

Relic gravitational waves in the accelerating Universe

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Abstract

Recent observations have indicated that the Universe at the present stage is in an accelerating expansion, a process that has great implications. We evaluate the spectrum of relic gravitational waves in the current accelerating Universe and find that there are new features appearing in the resulting spectrum as compared to the decelerating models. In the low-frequency range the peak of the spectrum is now located at a frequency $\nu_E \simeq \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \nu_H$, where ν_H is the Hubble frequency, and there appears a new segment of spectrum between ν_E and ν_H . In all other intervals of frequencies $\geq \nu_H$, the spectral amplitude acquires an extra factor $\frac{\Omega_m}{\Omega_\Lambda}$, due to the current acceleration; otherwise the shape of the spectrum is similar to that in the decelerating models. The recent WMAP result of CMB anisotropies is used to normalize the amplitude for gravitational waves. The slope of the power spectrum depends sensitively on the scale factor $a(\tau) \propto |\tau|^{1+\beta}$ during the inflationary stage with $\beta = -2$ for the exact de Sitter space. With increasing β , the resulting spectrum is tilted to be flatter with more power at high frequencies, and the sensitivity of the second science run of the LIGO detectors puts a restriction on the parameter $\beta \leq -1.8$. We also give a numerical solution which confirms these features.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The inflationary expansion of the early Universe can create a stochastic background of relic gravitational waves, which is important in cosmology and has been extensively studied in the past [1–5]. The spectrum of relic gravitational waves, as is to be observed today, depends not only on the details of the early stage of inflationary expansion, but also on the expansion behaviour of the subsequent stages, including the current one. The calculations of the spectrum

so far [6–11] have been done for the case that the current stage is in a decelerating expansion. The resulting spectrum has been put among the candidate list of sources for the gravitational wave detectors either in operation [12, 13], or under construction [14, 15]. The astronomical observations on SN Ia [16, 17] indicate that the Universe is currently under accelerating expansion, which may be driven by the cosmic dark energy ($\Omega_\Lambda \sim 0.7$) plus the dark matter ($\Omega_m \sim 0.3$) [18]. This is further supported by the recent WMAP results on CMB anisotropies [19] with $\Omega = 1$. So far, in the numerical codes for CMB, such as CMBFAST, the dark energy effects have been included in calculating CMB temperature anisotropies and polarizations, but the spectrum of relic gravitational waves has not been given, which can be important in producing the CMB anisotropies and polarizations. By the wave equation, the evolution of relic gravitational waves depends on the expanding spacetime background, and the wave amplitudes depend on whether the wavelengths are inside or outside the Hubble radius. The Universe under accelerating expansion has a Hubble radius as a function of time that differs from that in the conventional models of decelerating expansion. Therefore, one can expect that, if an earlier matter-dominated stage of decelerating expansion is followed by the current accelerating expansion, the outcome for the spectrum of relic gravitational waves will be altered.

In this paper we study the impact of the current accelerating expansion on the relic gravitational waves. We first sketch, as a setup, the well-known formulations of the gravitational waves in an expanding spacetime [20], and give the explicit scale factor $a(\tau)$ for a sequence of successive expanding epochs, including the current epoch of accelerating expansion. We then evaluate the power spectrum of relic gravitational waves and find that the current accelerating expansion does change the spectrum, including its shape and amplitude. Throughout the paper we work with units with $c = \hbar = 1$; otherwise it will be pointed out. We also use notation similar to that of Grishchuk [8] for convenience for comparison.

2. The gravitational wave equation

Incorporating the perturbations to the spatially flat Robertson–Walker spacetime, the metric is

$$ds^2 = a^2(\tau)[d\tau^2 - (\delta_{ij} + h_{ij}) dx^i dx^j], \quad (1)$$

where τ is the conformal time, the perturbations of spacetime h_{ij} is a 3×3 symmetric matrix containing generally the scalar, vector and tensor parts. The gravitational wave field is the tensorial portion of h_{ij} , which is transverse-traceless

$$\partial_i h^{ij} = 0, \quad \delta^{ij} h_{ij} = 0. \quad (2)$$

We are interested only in the creation of relic gravitational waves by the expanding spacetime background; the perturbed matter source is therefore not taken into account. Moreover, as the relic gravitational waves are very weak, in the sense that $h_{ij} \ll 1$, so one needs just the study of the linearized field equation:

$$\partial_\mu(\sqrt{-g}\partial^\mu h_{ij}(\mathbf{x}, \tau)) = 0. \quad (3)$$

In quantum theory of gravitational waves, the field h_{ij} is a field operator, which is written as a sum of the plane wave Fourier modes,

$$h_{ij}(\mathbf{x}, \tau) = \frac{\sqrt{16\pi}l_{\text{Pl}}}{(2\pi)^{\frac{3}{2}}} \sum_{\lambda=1}^2 \int_{-\infty}^{\infty} d^3\mathbf{k} \epsilon_{ij}^{(\lambda)}(\mathbf{k}) \frac{1}{\sqrt{2k}} [a_k^{(\lambda)} h_k^{(\lambda)}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a_k^{\dagger(\lambda)} h_k^{(\lambda)*}(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}], \quad (4)$$

where $l_{\text{Pl}} = \sqrt{G}$ is Planck's length, the two polarizations $\epsilon_{ij}^{(\lambda)}(\mathbf{k})$, $\lambda = 1, 2$, are symmetric and transverse-traceless $k^i \epsilon_{ij}^{(\lambda)}(\mathbf{k}) = 0$, $\delta^{ij} \epsilon_{ij}^{(\lambda)}(\mathbf{k}) = 0$, and satisfy the conditions

$\epsilon^{(\lambda)ij}(\mathbf{k})\epsilon_{ij}^{(\lambda')}(\mathbf{k}) = 2\delta_{\lambda\lambda'}$ and $\epsilon_{ij}^{(\lambda)}(-\mathbf{k}) = \epsilon_{ij}^{(\lambda)}(\mathbf{k})$, the creation and annihilation operators satisfy $[a_{\mathbf{k}}^{(\lambda)}, a_{\mathbf{k}'}^{\dagger(\lambda')}] = \delta_{\lambda\lambda'}\delta^3(\mathbf{k} - \mathbf{k}')$, and the initial vacuum state is defined as

$$a_{\mathbf{k}}^{(\lambda)}|0\rangle = 0 \tag{5}$$

for each \mathbf{k} and λ . As a matter of fact, this definition depends on the choice of the mode function $h_k^{(\lambda)}(\tau)$; different $h_k^{(\lambda)}(\tau)$ define different vacuum states, a point that will be further explained later. In the vacuum state the energy density of gravitational waves $t_{00} = \frac{1}{32\pi G}h_{ij,0}h^{ij}_{,0}$ gives the zero-point energy

$$\langle 0|\int t_{00}d^3\mathbf{x}|0\rangle = \frac{1}{2}\hbar\int_{-\infty}^{\infty}d^3\mathbf{k}\sum_{\lambda=1}^2\omega_k\langle 0|a_{\mathbf{k}}^{(\lambda)}a_{\mathbf{k}}^{(\lambda)\dagger}|0\rangle.$$

For a fixed wave number \mathbf{k} and a fixed polarization state λ , equation (3) reduces to the second-order ordinary differential equation [20]

$$h_k^{(\lambda)''}(\tau) + 2\frac{a'}{a}h_k^{(\lambda)'}(\tau) + k^2h_k^{(\lambda)}(\tau) = 0, \tag{6}$$

where the prime denotes $d/d\tau$. Since the equation of $h_{\mathbf{k}}^{(\lambda)}(\tau)$ for each polarization is the same, we denote $h_{\mathbf{k}}^{(\lambda)}(\tau)$ by $h_{\mathbf{k}}(\tau)$. One rescales the field $h_{\mathbf{k}}(\tau)$ as

$$h_{\mathbf{k}}(\tau) = \frac{\mu_{\mathbf{k}}(\tau)}{a(\tau)}, \tag{7}$$

and the equation for $\mu_{\mathbf{k}}$ is

$$\mu_k'' + \left(k^2 - \frac{a''}{a}\right)\mu_k = 0. \tag{8}$$

This equation can be regarded as the equation for a one-dimensional oscillator in a given effective potential barrier a''/a . For a given spacetime background with a generic power-law form of the scale factor

$$a(\tau) \propto \tau^\alpha, \tag{9}$$

where α is a real number, positive or negative, the general solution is a linear combination of Hankel's functions:

$$\mu_k(\tau) = A_k\sqrt{k\tau}H_{(\alpha-\frac{1}{2})}^{(1)}(k\tau) + B_k\sqrt{k\tau}H_{(\alpha-\frac{1}{2})}^{(2)}(k\tau). \tag{10}$$

Given a model of the expansion of Universe, consisting of a sequence of successive $a(\tau)$ as in equation (9) with different α , one can construct an exact solution $\mu_k(\tau)$ by matching its values and its derivatives at the joining points. One may also numerically solve equation (6) and give the corresponding power spectrum as we will present later in the paper. The vacuum state $|0\rangle$, determined by the mode function $h_k(\tau)$, is therefore fixed by a choice of coefficients A_k and B_k . In the limit $k \rightarrow \infty$, or $\tau \rightarrow \pm\infty$,

$$\sqrt{k\tau}H_{(\alpha-\frac{1}{2})}^{(1)}(k\tau) \rightarrow \sqrt{\frac{2}{\pi}}i^{-\alpha}e^{-ik\tau}, \quad \sqrt{k\tau}H_{(\alpha-\frac{1}{2})}^{(2)}(k\tau) \rightarrow \sqrt{\frac{2}{\pi}}i^\alpha e^{ik\tau},$$

approaching the positive and negative frequency modes, respectively. For instance, if we choose $B_k = 0$, then in the limit $k \rightarrow \infty$, or $\tau \rightarrow \pm\infty$,

$$h_{\mathbf{k}}(\tau) \propto e^{-ik\tau},$$

giving the positive frequency mode. This choice yields the so-called adiabatic vacuum [21] through (4) and (5). The amplitude of the power spectrum of the gravitational waves depends only on the initial value of $|h_{\mathbf{k}}(\tau)|$ at the horizon crossing.

The following two limiting cases are useful for an approximate evaluation of the spectrum. Outside the barrier $k^2 \gg \frac{a''}{a}$ (equivalent to $k \gg \frac{a'}{a}$) the gravitational wave field reduces to

$$h_k(\tau) \rightarrow A_k \frac{e^{ik\tau}}{a(\tau)} + B_k \frac{e^{-ik\tau}}{a(\tau)}, \quad (11)$$

having a decreasing amplitude

$$h_k(\tau) \propto 1/a(\tau). \quad (12)$$

Inside the barrier $k^2 \ll \frac{a''}{a}$ (equivalent to $k \ll \frac{a'}{a}$) the gravitational wave field

$$h_k(\tau) \rightarrow C_k + D_k \int^{\tau} \frac{d\tau'}{a^2(\tau')}. \quad (13)$$

The second term $D \int^{\tau} \frac{d\tau'}{a^2(\tau')}$ is small for the models that we shall study in the following, so the long wavelength limit of h_k is simply a constant:

$$h_k(\tau) = C_k. \quad (14)$$

Thus, as a function of τ , $h_k(\tau)$ has simple approximate behaviours in the two limiting cases, and we will use these to estimate the spectrum at the present stage.

3. Epochs of expanding Universe

The history of the overall expansion of the Universe can be modelled as the following sequence of successive epochs of power-law expansion.

The initial stage (inflationary)

$$a(\tau) = l_0 |\tau|^{1+\beta}, \quad -\infty < \tau \leq \tau_1, \quad (15)$$

where $1 + \beta < 0$, and $\tau_1 < 0$. The special case of $\beta = -2$ is the de Sitter expansion.

The z -stage

$$a(\tau) = a_z (\tau - \tau_p)^{1+\beta_s}, \quad \tau_1 \leq \tau \leq \tau_s, \quad (16)$$

where $\beta_s + 1 > 0$. Towards the end of inflation, during the reheating, the equation of state of energy in the Universe can be quite complicated and is rather model-dependent [22]. So this z -stage is introduced to allow a general reheating epoch, as has been advocated by Grishchuk [8].

The radiation-dominated stage

$$a(\tau) = a_e (\tau - \tau_e), \quad \tau_s \leq \tau \leq \tau_2. \quad (17)$$

The matter-dominated stage

$$a(\tau) = a_m (\tau - \tau_m)^2, \quad \tau_2 \leq \tau \leq \tau_E, \quad (18)$$

where τ_E is the time when the dark energy density ρ_Λ is equal to the matter energy density ρ_m . Before the discovery of the accelerating expansion of the Universe, the current expansion was usually taken to be in this matter-dominated stage, which is a decelerating one. Now, following the matter-dominated stage, we add an epoch of accelerating stage, which is probably driven by either the cosmological constant, or the quintessence, or some other kind of condensate [23]. The value of the redshift z_E at the time τ_E is given by $(1 + z_E) = a(\tau_H)/a(\tau_E)$, where τ_H is the present time. Since ρ_Λ is constant and $\rho_m(\tau) \propto a^{-3}(\tau)$, one has

$$1 = \frac{\rho_\Lambda}{\rho_m(\tau_E)} = \frac{\rho_\Lambda}{\rho_m(\tau_H)(1 + z_E)^3}.$$

If we take the current values $\Omega_\Lambda \sim 0.7$ and $\Omega_m \sim 0.3$, then it follows that

$$1 + z_E = \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/3} \sim 1.33.$$

The accelerating stage (up to the present)

$$a(\tau) = l_H |\tau - \tau_a|^{-1}, \quad \tau_E \leq \tau \leq \tau_H. \tag{19}$$

This stage describes the accelerating expansion of the Universe, which is the new feature in our model and will induce some modifications to the spectrum of relic gravitational waves. It should be mentioned that the actual scale factor function $a(\tau)$ differs from equation (19), since the matter component exists in the current Universe. However, the dark energy is dominant, so (19) is an approximation to the current expansion behaviour.

Given $a(\tau)$ for the various epochs, the derivative $a' = da/d\tau$ and the ratio a'/a follow immediately. Except for the β_s that is imposed upon as the model parameter, there are ten constants in the above expressions of $a(\tau)$. By the continuity conditions of $a(\tau)$ and $a(\tau)'$ at the four given joining points τ_1, τ_s, τ_2 and τ_E , one can fix only eight constants. The remaining two constants can be fixed by the overall normalization of a and by the observed Hubble constant as the expansion rate. Specifically, we put $|\tau_H - \tau_a| = 1$ as the normalization of a , which fixes the constant τ_a , and the constant l_H is fixed by the following calculation,

$$\frac{1}{H} \equiv \left(\frac{a^2}{a'}\right)_{\tau_H} = l_H, \tag{20}$$

so l_H is just the Hubble radius at present. Then everything is fixed up. In the expanding Robertson–Walker spacetime the physical wavelength is related to the comoving wave number by

$$\lambda \equiv \frac{2\pi a(\tau)}{k}, \tag{21}$$

and the wave number k_H corresponding to the present Hubble radius is

$$k_H = \frac{2\pi a(\tau_H)}{l_H} = 2\pi. \tag{22}$$

There is another wave number,

$$k_E \equiv \frac{2\pi a(\tau_E)}{1/H} = \frac{k_H}{1 + z_E},$$

whose corresponding wavelength at the time τ_E is the Hubble radius $1/H$.

By matching a and a'/a at the joint points, we have derived, for example,

$$l_0 = l_H b \zeta_E^{-(2+\beta)} \zeta_2^{\frac{\beta-1}{2}} \zeta_s^\beta \zeta_1^{\frac{\beta-\beta_s}{1+\beta_s}}, \tag{23}$$

where $b \equiv |1 + \beta|^{-(1+\beta)}$, which is defined differently from Grishchuk’s [08], $\zeta_E \equiv \frac{a(\tau_H)}{a(\tau_E)}$, $\zeta_2 \equiv \frac{a(\tau_E)}{a(\tau_2)}$, $\zeta_s \equiv \frac{a(\tau_2)}{a(\tau_s)}$ and $\zeta_1 \equiv \frac{a(\tau_s)}{a(\tau_1)}$. With these specifications, the functions $a(\tau)$ and $a'(\tau)/a(\tau)$ are fully determined. In particular, $a'(\tau)/a(\tau)$ rises up during the accelerating stage, instead of decreasing as in the matter-dominated stage. This causes the modifications to the spectrum of relic gravitational waves.

4. The spectrum of gravitational waves

The power spectrum $h(k, \tau)$ of relic gravitational waves is defined by the following equation,

$$\int_0^\infty h^2(k, \tau) \frac{dk}{k} \equiv \langle 0 | h^{ij}(\mathbf{x}, \tau) h_{ij}(\mathbf{x}, \tau) | 0 \rangle, \tag{24}$$

where the right-hand side is the vacuum expectation value of the operator $h^{ij}h_{ij}$. Substituting equation (4) into the above, and taking the contribution from each polarization to be the same, one reads the power spectrum

$$h(k, \tau) = \frac{4l_{\text{Pl}}}{\sqrt{\pi}} k |h_k(\tau)| \quad (25)$$

Once the mode function $h_k(\tau)$ is given, the spectrum $h(k, \tau)$ follows.

The initial condition is taken to be during the inflationary stage. For a given wave number k , its wave crossed over the horizon at a time τ_i , i.e. when the wavelength $\lambda_i = 2\pi a(\tau_i)/k$, is equal to $1/H(\tau_i)$, the Hubble radius at time τ_i . Equation (15) yields $1/H(\tau_i) = l_0 |\tau_i|^{2+\beta} / |1 + \beta|$, and for the exact de Sitter expansion with $\beta = -2$ one has $H(\tau_i) = l_0$. Note that a different k corresponds to a different time τ_i . Now choose the initial condition of the mode function $h_k(\tau)$ as

$$|h_k(\tau_i)| = \frac{1}{a(\tau_i)}. \quad (26)$$

Then the initial amplitude of the power spectrum is

$$h(k, \tau_i) = 8\sqrt{\pi} \frac{l_{\text{Pl}}}{\lambda_i}. \quad (27)$$

From $\lambda_i = 1/H(\tau_i)$ it follows that $\frac{a'(\tau_i)}{a(\tau_i)} = \frac{k}{2\pi}$. So the initial amplitude of the power spectrum is

$$h(k, \tau_i) = A \left(\frac{k}{k_H} \right)^{2+\beta}, \quad (28)$$

where the constant

$$A = 8\sqrt{\pi} b \frac{l_{\text{Pl}}}{l_0}. \quad (29)$$

For the case of $\beta = -2$ the initial spectrum is independent of k . The power spectrum for the primordial perturbations of energy density is $P(k) \propto |h(k, \tau_H)|^2$, and its spectral index n is defined as $P(k) \propto k^{n-1}$. Thus one reads off the relation $n = 2\beta + 5$. The exact de Sitter expansion with $\beta = -2$ will yield the so-called scale-invariant spectral index $n = 1$.

Once the initial spectrum is specified, we can derive the spectrum $h(k, \tau_H)$ at the present time τ_H .

For $k \leq k_E$, the wavelengths $\frac{2\pi a(\tau_H)}{k}$ in this range are even greater than the present Hubble radius l_H ; one has $k < a'/a$ throughout the whole expansion up to the present, so the amplitude remains the same constant as the initial one in equation (28):

$$h(k, \tau_H) = A \left(\frac{k}{k_H} \right)^{2+\beta}, \quad k \leq k_E. \quad (30)$$

During the whole period inside the barrier, the spectral amplitude remains $h(k, \tau_i)$ approximately until the wave leaves the barrier and begins to decrease as $1/a(\tau)$. Let $a_{**}(k)$ be the scale factor at this moment. For those very long wavelength modes with $k_E \leq k \leq k_H$, during the current epoch of accelerating expansion, $h_k(\tau)$ stops decreasing as soon as the barrier a'/a becomes higher than k at a time $\tau(k)$ earlier than τ_H , so $h_k(\tau)$ has decreased by a factor $\frac{a_{**}(k)}{a(\tau(k))}$, and the amplitude of the present spectrum is given by

$$h(k, \tau_H) = A \left(\frac{k}{k_H} \right)^{2+\beta} \frac{a_{**}(k)}{a(\tau(k))}. \quad (31)$$

The decreasing factor is written as

$$\frac{a_{**}(k)}{a(\tau(k))} = \frac{a_{**}(k)}{a(\tau_E)} \frac{a(\tau_E)}{a(\tau(k))}.$$

During the matter-dominated stage one has $a_{**}(k) \propto \tau^2 \propto k^{-2}$ and $a(\tau_E) \propto \tau_E^2 \propto k_E^{-2}$, and during the accelerating stage one has $a(\tau_E) \propto \tau_E^{-1} \propto k_E$ and $a(\tau(k)) \propto \tau^{-1} \propto k$, so the decreasing factor is

$$\frac{a_{**}(k)}{a(\tau(k))} = \left(\frac{k_E}{k}\right)^3.$$

Thus one gets

$$h(k, \tau_H) = A \left(\frac{k}{k_H}\right)^{2+\beta} \left(\frac{k_E}{k}\right)^3 = A \left(\frac{k}{k_H}\right)^{\beta-1} \frac{1}{(1+z_E)^3}, \quad k_E \leq k \leq k_H. \quad (32)$$

where the relation $\frac{k_E}{k_H} = \frac{1}{1+z_E}$ has been used. The spectrum has a rather stiff slope with the power-law index $\beta - 1$. The occurrence of this segment of power spectrum is a new feature of the model of accelerating expansion that is absent in the decelerating model. The wavelengths corresponding to this (k_E, k_H) segment are very long, comparable to the present Hubble radius, and can only possibly be observed through the CMB anisotropies at low multipoles.

For all the wave numbers $k > k_H$, as soon as the waves leave the barrier at $a_{**}(k)$, the modes $h_k(\tau)$ decrease all the way up to the present time τ_H . So it has been reduced by a factor $\frac{a_{**}(k)}{a(\tau_H)}$, and the amplitude of the present spectrum is given by

$$h(k, \tau_H) = A \left(\frac{k}{k_H}\right)^{2+\beta} \frac{a_{**}(k)}{a(\tau_H)}. \quad (33)$$

We use this formula to obtain the following result for the spectrum of gravitational waves in the remaining range of wave numbers.

For $k_H \leq k \leq k_2$, the wave number does not hit the barrier, so

$$\frac{a_{**}(k)}{a(\tau_H)} = \frac{a_{**}(k)}{a(\tau_E)} \frac{a(\tau_E)}{a(\tau_H)} = \left(\frac{k_E}{k}\right)^2 \frac{1}{(1+z_E)},$$

and one obtains

$$h(k, \tau_H) = A \left(\frac{k}{k_H}\right)^\beta \frac{1}{(1+z_E)^3}, \quad k_H \leq k \leq k_2. \quad (34)$$

The spectrum in this interval differs from that of the matter-dominated model by an extra factor $\frac{1}{(1+z_E)^3} = \frac{\Omega_m}{\Omega_\Lambda} \sim 0.43$. The wavelengths in this range are very long, but are still shorter than l_H . The spectrum in this interval may contribute to, and, therefore, have its imprints in CMB anisotropies. Let us estimate the value k_2 . Assuming that the equality of radiation and matter occurred at the redshift $z_2 \simeq 3454$, as indicated by the WMAP observation [19], one has

$$\frac{k_2}{k_E} = \left(\frac{a(\tau_E)}{a(\tau_2)}\right)^{\frac{1}{2}} = \left(\frac{a(\tau_H)}{a(\tau_2)}\right)^{\frac{1}{2}} \left(\frac{a(\tau_E)}{a(\tau_H)}\right)^{\frac{1}{2}} \simeq \sqrt{3454} \frac{1}{\sqrt{1.33}} \simeq 51.$$

For $k_2 \leq k \leq k_s$, the calculation is similar to the previous case with the result

$$h(k, \tau_H) = A \left(\frac{k}{k_H}\right)^{1+\beta} \left(\frac{k_H}{k_2}\right) \frac{1}{(1+z_E)^3}, \quad k_2 \leq k \leq k_s. \quad (35)$$

Again the extra factor $1/(1+z_E)^3$ appears. Note that this range of frequency covers the one to which the detectors of LIGO and LISA are sensitive.

For $k_s \leq k \leq k_1$, the calculation yields

$$h(k, \tau_H) = A \left(\frac{k}{k_H} \right)^{1+\beta-\beta_s} \left(\frac{k_s}{k_H} \right)^{\beta_s} \frac{k_H}{k_2} \frac{1}{(1+z_E)^3}, \quad k_s \leq k \leq k_1. \quad (36)$$

It also contains an extra factor $\frac{1}{(1+z_E)^3}$. This is the high-frequency range which is still beyond current detection.

5. Determining the parameters

To completely determine the spectrum, we also need to specify the values of ν_1 , A , ν_s , β_s , which appear in the expressions for the spectrum $h(k, \tau_H)$. Since the wave number is proportional to the frequency, $k \propto \nu$, the ratios of the wave numbers can be replaced by those of the frequencies, e.g., $k/k_H = \nu/\nu_H$, etc, in the above formulae of the spectrum $h(k, \tau_H)$. The Hubble frequency $\nu_H = 1/l_H = H \simeq 2 \times 10^{-18}$ Hz.

An estimate of the highest frequency ν_1 can be made as given in [8]. From expression (36) one has

$$h(k_1, \tau_H) = 8\sqrt{\pi} \frac{l_{Pl}}{l_H} \left(\frac{\nu_1}{\nu_H} \right) = 8\sqrt{\pi} \frac{l_{Pl}}{l_1}, \quad (37)$$

where expressions (23) for b/l_0 and (29) for A have been used. The spectral energy density parameter $\Omega_g(\nu)$ of gravitational waves is defined through the relation $\rho_g/\rho_c = \int \Omega_g(\nu) \frac{d\nu}{\nu}$, where ρ_g is the energy density of the gravitational waves and ρ_c is the critical energy density. One reads

$$\Omega_g(\nu) = \frac{\pi^2}{3} h^2(k, \tau_H) \left(\frac{\nu}{\nu_H} \right)^2.$$

If it is imposed that at the highest frequency ν_1 the value $\Omega_g(\nu_1)$ does not exceed the level of 10^{-6} , as required by the rate of the primordial nucleogenesis, then one gets $\nu_1 = 3 \times 10^{10}$ Hz.

Next let us estimate the overall factor A in the spectrum $h(k, \tau_H)$. If the CMB anisotropies at low multipoles are induced by the gravitational waves, or if the contributions from the gravitational waves and from the density perturbations are of the same order of magnitude, we may assume $\Delta T/T \simeq h(k, \tau_H)$. This will determine A . The observed CMB anisotropies [19] at lower multipoles is $\Delta T/T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$, which corresponds to the largest scale anisotropies that have observed so far. Taking this to be the perturbations at the Hubble radius $1/H$ yields

$$h(k_H, \tau_H) = A \frac{1}{(1+z_E)^3} = 0.37 \times 10^{-5}. \quad (38)$$

Actually the observed $\Delta T/T$ varies with l , for instance, $\Delta T/T \simeq 1.04 \times 10^{-5}$ at $l \sim 10$, which would give an outcome for A greater than that in equation (38) by a factor ~ 2.8 . However, if the contributions from relic gravitational waves to $\Delta T/T$ are less than those from the density perturbations, the normalization of A should be chosen less than that in equation (38) accordingly.

However, there is a subtlety here in the interpretation of $\Delta T/T$ at low multipoles, whose corresponding scale is very large $\sim l_H$. At present the Hubble radius is l_H , and the Hubble diameter is $2l_H$. On the other hand, the smallest characteristic wave number is k_E , whose corresponding physical wave length at present is $\frac{2\pi a(\tau_H)}{k_E} = l_H(1+z_E) \simeq 1.32l_H$, which is within the Hubble diameter $2l_H$, and is theoretically observable. So, instead of (38), if $\Delta T/T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ were taken as the amplitude of the spectrum at ν_E , one would have $h(k_E, \tau_H) = A \frac{1}{(1+z_E)^{2+\beta}} = 0.37 \times 10^{-5}$, yielding a smaller A than that in equation (38) by a factor $(1+z_E)^{1-\beta} \sim 2.2$.

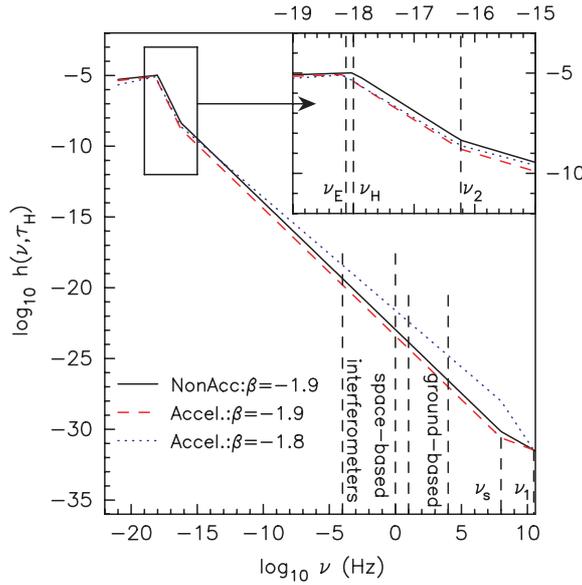


Figure 1. The spectrum $h(k, \tau_H)$ as a function of the frequency ν in the present Universe in both accelerating and decelerating expansion.

We now check the range of β allowed. During the inflationary expansion when the k -mode wave enters the barrier with $\lambda_i = 1/H(\tau_i)$, it follows that $\lambda_i = \frac{l_0}{b} \left(\frac{k_H}{k}\right)^{2+\beta}$. For the classical treatment of the background gravitational field to be valid, this wavelength should be greater than the Planck length, $\lambda_i > l_{Pl}$, so

$$\left(\frac{\nu}{\nu_H}\right)^{2+\beta} < \frac{8\sqrt{\pi}}{A}.$$

At the highest frequency $\nu = \nu_1$, this yields a constraint

$$\beta < -2 + \ln\left(\frac{8\sqrt{\pi}}{A}\right) / \ln\left(\frac{\nu_1}{\nu_H}\right), \tag{39}$$

which depends on A . Thus, for A given in (38), one obtains the upper limit $\beta < -1.78$. We remark that this range is larger than that in the decelerating model [8], which allows for only $\beta < -1.9$.

Finally, we give an estimate of the allowed values of β_s and ν_s . Plugging b/l_0 given by (23) into $A = 8\sqrt{\pi}b\frac{l_{Pl}}{l_0}$, using $\nu_2/\nu_H = 58.8$ and $l_H/l_{Pl} = 1.238 \times 10^{61}$, one has

$$1.484 \times 10^{58} \frac{A}{(1+z_E)^3} = \left(\frac{\nu_1}{\nu_H}\right)^{-\beta} \left(\frac{\nu_1}{\nu_s}\right)^{\beta_s}. \tag{40}$$

Given a set values of A and β , one can take ν_s and β_s to satisfy this relation. From equation (36) it is seen that, for a fixed ν_s , a smaller β_s tends to slightly increase the amplitude $h(\nu, \tau_H)$. For definiteness, we take $\nu_s = 10^8$ Hz as an example in the following. For A given in (38), taking $\beta = -1.8, -1.9$, then $\beta_s = 0.598, -0.552$, respectively.

With these results we plot the spectrum $h(k, \tau_H)$ versus the frequency ν for two values of the parameter $\beta = -1.9, -1.8$ in figure 1. Besides, for convenience of comparison, the spectrum for the non-accelerating model [8] is also plotted for $\beta = -1.9$. It is clearly seen that, for the same parameter $\beta = -1.9$ in the whole range $\nu \geq \nu_H$, the accelerating model has

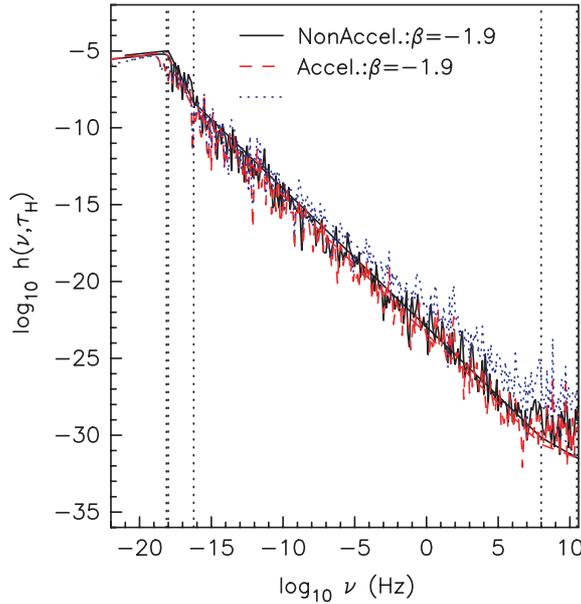


Figure 2. The spectrum $h(k, \tau_H)$ from the numerical calculation.

an amplitude lower by a factor of ~ 2.2 than that of the decelerating model. This is due to the extra factor $\frac{1}{(1+z_E)^3} = \frac{\Omega_m}{\Omega_\Lambda}$, as has been demonstrated in the previous expressions (34), (35) and (36). Interestingly, the larger value $\beta = -1.8$ gives a flatter spectrum $h(\nu, \tau_H)$ with an overall higher amplitude in the range $\nu \geq \nu_2$. In particular, in the higher frequency range (10^{-4} – 0) Hz covered by LISA, the amplitude is $h \simeq 10^{-19}$ – 10^{-22} for the $\beta = -1.8$ case, about 10 to 10^2 higher than the $\beta = -1.9$ case. In the even higher frequency range (10 – 10^4) Hz covered by LIGO, the amplitude is $h \simeq 10^{-23}$ – 10^{-25} for the $\beta = -1.8$ case, also 10 to 10^2 higher than the $\beta = -1.9$ case. Thus an inflationary model of larger index $n = 2\beta + 5$ will predict a stronger signal of relic gravitational waves in higher frequencies.

The recent second science run of the LIGO interferometric detectors [24] gives a sensitivity 3×10^{-24} to 3×10^{-23} in the frequency range (10^2 – 10^3) Hz. The best sensitivity is given by the 4 km arm L1 detector located at Livingstone, which is about 3×10^{-24} near a frequency ~ 300 Hz. Our calculation for the $\beta = -1.9$ case yields an amplitude, $h \simeq 10^{-26}$, much smaller than this sensitivity of the second run. However, the most interesting case is the model of $\beta = -1.8$, in which the amplitude of the gravitational waves just falls into the sensitivity of the L1 detector. Since the second run of LIGO has not observed any signal of stochastic gravitational waves in this frequency range, we arrive at a constraint $\beta \leq -1.8$ on the model parameter of the inflationary expansion. By the way, if in future LIGO does not detect any signals of relic gravitational waves at a higher sensitivity of 10^{-25} , then the constraint will be $\beta \leq -1.9$. Note that this constraint is consistent with, though more stringent than, $\beta < -1.78$, as has been imposed, through equation (39), from the observed CMB anisotropies of WMAP. Thus, in regard to the index β of the inflationary expansion, so far both observations from CMB and from relic gravitational waves have given a consistent restriction.

As a double check, we also have numerically solved the differential equation (6) of the gravitational waves, and found the resulting power spectrum $h(\nu, t)$ from the numerical solution, which is plotted in figure 2. Since the numerical result that we have plotted carries the oscillating factor $\cos(2\pi\nu(t - t_\nu))$, its curve shows an extra small zigzag, as expected.

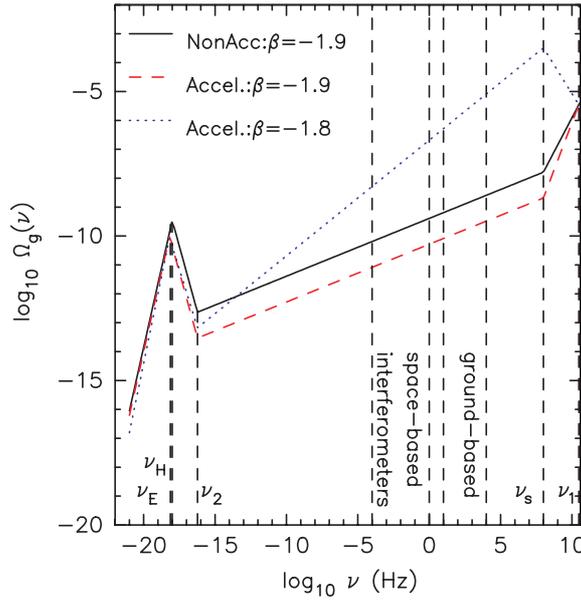


Figure 3. The energy spectrum $\Omega_g(\nu)$ in the present Universe in both accelerating and decelerating expansion.

Moreover, for the range of small frequencies, $\nu < \nu_H$, $\cos(2\pi\nu(t - t_\nu)) \simeq 1$, the oscillating amplitude is very small, confirming an observation made in [8]. Except for this oscillating effect, the numerical result of the spectrum in figure 2 agrees with the analytical one in figure 1.

We also plotted the spectral energy density $\Omega_g(\nu)$ from the analytic solution in figure 3. It is similar to the known result, except for the obvious distortions caused by the acceleration of the Universe expansion in the low-frequency range.

There is one more consistency condition to be satisfied. Since the spacetime is assumed to be spatially flat ($k = 1$) with $\Omega = 1$, the fraction density of relic gravitational waves should be less than 1, $\rho_g/\rho_c < 1$. Using the normalization of A in equation (38), we have integrated $\int \Omega_g(\nu) \frac{d\nu}{\nu}$ up to the frequency ν_1 , giving the present values

$$\begin{aligned} \rho_g/\rho_c &= 2.7 \times 10^{-6} & \text{for } \beta = -1.9, \\ \rho_g/\rho_c &= 1.2 \times 10^{-3} & \text{for } \beta = -1.8, \end{aligned}$$

both being less than 1, so the models are consistent. But if a limit of $\rho_g/\rho_c \leq 10^{-4}$, *a priori*, is imposed, then the model $\beta = -1.8$ is ruled out. However, in most cosmic scenarios the contribution from the relic gravitational waves to CMB anisotropies at low multipoles is approximately $\leq 1/4$ of that from the density perturbation; the normalization of A in equation (38) should be reduced correspondingly, and $\rho_g/\rho_c \leq 3 \times 10^{-4}$ for the model $\beta = -1.8$, which will be barely saved.

6. Conclusion

We have presented a calculation of the spectrum of relic gravitational waves in the present Universe in accelerating expansion. The recent WMAP result of $\Delta T/T$ has been used to normalize the amplitude of relic gravitational waves. In comparison with the decelerating

models, the spectrum has been modified due to the particular form of the barrier function a'/a during the acceleration of current expansion. Specifically, in the very low frequency range ($< \nu_H$) the spectrum has been changed with the peak of spectrum now being located at ν_E , and there appears a new segment of spectrum from ν_E to ν_H . These very long wavelength features can be only possibly be detected by the CMB anisotropies at low multipoles. In the higher frequency range ($\geq \nu_H$) the spectral amplitude acquires an overall factor $\frac{\Omega_m}{\Omega_\Lambda}$ as compared with the decelerating model. This higher frequency range is pertinent to detection projects such as LISA and LIGO. A larger value of β yields a flatter spectrum $h(\nu, \tau_H)$ as a function of ν , producing more power in the higher frequencies. The resulting sensitivity of the second scientific run of the LIGO detectors has put a restriction on the model parameter $\beta \leq -1.8$, consistent with that imposed by the normalization at low multipoles of CMB $\Delta T/T$.

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