad r=16\epsilon_{_V},\\ &\text{and}\quad P_\mathrm{S}(k_0)=\frac{V}{24\pi^2\epsilon_{_V}},\end{aligned} \tag{33}$$

which follows that

$$\begin{aligned}&\phi_\mathrm{ini}=\sqrt{8/(1-n_\mathrm{s})},\quad r=4(1-n_\mathrm{s})\\ &\Lambda^4/\mu^2=0.75\pi^2P_\mathrm{S}(k_0)(1-n_\mathrm{s})^2.\end{aligned} \tag{34}$$

Inserting these into the Hamilton–Jacobi formula,

$$2[H'(\phi)]^2-3H^2(\phi)=-V(\phi)\ , \tag{35}$$

one can immediately obtain the function $H(\phi)$ by the numerical calculation. We define the e-fold number N, and the scale factor as $a=a_\mathrm{ini}e^N$. When k_0 crosses the horizon, we set the scale factor $a=a_\mathrm{ini}=1$ i.e. $N=0$. The relation between N and ϕ is

$$\frac{\mathrm{d}\phi}{\mathrm{d}N}=-2\frac{H'}{H}\ , \tag{36}$$

where H is the Hubble parameter during inflation, and $H'\equiv\mathrm{d}H/\mathrm{d}\phi$. One can define a Hubble slow-roll parameter $\epsilon\equiv2(H'/H)^2$, so the primordial power spectrum of RGW is (to the first slow-roll order)[66]

$$P_\mathrm{T}(k)=\frac{2}{\pi^2}\Big[1-\frac{c+1}{4}\epsilon\Big]H^2\ \Big|_{k=aH}, \tag{37}$$

where $c=4(\ln2+\gamma)-5\simeq0.0814514$ (with γ being the Euler–Mascheroni constant) is a constant. Using Eqs.(35) and (36), one can numerically calculate $H(N)$. Inserting it into expression (37), one can have the primordial spectrum of RGW, at the same time the total e-fold N is also obtained. The value of Ω_g is also obtained by using expression (26). We have plotted $\log\Omega_\mathrm{g}$ versus $\log r$ in Fig.2, where we have chosen $n_\mathrm{s}\in[0.94,\ 0.98]$. It is easily found that the value of r is in the range $r\in[0.08,\ 0.24]$. Compared with the value from approximate formula (31), the numerical value is much small: When $n_\mathrm{s}=0.951$, the value is only one-third of the approximate value.

Now we consider another model $V(\phi)=\Lambda^4[1-(\phi/\mu)^2]$, which belongs to the small-field model, and it predicts a very small r.[63−65] From expression (6), one achieves

$$\epsilon_{_V}=\frac12\Big[\frac{2x/\mu}{1-x^2}\Big]^2,\quad \eta_{_V}=\frac{2/\mu^2}{x^2-1},\quad \text{and}\quad \xi_{_V}=0, \tag{38}$$

where $x\equiv\phi/\mu$. At the end of inflation, $\phi_\mathrm{end}=\mu$, i.e. $x_\mathrm{end}=1$, where $V=0$ is satisfied. The initial value of x must be very small to account for the slow-roll condition. Since it cannot be obtained from the observed n_s and $P_\mathrm{S}(k_0)$, we must set it before the calculation. First we consider the model with $x_\mathrm{ini}=0.1$ (Mod.2.1), using expression (33), we immediately have

$$\mu^2=4.89746/(1-n_\mathrm{s}),\quad r=0.06667(1-n_\mathrm{s}),\quad \text{and}\quad \Lambda^4=0.99693(1-n_\mathrm{s})P_\mathrm{S}(k_0). \tag{39}$$

Second we consider the model with $x_{\rm ini} = 0.2$ (Mod.2.2), which follows that

$$\mu^2 = 6.07693/(1-n_{\rm s}), \quad r = 0.22857(1-n_{\rm s}), \quad \text{and} \quad \Lambda^4 = 3.52486(1-n_{\rm s})P_{\rm S}(k_0). \tag{40}$$

The third model has $x_{\rm ini} = 0.3$ (Mod.2.3), which follows that

$$\mu^2 = 7.72854/(1-n_{\rm s}), \quad r = 0.45(1-n_{\rm s}), \quad \text{and} \quad \Lambda^4 = 7.32086(1-n_{\rm s})P_{\rm S}(k_0). \tag{41}$$

Then using the Hamilton–Jacobi formula in Eq.(35) and the relation between N and ϕ which here becomes $\mathrm{d}\phi/\mathrm{d}N = 2H'/H$, one can also achieve the function $H(N)$. Using the formula (37), the values of r, $P_{\rm T}(k)$, $\Omega_{\rm g}(k)$ and N are also acquired, which are plotted in Figs.2 and 3. From Fig.3, one finds that a larger $n_{\rm s}$ leads to a larger N, which holds for all these four inflationary models. When $n_{\rm s} = 0.951$, $N = 41.96$ for the Mod.1.1, and $N = 62.47$ for the Mod.2.3, which are in the region of $N \in [40,\ 70]$ and acceptable.[67,68] But for the Mod.2.1, $N = 97.90$, and for the Mod.2.2, $N = 74.59$, which are too large to be acceptable. From Fig.2, one finds that when $n_{\rm s} \in [0.94,\ 0.98]$, $r < 0.02$ is satisfied for Mod.2.1, Mod.2.2 and Mod.2.3 at the very small values. And values of $\Omega_{\rm g}$ are exactly the same as the approximate results. So one cam draw a conclusion: when r is small, formula (31) is a very good approximation, but when r is larger ($r > 0.1$), the approximate formula (31) is not very good, and the numerical calculation is necessary to be carried out.

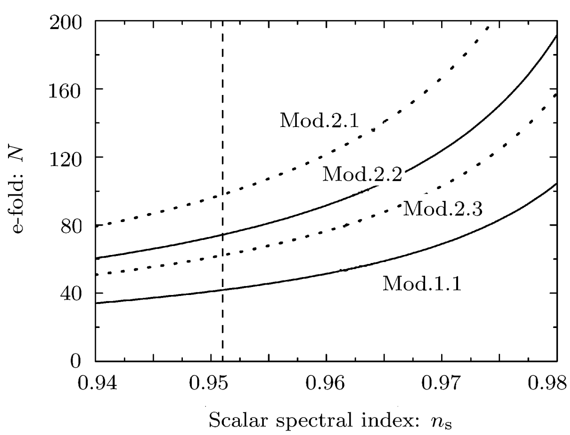

Fig.3. The evolution of the value of the e-fold N with the scalar spectral index $n_{\rm s}$ for the inflationary models. The dot line denotes the curve with $n_{\rm s} = 0.951$.

5. Conclusion and discussion

Inflation has received strong supports from the observations of the CMB and LSS. As a key prediction of inflationary models, the detection of RGW can provide incontrovertible evidence that the inflation actually occurred and set strong constraints on the dynamics of inflation. A lot of experiments are under development for the RGW detection, which mainly include two kinds: The CMB experiments, including Planck, Clover, and others; the laser interferometers, including BBO, DECIGO and so on. For investigating the detection abilities of these two kinds of experiments, it is convenient to study the distribution of the inflationary models in the $r-\Omega_{\rm g}$ diagram. So it is necessary to accurately calculate the RGW in the whole frequency range. In this paper, we have improved the previous calculation in two aspects. First, we have studied the transfer function by taking into consideration the redshift-suppression effect, the accelerating expansion effect, the damping effect of free-streaming relativistic particles, and the damping effect of cosmic phase transition, and given a simple approximate formula of the transfer function, which applies to the waves with $k > k_{\rm eq}$. This function depends on the values of the cosmic parameters: $\Omega_{\rm m}$, Ω_Λ, H_0, $k_{\rm eq}$, τ_0, g_*, $g_{*\rm s}$, and f_k. Second, we have developed a numerical method of calculating the primordial power spectrum of RGW, especially at high frequencies, where the observed constraints on $n_{\rm s}$ and $P_{\rm S}(k_0)$ and the Hamilton–Jacobi equation are used. We have applied this method to two kinds of inflationary models, which satisfy the current constraints on $n_{\rm s}$, α and r.

From Fig.3, one can find that in all these inflationary models, a larger $n_{\rm s}$ follows a larger N. For the first kind of model, when $n_{\rm s} > 0.97$, the value of $N > 70$ is satisfied, which is unsuitable. To account for the constraint of $N \in [40,\ 70]$, $n_{\rm s}$ can be only in a very narrow region $n_{\rm s} \in [0.948,\ 0.970]$. For the second kind of model, the initial conditions of $x_{\rm ini} = 0.1$ and $x_{\rm ini} = 0.2$ are not acceptable, which predict too large an e-fold. The condition of $x_{\rm ini} = 0.3$ is suitable, which predicts $N = 62.47$ when $n_{\rm s} = 0.951$. But to account for the constraint $N < 70$, $n_{\rm s} < 0.956$ must be satisfied. From Fig.2, one can find that for Mod.1.1, when $n_{\rm s} \in [0.94,\ 0.98]$, the value of r is in the region $r \in [0.08,\ 0.24]$, which is mostly in the sensitive region of Planck satellite. The value of $\Omega_{\rm g}$ is in the region from 5.6×10^{-17} to 2.2×10^{-18}. In most of region of $n_{\rm s}$, a larger $n_{\rm s}$ follows a smaller r, and corresponds to a larger $\Omega_{\rm g}$, which is an unexpected result. This is obviously different from the result of the approximate formula. When $n_{\rm s} = 0.951$, $\Omega_{\rm g} = 1.3 \times 10^{-17}$, which is in the sensitive region of ultimate DECIGO, but beyond the sensitive limit of BBO. This value is only one third of the value from the approximate formula in expression (31). For Mod.2.1, Mod.2.2 and Mod.2.3, a larger $n_{\rm s}$ follows a smaller r and a smaller $\Omega_{\rm g}$. These models predict a very small r, when $n_{\rm s} \in [0.94,\ 0.98]$, $r < 0.02$ is always satisfied, and the value of $\Omega_{\rm g}$ is exactly the same as the value from the approximate formula in expression (31). For Mod.2.3, which predicts an acceptable e-fold, the values of r are all in the sensitive region of Clover, but beyond which of Planck; the values of $\Omega_{\rm g}$ are all in the sensitive region of ultimate DECIGO, but beyond that of BBO.

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