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A Test of the MOND Theory and the Model of Dark Matter† *

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Abstract The predictions of the modified gravitational model MOND and the model of dark matter for the earth's gravitational system are examined. We focus on the commonly used MOND models. For the simplest of these models, we present a general expression for the gravitational potential in the case of spherical symmetry, and calculate the predicted angular velocity of a satellite moving in the field. It is found that the angular velocity is different in the different MOND models, that the differences between the predictions of these models and the Newtonian theory are very small, but that the difference in the simplest case ($n = 1$) is larger. In the case of the moon, actual measurements with existing techniques can be possibly made for this difference in angular velocity. We have also estimated the influence on the angular velocity of the moon due to the model of dark matter, and found it to be far smaller than the effect of the MOND theory. Therefore, the measurement of this difference in angular velocity may provide a criterion for discriminating the MOND theory and the model of dark matter.

Key words: gravitation—dark matter—celestial mechanics

1. INTRODUCTION

It is well known that the classical Newtonian theory of gravitation, as the weak-field approximation of the general theory of relativity, has been confronted with great difficulties in interpreting the curve of rotation of spiral galaxies. For instance, for spiral galaxies the curve of rotation, which shows the relation between the observed velocity of rotation v and

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the radial distance r, does not possess the Keplerian range of descent predicted by Newtonian dynamic theory, instead, it is an extended smooth curve. For clusters of galaxies, the virial masses of galaxies yielded by dynamic methods in Newtonian theory are far larger than those estimated with the luminosities of the galaxies [1–5]. The answers given by the model of dark matter to these problems are as follows [6–9]. Newtonian dynamics is still valid, but it is necessary to suppose that in cosmos there exists an immense amount of diffuse dark matter in the form of dark halos around galaxies and clusters of galaxies. The spatial scales may be many times larger than those of the luminous galaxies. Moreover, it is suggested that the particles of dark matter participate only in gravitational interaction, so their existence can be detected only via gravitational effect. By means of some appropriate distribution of dark matter in space, the above-mentioned smooth curve of rotation may be interpreted. There is another possibility, however. Namely, in the vicinity of galaxies there does not exist the so-called invisible dark matter, but instead, on the scale of galaxies, the Newtonian theory itself does not hold. In other words, the gravitational behavior might deviate from the general theory of relativity. This second possibility has been explored by many authors, in particular, Milgram et al.[10–15] made a most detailed study in the 1980s. These authors proposed a Modified Newtonian Dynamics (hereafter abbreviated to MOND). This theory revises Newton's second law or his gravitational theory, thereby accounting for the rotation curves of galaxies without invoking dark matter. Present observations on the scales of galaxies and clusters of galaxies cannot discriminate between these two theories. This implies that they are degenerate theories. Until the present, all the studies of the observational effect of the MOND theory have been focused on large-size bodies such as galaxies. Now we like to probe whether the effect of this theory can be directly observed in the terrestrial gravitational system on a rather smaller scale, and whether it could be compared with the effect of dark matter. If the effect can actually be measured and the two theories could be discriminated, then this would be of great significance for gravitational theory and astronomy. With the continual development of techniques of time determination, the accuracy of time measurement is being greatly raised all the time, and the detection of various subtle effects has become possible. Nowadays the stability of cesium atomic clocks has attained 10^{-15} s. With the method of time determination with atomic interference [16,17] currently being developed, the accuracy may attain 4×10^{-17} s within the next few years. Therefore the effect under investigation may be directly observed in the near future.

2. EQUATION OF MOTION IN MOND THEORY

As stated above, direct application of Newtonian dynamical theory to spiral galaxies has met with some difficulties. In order to overcome them, the existence of certain dark matter has been invoked. Up to now all the lines of evidence for the existence of dark matter are indirect ones. On the other hand, according to the MOND theory [10–15], it is not at all needed to introduce dark matter. According to it, Newtonian dynamical theory holds only in the extreme case of large acceleration $a \gg a_0$, and does not hold in the extreme case of small acceleration (or large scales) $a \ll a_0$, the constant acceleration $a_0 \approx 2 \times 10^{-8}(H_0/50\,\mathrm{km\,s^{-1}\,Mpc^{-1}})\,\mathrm{cm\,s^{-2}}$, marking the transition is a parameter obtained by fitting the observed rotation curves of galaxies (H_0 is the Hubble constant in cosmology).

At present all the researches on the gravitational effect of MOND theory are made for large-scale galaxies and clusters of galaxies. Moreover, both the MOND theory and model of dark matter can satisfactorily interpret the rotation curves and other observations. Now we like to discuss the MOND theory, for the following reasons. In the existing literature, this theory has not been applied to gravitational system of the earth and to its actual observations. Also, present studies have not made any discrimination of various models of MOND theory, or pointed out any possible observational differences. What is more important is that in the earth gravitational system the effects of dark matter and MOND theory are different, and this point will be proved in this paper. Specifically, the effect predicted by the MOND theory cannot be ignored and, on the technical level of time determination in the future, it might be actually observed. By contrast, for the same system the effect of dark matter is extremely trivial. This is important for a check of the MOND theory and model of dark matter. Now, let's briefly introduce the equation of motion in the MOND theory. According to its usual formulation, either the Newtonian gravitational theory (the inverse square law of gravity) or the second law in Newtonian mechanics should be modified. Maybe both have to be modified. In practice, when a gravitational field exists, the two alternatives are equivalent. Here, we do not intend to make a detailed study of the revision of gravitational theory or the technical details on the revision of the dynamical theory. What will be investigated is whether it is possible to discriminate the model of dark matter and the MOND theory via an analysis in the system of the earth's gravitation. According to the usual Newtonian theory of gravitation, the acceleration gained by a test particle in the gravitational field is equal to the gravitational acceleration $\boldsymbol{a} = \boldsymbol{g}_N$. However, according to the MOND theory, the acceleration gained by a test particle of unit mass in a given gravitational field, $\boldsymbol{a}$, is specified by the following dynamical equation:

$$\mu(a/a_0)\boldsymbol{a} = \boldsymbol{g}_N \,. \tag{1}$$

Here a_0 is the constant acceleration parameter of MOND theory, and $\boldsymbol{g}_N$ is the gravitational acceleration in the usual Newtonian theory. In the case of a symmetrical gravitational field with total mass M, we have $\boldsymbol{g}_N = -(GM\boldsymbol{r}/r^3)$, and the function $\mu(x)$ represents just the difference between the MOND and Newtonian theories. Because the study of the theoretical structure of MOND theory is still not so mature as that of the general theory of relativity, the function $\mu(x)$ in principle may be expressed in many different ways. For instance, one way is

$$\mu(x) = x(1+x^n)^{-1/n} \,, \tag{2}$$

with n an arbitrary positive number. In the literature, it is commonly taken to be either 1 or 2. So we have

$$\mu(x) = x(1+x)^{-1} \,, \tag{3}$$

$$\mu(x) = x(1+x^2)^{-1/2} \,. \tag{4}$$

In the extreme case of $x \gg 1$, all Eqs.(2)-(4) have $\mu(x) \approx 1$, then the MOND theory returns to the Newtonian theory. However, in the extreme case of $x \ll 1$, all the equations have $\mu(x) \approx x$, then the MOND theory deviates greatly from the Newtonian theory. But, besides these differences between the two extreme cases, the functions $\mu(x)$ for different values of n also lead to different forms of Eq.(1). In the following we shall spell out their differences in physical prediction. Let us substitute in Eq.(1) the Eq.(3) in the simplest case of $n = 1$ in MOND theory, then we have the solution for the acceleration, $a/a_0 = (g_N/a_0 + \sqrt{(g_n/a_0)^2 + g_N/a_0})/2$. Here $g_N = GM/r^2$. Moreover, we use the relation between the acceleration a and the equivalent gravitational potential φ, $a = -\nabla\varphi$, and make a direct integration in the radial direction, and we obtain the following usual expression of equivalent gravitational potential j in the case of spherical symmetry,

$$\varphi(r) = -GM/2r - GM\sqrt{(1 + 4a_0/g_N)}/2r + \sqrt{(a_0/GM)}\ln(\sqrt{(a_0/g_N)} + \sqrt{(1 + 4a_0/g_N)}) + c, \tag{5}$$

where c is a constant of integration. When $a_0/g_N \ll 1$, Eq.(5) returns to the usual Newtonian potential, i.e. $\varphi(r) \approx -GM/r$. When $a_0/g_n \gg 1$, $\varphi(r) \approx (MGa_0)^{1/2}\ln(r/r_0)$, where r_0 is an arbitrary radial distance, and this is just the extreme case formula given by Milgrom [10]. If the MOND theory is like the theory of general relativity and a metric can be used to express gravitational field, then the $\varphi(r)$ thus obtained can be used to calculate the light-time delay, etc. Here, we do not wish to discuss this problem in detail.

3. SATELLITE MOTION IN THE MOND THEORY

We apply the equation of motion (1) in the MOND theory to a gravitational system with the scale of the earth, propose a theoretical prediction and examine whether the effect may be observed. Specifically, we first calculate the theoretical value of the angular velocity of an earth's satellite according to Eq.(1), then we compare it with the calculated result of the Newtonian theory. We carry out calculations for the two MOND models with $n = 1$ and $n = 2$ as represented by Eqs.(3) and (4). The earth with mass M is taken to be the source of gravitation with spherical symmetry. We first consider a satellite moving in a circular orbit of radius r around the earth. Substituting Eqs.(3) and (4) into Eq.(1), we get, respectively,

$$a = \frac{g_N}{2} + \frac{g_N}{2}\sqrt{1 + 4a_0/g_N}\,, \tag{6}$$

$$a = \frac{g_N}{2}\sqrt{1 + \sqrt{1 + 4a_0^2/g_N^2}}\,. \tag{7}$$

In the above, the earth 's radius is taken to be $r = 6.4\times10^3$ km. Then we get $a_0/g_N \ll 1$. This implies that for the earth-satellite system the MOND theory differs very little from the Newtonian theory. Taking a_0/g_N as a small quantity, making Taylor expansions of Eqs.(6) and (7) and retaining only the lowest order terms, we have

$$a = g_N(1 + a_0/g_N)\,, \tag{8}$$

$$a = g_N(1 + a_0^2/2g_N^2)\,. \tag{9}$$

Eqs.(8) and (9) are the expressions for the acceleration, and they contain terms of revision to different powers of the small quantity a_0/g_N. The centripetal acceleration is given by a formula of dynamics, i.e. $a = \omega^2 r$, where ω is the angular velocity of the satellite. Substituting Eqs.(8) and (9) into this formula and retaining terms up to the respective lowest orders, we get the following expressions of angular velocity for the two models with $n = 1$ and $n = 2$ of the MOND theory,

$$\omega = \omega_0(1 + a_0 r^2/2GM)\,, \tag{10}$$

$$\omega = \omega_0(1 + (a_0 r^2/2GM)^2)\,, \tag{11}$$

where ω_0 is the angular velocity given by Newtonian gravitational theory, i.e. $\omega_0 = (GM/r^3)^{1/2}$. Then Eqs.(10) and (11) may be written as:

$$(\omega - \omega_0)/\omega_0 = a_0 r^2/2GM\,, \tag{12}$$

$$(\omega - \omega_0)/\omega_0 = (a_0 r^2/2GM)^2\,. \tag{13}$$

These are the explicit expressions for the difference in angular velocity between the Newtonian theory and the two models of the MOND theory. For the ordinary model represented by Eq.(2) and with an arbitrary n, we can get by the above method of deduction:

$$(\omega - \omega_0)/\omega_0 = (a_0 r^2/2GM)^n/2n\,. \tag{14}$$

It may be seen that the larger n is, the smaller the difference in the angular velocity will be. Moreover, the larger the radius r of the satellite's orbit, the larger the difference. This is precisely the characteristic of the MOND theory: it clearly differs from the Newtonian theory on large scales.

Substituting the orbital radius r and the earth's mass M into Eq.(12), we get $(\omega - \omega_0)/\omega_0 \approx 10^{-11}$. Substituting these into Eq.(13), we have $(\omega - \omega_0)/\omega_0 \approx 10^{-21}$. Again substituting the last into Eq.(14), we obtain $(\omega - \omega_0)/\omega_0 \approx 10^{-11n}/n$. This implies that the model with $n = 1$ has the largest difference of angular velocities. For the other models, the value decreases by a factor of 10^{-11} at each step. As may be seen from Eq.(12), in order to enlarge the difference of angular velocities, we need to increase the value of r. As a satellite of the earth, the moon may be thought to revolve in an approximately circular orbit. Substituting the earth-moon distance $r = 3.8 \times 10^5$ km into, respectively, Eqs.(12) and (13), the difference of the moon's angular velocities given by the MOND theory and Newtonian theory can be obtained, and they may be transformed to the equivalent differences of periods, i.e.

$$(T_0 - T)/T_0 = 3.5 \times 10^{-8}\,, \tag{15}$$

$$(T_0 - T)/T_0 = 1.2 \times 10^{-15}\,. \tag{16}$$

Both these values are very small, and they are equivalent to discrepancies of, respectively, 10^{-3} s and 10^{-10} s in one revolution of the moon. Nowadays the accuracy of actual measurement of the period of the moon's revolution has attained the level of 10^{-9} s [18]. The period difference given by Eq.(15) is the largest, and it lies in the range of measurements and can be directly measured. The period difference given by Eq.(16) is far smaller than the accuracy of measurements. For the model with $n > 2$, the difference of periods will be smaller still. Therefore, if a sufficiently large n is taken in the MOND model then a null result in the period will not negate the MOND theory. In the deduction of Eqs.(12) and (13) we assumed the values of M and r to be accurate, but in practice they themselves possess some errors. As these involve problems of environmental influence and techniques, they will not be discussed here. We believe that with the progress of time determination with atomic clocks in space [16,17], it will be possible in the foreseeable future to carry out practical observations. Let's suppose that within the accuracy of observation such a difference is not discovered, then MOND models with $n > 1$ will still not be denied. At the same time, the dark matter model remains to be viable. However, if the frequency difference is actually observed, then the dark matter model will have difficulties and the MOND theory will be extremely worthy of investigation.

4. INFLUENCE OF MODEL OF DARK MATTER ON EARTH-MOON SYSTEM

It is well known that up to the present the existence of dark matter is still a theoretical hypothesis, and the particles of dark matter have not yet been directly observed. So the model of dark matter is in the same status as that of the MOND theory, that is, both are still theoretical models awaiting observational confirmation. Evidence for the existence of dark matter comes mainly from the rotation curves of spiral galaxies, from the mass-luminosity ratios of clusters of galaxies [20,21], of the central nuclei of elliptical galaxies [22] and of the solar neighborhood of the Galactic disk [23], and from a necessity in the theory of formation of large-scale structures in the universe [24]. Hence it may be inferred that if cosmic dark matter really exists, then it will be widespread around various cosmic bodies, especially in the vicinity of the Galaxy. So in the neighborhood of the earth there should also be dark matter. However, it may be predicted that any dark matter around the earth will be extremely dilute, and only by means of theoretical analysis can we predict whether this dark matter may be directly observed. In the existing models of dark matter, the spatial distribution of the energy density of dark matter has not yet been determined, and this is a topic of general interest. Now we like to adopt a rather simple model and assume that the energy density ρ of the halos of dark matter in the Galaxy exhibits the following dependence on the Galactocentric distance r,

$$\rho(r) = \rho_0 \frac{r_2^2 + b^2}{r^2 + b^2}\,. \tag{17}$$

Here, the parameter $r_2 \approx 8.5$ Kpc is Galactic radius of the earth, and $b \approx 50$ Kpc stands for the radius of the Galactic halo, and ρ_0 is the energy density of dark matter in the neighborhood of the earth. From the existing data of astronomical observations it may be inferred that $\rho_0 = 7.1 \times 10^{-25}\,\mathrm{g\,cm^{-3}}$. Here we concretely investigate the earth-moon gravitational system in the framework of Newtonian gravitational theory and estimate the influence of dark matter on the orbit of the moon around the earth. The model represented by Eq.(17) can approximately illustrate the overall distribution of dark matter in the interior of the Galaxy. Nevertheless, we know nothing about the spatial distribution of local dark matter near the earth. According to a physical conjecture, due to the gravitational attraction of the earth the density of dark matter is larger near the earth than in the space far away. Moreover, the former may be spherically symmetrical. For the sake of simplicity, we take the energy density of dark matter in the vicinity of the earth to be a constant ρ_d. According to the Newtonian mechanical theory, the gravitational force of dark matter on the moon is equivalent to the attraction produced by all the dark matter inside the sphere of the lunar orbit and put together at the earth's center. Then the moon's equation of motion is

$$m_m \omega_1^2 r_1 = G m_m (M + 4\pi_1^3 \rho_d/3)/r_1^2\,. \tag{18}$$

Here m_m, M and r_1 are, respectively, the moon's mass, the earth's mass and the distance between their centers, and ω_1 is the moon's angular velocity around the earth. As may be learned from Eq.(18), the gravitational effect of the dark matter is superimposed on that of the earth, and it is equivalent to a small increment of the earth's mass, i.e. $4\pi r_1^3 \rho_d/3$. The magnitude of this increment depends on the density of dark matter near the earth. If its value is the ρ_0 given by Eq. (17), then this increment, $4\pi r_1^3 \rho_d/3 \approx 10^{-21} M$, is extremely trivial. Even if ρ_d is by several orders of magnitude larger than ρ_0, there can still be hardly any observable effect. In short, in the case of spherically symmetric distribution, it is difficult to discriminate the gravitational effects of dark matter and the earth. If the distribution is not spherically symmetrical, then the dark matter's gravitational effects at different places of the moon's orbit will be different and the motion of the moon will change according to its location in the orbit. The density distribution of dark matter represented by Eq.(17) is asymmetrical for the earth: on the moon's orbit, the density on the side toward the Galactic center is higher, while that on the outward side is lower. However, relative to the scale of the Galaxy (10 Kpc), the moon's orbit around the earth is too small and the difference of densities on two sides, i.e. $\Delta\rho/\rho$, is merely 10^{-17}. Moreover, the density of dark matter near the earth is actually quite small. Hence the influence of $\Delta\rho$ on the moon's motion is exceedingly trivial, and at present it still cannot by any means be detected. What appears to be important is that this effect is far smaller than the amount of revision of the moon's orbital angular momentum given by Eqs.(10) and (11) by the MOND models. So this effect of dark matter cannot be directly observed on the earth even in the foreseeable future, unless we suppose that the density of dark matter near the earth is many orders of magnitude larger than the average density of dark matter in the Galactic System and that its distribution is extremely asymmetrical; otherwise, the dark matter in the vicinity of the earth has no observable effect. Of course, this discussion is an exceedingly idealized one. We have not yet taken into account the effects of other factors, such as the existence of other gases between the earth and moon, stochastic shakings on their surfaces, etc. In short, we think that for the terrestrial gravitational system the effect of dark matter may be completely ignored. However, the effect of the model with $n = 1$ in the MOND theory is not negligible, and this provides a new observational criterion for discriminating the model of dark matter and

MOND theory. This has very important significance for both astronomical research and the development of new theory of gravitation.

5. CONCLUSIONS

For the earth's gravitational system, the angular velocity of a satellite around the earth predicted by the MOND theory differs slightly from that predicted by the Newtonian theory. Moreover, the angular velocities yielded by various models in MOND theory are also different. Among these, the simplest model, namely the one with $n = 1$, gives rise to the largest difference in the angular velocity, and it is of the order of magnitude $(\omega-\omega_0)/\omega_0 \approx a_0^2/2GM$. For the moon's revolution around the earth, this difference amounts to 10^{-11} and lies in the range of the present time determination and may be actually observed. The differences of angular velocities yielded by other models are higher-order powers of $a_0^2/2GM$, and cannot by actually measured. This provides an observational criterion for discriminating different models in the MOND theory. We have also in this work made an estimation of the dark matter that possibly exists in the earth's gravitation system. We have shown that the influence of dark matter on the angular velocity of satellite is extremely trivial, and it is far smaller than the effect of MOND theory: it cannot be observed by any present-day means. This provides a new observational criterion for discriminating the MOND theory and the model of dark matter in the terrestrial gravitational system. If this effect is actually measured in the future, it is needless to say that this effect possesses important significance for astronomy and theoretical research of gravitation.

References

1 Oort J.H., ApJ, 1940, 91, 273
2 Faber S.M., Gallagher J.S., ARA&A, 1979, 17, 135
3 Binney J., ARA&A, 1982, 20, 399
4 Rubin V.C., Burstein D., Ford W.K., et al., ApJ, 1985, 289, 81
5 Sofue Y., Robin V., ARA&A, 2001, 39, 137
6 Zwicky F., Helvetica Physica Acta, 1933, 6, 110
7 Ostriker J.P., Peebes P.J.E., Yahil A., ApJ, 1971, 193, L1
8 Einasto J., Kassik A., Saar E., Nature, 1974, 250, 309
9 Trimble V., ARA&A, 1987, 25, 425
10 Milgrom M., ApJ, 1983, 270, 365, 371, 384; ApJ, 1984, 287, 571; ApJ, 1986, 302, 617
11 Bekenstein J., Milgrom M., ApJ, 1984, 286, 7
12 Sanders R¿H¿, ApJ, 1996, 473, 117; ApJ, 1997, 480, 492; A&AR, 1990, 2,1
13 Begeman K.G., Broeil A.H., Sanders R.H., MNRAS, 1991, 249, 523
14 Sanders R.H., Verheijen M.A.W., ApJ, 1998, 503, 97
15 de Blok W.J.G., Mcgaugh S.S., ApJ, 1998, 508, 132
16 Wilpers G., et al., ArXiv:physics/0205049
17 Borde C.J., Comptes Rendus IV Physique Astrophysique, 2001, 2, 509
18 Cox A.N., Allen' Astrophysical Quantities (AIP Press, Springer, 4th ed.), 2000
19 Freeman K.C., ApJ, 1970, 160, 881
20 Zwicky F., Helvetica Physica Acta, 1933, 6, 110

21 Smith S., ApJ, 1936, 83, 23
22 Lauer T.R., ApJ, 1985, 292, 104
23 Bahcall J.N., ApJ 1984, 276, 169; ApJ, 1984, 287, 926
24 Blumenthal G.R., Faber S.M., Primack J.R., et al., Nature, 1984, 311, 517
25 Kamionkowski M., Dark Matter, 2nd Int. Conf. in Astro. and Particle Phys. (Heidelberg), 1998