ELSEVIER

Chinese Astronomy and Astrophysics 29 (2005) 250–262

CHINESE
ASTRONOMY
AND ASTROPHYSICS

An Approximate Analytic Formula for the Polarization of Cosmic Microwave Background Radiation† *

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Abstract In the decoupling of the early universe, photons interacted with electrons via Thompson scattering, whereby any anisotropy in the spatial distribution of the photon gas will lead to polarization of the cosmic microwave background (CMB) radiation which has recently been observed by WMAP . In this paper, starting from the Boltzmann equation of the photon gas and adopting the general expression of the optical depth function, we have derived separate, approximate analytic solutions for the polarization of the CMB radiation induced by primordial density perturbation and remnant gravitational waves, solutions that are valid for the general recombination process. For the scalar type of perturbations F_s, the derived approximate analytic expression of polarization is $\beta_s \simeq -CF_s(\tau_d)\Delta\tau_d$, where τ_d and $\Delta\tau_d$ are the decoupling time and the decoupling duration, respectively, and $C \simeq (0.08 \sim 0.12)$ depending on the recombination model For the tensorial type of perturbations F_T, the calculation is made in the long-wave approximation By expanding the perturbation function in terms of the wave number and keeping the first two terms $F_T \simeq F_T^{(1)} + F_T^{(2)}$, we obtain the analytic expression of polarization $\beta_T \simeq -[CF_T^{(1)}(\tau_d)+DF_T^{(2)}(\tau_d)]\Delta\tau_d$, where $D \simeq (0.22 \sim 0.32)$, again depending on the recombination model. Our results may help to interpret the observed temperature-polarization cross correlation and to detect the contribution of remnant gravitational waves to the anisotropy of the CMB radiation.

Key words: cosmic microwave background —early universe—cosmological parameters

† Supported by National Natural Science Foundation

Received 2004-03-25, revised version 2004-06-09

* A translation of *Acta Astron Sin* Vol 46, No 1, pp 1–11, 2005

DOI. 10.1016/j.chinastron 2005 06.003

1. INTRODUCTION

To study the cosmic microwave background (CMB) radiation is not only necessary for exploring the evolution of the universe, and it is also important for the other branches of astronomy and particle physics. In the recent ten years or more, many achievements have been made in the observational and theoretical research of the CMB radiation. The large-scale inhomogeneity of the microwave background radiation $\Delta T/T \simeq 2 \times 10^{-5}$ was detected for the first time by COBE (Cosmic Background Explorer)[1], since then, observations are gradually moving into smaller scales[2–4] Recently, the observations of WMAP (Wilkinson Microwave Anisotropy Probe) have not only obtained a rather complete curve (the heights and positions of the first three peaks) for the power spectrum C_l of $\Delta T/T$[5,6], but also observed clearly the polarization of the CMB radiation[7–9], and calculated the temperature-polarization cross correlation, which indicates that a reionization happened in the early universe. These new observational results are of important significance.

The mechanism leading to the polarization of the CMB radiation is the recombination process in which protons and electrons in the ionized state were combined into neutral hydrogen in the early universe at redshift $z \sim 1000$. At that time a large number of free electrons still existed and, as the gas temperature was far less than the rest mass of the electron, the electrons were non-relativistic. When photons and electrons interacted with each other, Thompson scattering took place. If the photon gas was completely homogeneous and isotropic in space, then the Thompson scattering will not make the photon gas polarized. In fact, in the universe the spatial distribution of the photon gas is inhomogeneous, containing particularly the quadrupole components of inhomogeneity $\propto Y_{lm}(\theta, \phi), l = 2$, so after the Thompson scattering the photon gas became linearly polarized. After the recombination process the protons and electrons in the universe were combined into neutral hydrogen, and after the decoupling of the photons and electrons, the polarization of photons remains to be the observed polarization of the CMB radiation now. The observations of WMAP indicate that after the decoupling the reionization process took place which further affected the polarization of the CMB radiation At present, the mechanism of reionization is not yet clear and still awaiting further study We will not discuss it any more in this paper

In the theoretical study of the CMB, we mainly deal with the group of Boltzmann equations of photons, baryons, neutrinos and other probably existing dark matter in the expanding Robertson-Walker spacetime. Not only does this group of equations contain couplings among the different components, a small perturbation of spacetime itself will enter directly into the equations of the different components to affect the perturbations of the different components. Therefore, this group of equations involve a number of physical quantities and parameters, and their solution becomes rather complicated. Some solution can be made by some numerical method (such as cmbfast and so on) or some approximate analytic method[10]. The latter is very important because it can display clearly the roles of the different parameters in the cosmological model, so leading to a better understanding of the various physical mechanisms and characteristics of the CMB. An important physical quantity in the polarization theory of the CMB is the function β, i.e. the Fourier component of linear polarization in the momentum space, by which the autocorrelation function of polarization, temperature-polarization cross correlation and other important observable quantities can be built up References [11,12] have given the approximate analytic expression $\beta \simeq -0.17F(\tau_d)\Delta\tau_d$, in which $F(\tau_d)$ is the value of the primordial perturbation at the time of decoupling τ_d, and $\Delta\tau_d$ is the duration of the decoupling. But in the derivation of Reference [11], as an approximation, the integral was simply taken to be the product of the peak value of a Gaussian approximation of the curve of visibility function and the decoupling duration (width of the peak). Generally, the curve of the visibility function for the recombination process consists of a very sharp peak and a rather broad wing on each side, the result obtained by the above direct multiplication is obviously greater than the actual value, even by a factor of several units. Besides, in this way the particular optical depth function during the recombination process and the effect of the baryon density Ω_b can not be seen

In this paper, the polarization formula will be re-derived by adopting a general optical-depth function for the recombination process and by improving the integration process. In addition, the values of the decoupling time τ_d and the decoupling duration $\Delta\tau_d$ will be given in a conformal time-coordinate by adopting the newest WMAP data. The result fits the general recombination process, and it shows obviously the effect of the cosmological parameter, the baryon density Ω_b, on the polarization. We will give the approximate analytic solution of the polarization by integrating separately primordial perturbations of the scalar type (density) and tensorial type (gravitational wave) .

2. MODEL OF THE OPTICAL DEPTH OF THE RECOMBINATION PROCESS

As we have mentioned, the formation of the CMB was performed in an expanding spacetime background, affected by perturbations of the spacetime metric and by anisotropy of the photon gas. We take the Robertson-Walker metric of flat space as the spacetime background,

$$\mathrm{d}s^2 = a^2(\tau)[-\mathrm{d}\tau^2 + (\delta_{ij} + h_{ij})\mathrm{d}x^i\mathrm{d}x^j]\,, \tag{1}$$

in which h_{ij} is the spacetime perturbation, representing the inhomogeneity and anisotropy of generally both scalar and tensorial types; τ is the conformal time—it is convenient to use the conformal time coordinate system The recombination process, during which polarization is produced, is in the matter-dominant period The scale factor is $a(\tau) = (\tau/\tau_0)^2$, in which τ_0 is the present time, and it is equivalent to normalizing the scale factor by $a(\tau_0) = 1$. The redshift is defined by $a(\tau) = a(\tau_0)/(1+z)$, from which we can obtain the relation between the conformal time and the redshift in the matter-dominant period: $\tau = \tau_0/\sqrt{1+z}$ and $d\tau = -\frac{\tau_0}{2}(1+z)^{-3/2}dz$.

The details of the recombination process, forming neutral hydrogen atoms from protons and electrons, affect directly the production of polarization, in which the major relevant physical quantity is the optical depth function $\kappa(\tau)$, which has been studied in many papers [13−15]. The optical depth function from the present time τ_0 ($z_0 = 0$) to a past time τ (redshift z) is defined as.

$$\kappa(\tau) \equiv \kappa(\tau_0, \tau) \equiv \int_\tau^{\tau_0} q(\tau')\mathrm{d}\tau'\,, \tag{2}$$

in which $q(\tau)$ is the differential optical depth, proportional to the degree of ionization, the number density of electrons and the Thompson scattering cross-section. q depends on the total amount of matter Ω and other cosmological parameters, and it is generally defined by the adopted cosmological model and the particular recombination model By definition we have

$$\kappa(\tau,\tau') = \kappa(\tau') - \kappa(\tau)\,. \tag{3}$$

Instead of τ, the redshift z is generally taken as the argument of the optical depth function, and is denoted by $\kappa(0,z)$ Studies indicate that for the general recombination process in the early universe, the optical depth function can be expressed in the following power-law form.

$$\kappa(z) = \kappa(0,z) = b\left(\frac{z}{1000}\right)^c\,, \tag{4}$$

in which b and c are dimensionless parameters to be determined by the particular model. The differential optical depth is

$$q = -\frac{\mathrm{d}\kappa(\tau)}{\mathrm{d}\tau} = -\frac{\mathrm{d}z}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}z}\kappa(z)\,. \tag{5}$$

The quantity $e^{-\kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}z}$, which appears very often in the discussion of the recombination process, is called the visibility function, and its physical meaning is the probability that the photon was last scattered at redshift z The curve of the visibility function is single-peaked, close to a Gaussian distribution But for the recombination process, the curve of its visibility function has a very high and sharp peak, as well as rather broad gentle slopes on the two sides of the peak. In the discussions of this paper the following two commonly used models will be used

(1) Solving the equation of the degree of ionization with the calculation of the recombination process, and fitting to observational data, Jones et al [14] obtained the optical depth function:

$$\kappa(0,z) = 0\ 37\left(\frac{z}{1000}\right)^{14\ 25}\,, \quad 800 < z < 1200\,. \tag{6}$$

The corresponding parameters are $b = 0\ 37$, $c = 14\ 25$.

(2) The optical depth is related to the baryon density Ω_b in the universe. Especially for the dark matter model, in which $\Omega_b \ll \Omega$, the above recombination model has to be revised Considering the correction for the baryon composition Ω_b in the model of $\Omega = 1$, Hu et al [10] have presented the following optical depth formula·

$$\kappa(0,z) = \Omega_b^{0\ 43}\left(\frac{z}{1000}\right)^{16+1\ 8\ln\Omega_b}\,, \quad 800 < z < 1200\,, \tag{7}$$

corresponding to $b = \Omega_b^{0\ 43}, c = 16 + 1\ 8\ln\Omega_b$. This model is quite different from the Jones-Wyse model If we take the baryon density $\Omega_b = 0.045$, then the two models will have a difference of about 60% in the peak value and width of the visibility function curve, and a corresponding difference between the CMB polarizations given by these two recombination models is expected

3. BOLTZMANN EQUATION FOR THE CMB POLARIZATION

For the study of the CMB, the usual way is to obtain a series of equations of multipole moments by spherical-harmonic expansion, then to solve the equations according to the order But for an approximate analytic solution of polarization, it is simpler mathematically to adopt the Basko-Polnarev method[16,17] The distribution function $\boldsymbol{n}$ of polarized photons satisfies the Boltzmann equation[18]

$$\left(\frac{\partial}{\partial\tau} + e^i \frac{\partial}{\partial x^i}\right) \boldsymbol{n} = -\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\frac{\partial \boldsymbol{n}}{\partial\nu} - q(\boldsymbol{n} - \boldsymbol{J}) , \tag{8}$$

where

$$\boldsymbol{J} = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} P(\mu, \phi, \mu', \phi')\boldsymbol{n}\mathrm{d}\mu'\mathrm{d}\phi' , \tag{9}$$

in which e^i is the unit vector in the propagating direction of photons, specified in the given spherical coordinate by (θ, ϕ). The differential optical depth is $q = \sigma_T N_e a$, σ_T and N_e are the Thompson scattering cross-section and the number density of free electrons, respectively. $P(\mu, \phi, \mu', \phi')$ is the square matrix describing the effect of the electronic gas component on the photons, i.e , the effect of the Thompson scattering on the polarization[18], and $\mu = \cos\theta$ $-\frac{\mathrm{d}\nu}{\mathrm{d}\tau}\frac{\partial \boldsymbol{n}}{\partial\nu}$ on the right side of the equation represents the variation of the photon frequency ν caused by the perturbation of spacetime metric. More specifically, it is reflected by the Sachs-Wolf effect:

$$\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = \frac{1}{2}\frac{\partial h_{ij}}{\partial\tau} e^i e^j$$

Here, the metric perturbation h_{ij} may be a density perturbation or gravitational waves The most common distribution function of polarized photons $\boldsymbol{n}$ is the column matrix $\boldsymbol{n}(\nu, \theta, \phi) = (I_l, I_r, U, V)$, whose components are the 4 Stokes parameters. In this paper we study the Thompson scattering which induces only a linear polarization, and the parameter of circular polarization is $V = 0$. The radiation intensity is $I = I_l + I_r$, and $Q = I_l - I_r$ describes the linear polarization. For polarization induced by density perturbations, we can take $U = 0$. But for polarization induced by gravitational waves, we have generally $U \neq 0$ Adopting the Basko-Polnarev method, we assume that $\boldsymbol{n}$ is divided into two parts, i.e., a non-polarized and a polarized part.

$$\boldsymbol{n} = n_0 \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \boldsymbol{n}_1\right] ,$$

in which n_0 is the Planck spectrum, dependent only on the frequency ν, and independent of the direction. $\boldsymbol{n}_1$ is the polarized part that we are mostly concerned with. After the Fourier transformation in space domain, its components satisfy the following equation

$$\frac{\partial \boldsymbol{n}_1(k)}{\partial\tau} + ik\mu\boldsymbol{n}_1(k) = \gamma\mu^2 F - q(\boldsymbol{n}_1(k) - \boldsymbol{J}(k)) , \tag{10}$$

in which $\gamma \equiv \dfrac{\mathrm{d}\ln n_0}{\mathrm{d}\ln\nu} \simeq 1$ in the Rayleigh-Jeans low-frequency region, $\mu^2 F$ is the Fourier component of the Sachs-Wolf effect term $(-\frac{1}{2}\dot{h}_{ij}e^i e^j)$, and F is the primordial perturbation source For the density perturbation, let

$$n_1(k) = \alpha(\mu^2 - \frac{1}{3}) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta(1-\mu^2) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

for the perturbation of gravitational waves, let

$$n_1(k) = \frac{\alpha}{2}(1-\mu^2)\cos 2\phi \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} (1+\mu^2)\cos 2\phi \\ -(1+\mu^2)\cos 2\phi \\ 4\mu \sin 2\phi \end{pmatrix},$$

in which α and β are functions of time to be defined, depending on the wave number k, and β is the polarization Thus the complicated Boltzmann equation is reduced to the following group of 2-element first-order differential equations with the same coupling[11,16,17,19,20]

$$\frac{\partial \xi}{\partial \tau} + q\xi = F,$$

$$\frac{\partial \beta}{\partial \tau} + \frac{3}{10} q\beta = -\frac{1}{10} q\xi,$$

in which $\xi \equiv \alpha + \beta$. By formal integration, we can obtain the following formulae.

$$\xi(\tau) = \int_0^\tau F(\tau')e^{-\kappa(\tau,\tau')}\mathrm{d}\tau', \tag{11}$$

$$\beta(\tau) = -\frac{1}{10}\int_0^\tau q(\tau')\xi(\tau')e^{-\frac{3}{10}\kappa(\tau,\tau')}\mathrm{d}\tau'. \tag{12}$$

From these two integration expressions, we see that the optical depth function κ representing the recombination process, as well as the primordial perturbation F, will affect the resulting polarization The procedure of calculating the polarization is to obtain first $\xi(\tau)$ by integration, and then inserting it into the second integration to obtain $\beta(\tau)$.

4. THE CMB POLARIZATION INDUCED BY DENSITY PERTURBATIONS

According to the current understanding of cosmology, in the inflationary stage of the early universe, quantum zero-point vibrations of the quantum field and gravitational field of the dominant matter will be elongated by the expansion of the universe, and this makes the large-scale matter distribution in the universe and the spacetime metric itself inhomogeneous and anisotropic As indicated by the linearized Einstein equations, the inhomogeneity and anisotropy of the matter distribution and those of the metric itself are interrelated and interactive[21]. When fluctuations corresponding to the inhomogeneity and anisotropy enter again the visual field of the universe, they become the tiny primordial perturbations[22] As mentioned before, the primordial perturbations can be divided into two types· density perturbation F_S and the perturbation of the remnant gravitational waves F_T, and they affect the photon gas via the Sachs-Wolf effect. In this paper, we assume that the primordial perturbations are given. Next, we will calculate first the polarization induced by the density perturbation. The calculation is basically similar for the perturbation of remnant gravitational waves

The density perturbation source is[21]

$$F_S(\tau) = \frac{1}{15}\zeta(k)k^2\tau\,. \tag{13}$$

This is a linear function of the time τ, in which $\zeta(k)$ is a function of the wave number k, independent of τ, and can be determined by the inflation model.

Let us calculate $\xi(\tau)$. Substituting $F_S(\tau)$ into Eq (11) and from Eq.(13), we have

$$\xi_S(\tau) = \frac{1}{15}\zeta(k)k^2 e^{\kappa(\tau)}A(\tau)\,, \tag{14}$$

in which

$$A(\tau) = \int_0^{\tau} \tau' e^{-\kappa(\tau')}\mathrm{d}\tau'\,. \tag{15}$$

This integration is difficult to yield a strict analytic solution. Reference [11] made an approximation for the integration $\xi_S(\tau)$ as follows: first, replace the factor $F(\tau)$ of the integrand with its value at the time of decoupling, i e , $F(\tau_d)$, and it is equivalent to $A(\tau) \simeq \tau_d \int_0^{\tau} e^{-\kappa(\tau')}\mathrm{d}\tau'$ Then, make the approximation $\frac{\mathrm{d}\kappa(\tau')}{\mathrm{d}\tau'} \simeq -\frac{\kappa(\tau')}{\Delta\tau_d}$. In fact, near the decoupling time we have $(-\frac{\kappa(\tau')}{\Delta\tau_d})/(\frac{\mathrm{d}\kappa(\tau')}{\mathrm{d}\tau'}) \simeq z_d/\Delta z_d \sim 5 \gg 1$. So the two approximations have rather large errors, and they can not reflect the effects of different recombination models. The following is our improved approximation for the integration Replacing the variable of integration with z and inserting the general optical depth formula $\kappa(\tau')$ into Eq.(15), we have

$$A(\tau) = -\frac{\tau_0^2}{2}\int_{\infty}^{z}(1+z)^{-2}\exp\left(-b(\frac{z}{1000})^c\right)\mathrm{d}z\,.$$

The lower limit of this integration $z = \infty$ corresponds to $\tau = 0$. Our final purpose is to obtain β_S by integration. The factor $q(\tau')e^{-\frac{3}{10}\kappa(\tau,\tau')}$ in the integrand is similar to the visibility function $q(\tau')e^{-\kappa(\tau,\tau')}$, and it makes its contribution mainly around the decoupling time $z \approx (800 \sim 1400)$. As $A(\tau)$ appears in integrand of β_S and it is multiplied with the factor $q(\tau')e^{-\frac{3}{10}\kappa(\tau,\tau')}$, so $A(\tau)$ will make its contribution around the decoupling time as well Thus we can make an approximation for $A(\tau)$ by $(1+z)^{-2} \sim z^{-2}$. Introducing the variable of integration $x \equiv (z/1000)^c$, the optical depth function becomes $\kappa(\tau) = bx$. And we have

$$A(\tau) \simeq -\frac{\tau_0^2}{2}10^{-3}c^{-1}\int_{\infty}^{x} x^{-(1+\frac{1}{c})}e^{-bx}\mathrm{d}x\,.$$

The integration in the above formula is an incomplete Γ function (refer to p 317 of Reference [23]), so

$$A(\tau) \simeq \frac{\tau_0^2}{2}10^{-3}c^{-1}\frac{1}{b^c}\Gamma(-\frac{1}{c}, bx)\,. \tag{16}$$

And we obtain ξ_S to be

$$\xi_S(\tau) \simeq B(k)e^{bx}\Gamma(-\frac{1}{c}, bx)\,, \tag{17}$$

in which

$$B(k) \equiv \frac{1}{15}\zeta(k)k^2\frac{\tau_0^2}{2}10^{-3}\frac{1}{c\,b^{1/c}} \tag{18}$$

From the result obtained by WMAP recently[24], the redshift at the decoupling time is $z_d = 1089 \pm 1$, the decoupling duration is $\Delta z_d = 195 \pm 2$, and these are equivalent to $\tau_d \simeq 3 \times 10^{-2}\tau_0$, $\Delta\tau_d \simeq 2.7 \times 10^{-3}\tau_0$ Then, by replacing τ_0 in the above formula with τ_d and $\Delta\tau_d$, we obtain

$$B(k) \simeq \frac{50}{8\,13c\,b^{1/c}}F_S(\tau_d)\Delta\tau_d\,. \tag{19}$$

Next, we will do the integration with the obtained ξ_S to calculate the polarization $\beta(\tau')$.

From Eq (3), the integration for $\beta_S(\tau)$ can be reduced to

$$\beta_S(\tau) = -\frac{1}{10}e^{\frac{3}{10}\kappa(\tau)}\int_0^{\tau} q(\tau')\xi_S(\tau')e^{-\frac{3}{10}\kappa(\tau')}\mathrm{d}\tau'\,.$$

Inserting into $\xi_S(\tau)$ and using $q(\tau')d\tau' = -d\kappa = -bdx$, we obtain

$$\beta_S(\tau) \;=\; \frac{1}{10}e^{\frac{3}{10}\kappa(\tau)}bB(k)\int_{\infty}^{x}\Gamma(-\frac{1}{c}, bx)e^{\frac{7}{10}bx}\mathrm{d}x\,. \tag{20}$$

This is the result for arbitrary time. Taking $\tau = \tau_0$, $x = 0$, and from $\kappa(\tau_0) = 0$, we obtain the polarization:

$$\beta_S \equiv \beta_S(\tau_0) = -\frac{1}{10}bB(k)I(0)\,. \tag{21}$$

Here (refer to p.663 of Reference [23])

$$I(0) \equiv \int_0^{\infty}\Gamma(-\frac{1}{c}, bx)e^{\frac{7}{10}bx}\mathrm{d}x = b^{-1}\Gamma(1-\frac{1}{c})\,F(1-\frac{1}{c}, 1; 2; \frac{7}{10}) \tag{22}$$

By substituting $I(0)$ and $B(k)$ into β_S, we obtain the final approximate analytic expression for the polarization caused by the density perturbation:

$$\beta_S = -CF_S(\tau_d)\Delta\tau_d\,, \tag{23}$$

in which the coefficient C is·

$$C \equiv \left(\frac{5}{8.13}\frac{1}{c\,b^{1/c}}\right)\Gamma(1-\frac{1}{c})\,F(1-\frac{1}{c}, 1; 2, \frac{7}{10})\,. \tag{24}$$

This result suits the general recombination models, and the effects of the cosmological parameters on the polarization can be found directly from b and c. With b fixed and c increasing, the proportionality coefficient C and therefore the amplitude of β_S decrease This is the physical behavior that we expect: the greater the value of c, the narrower the peak of the curve of the visibility function $qe^{-\kappa}$, the shorter the time interval that photons undergo the Thompson scattering, and the smaller the amplitude of the polarization produced. In the Jones-Wyse model,

$$\beta_S \simeq -0.075 F_S(\tau_d)\Delta\tau_d \tag{25}$$

The parameters b and c of the Hu-Sugiyama model depend on the baryon density Ω_b. If we take $\Omega_b = 0.045$, then $b = 0.26$, $c = 10.42$, and

$$\beta_S \simeq -0.119 F_S(\tau_d)\Delta\tau_d\,. \tag{26}$$

Harari and Zaldarriaga[11] have given a simple approximate formula: $\beta_S \simeq -0.17F_S(\tau_d)\Delta\tau_d$, its amplitude is 1.5~2.3 times our result. Fig.1 shows the coefficient C in the Jones-Wyse model as a function of c. Fig 2 gives the coefficient C in the Hu-Sugiyama model as a function of Ω_b. It is obvious that C decreases with increasing c and Ω_b, respectively.

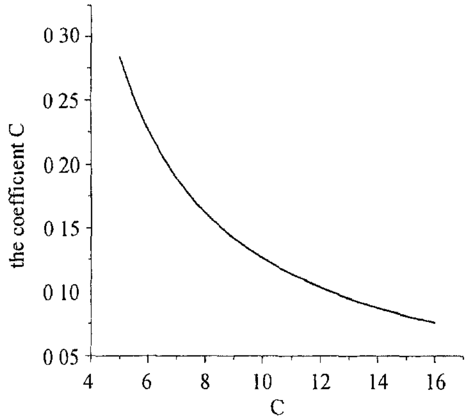

Fig 1 The curve of the polarization coefficient C as a function of the power exponent c, as given in Eq (24) for the Jones-Wyse model

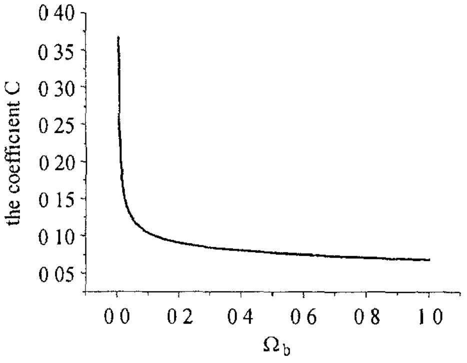

Fig 2 The curve of the polarization coefficient C as a function of the baryon density Ω_b, as given in Eq (24) for the Hu-Sugiyama model, in which $c = 16 + 1.8 \ln \Omega_b$

5. THE CMB POLARIZATION INDUCED BY GRAVITATIONAL WAVES

The primordial perturbation source corresponding to remaining gravitational waves is[25,26]

$$F_T(\tau) = \frac{1}{2}h(k)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{3j_1(k\tau)}{k\tau}\right)\,. \tag{27}$$

Here $h(k)$ is the amplitude function of remnant gravitational waves, depending on the wave number k, and $j_1(x) = -\frac{d}{dx}(\frac{\sin x}{x})$ is the spherical Bessel function. For this kind perturbation source, direct integration is difficult. The effect of remnant gravitational waves on the CMB polarization is relatively remarkable at the long-wave stage, so we can make the calculation on the long-wave approximation. In this case $k\tau \ll 1$, and, reserving the first two terms, we have

$$F_T(\tau) \simeq -\frac{1}{10}h(k)k^2\tau + \frac{1}{140}h(k)k^4\tau^3 . \tag{28}$$

The first term of the perturbation source $F_T^{(1)} = -\frac{1}{10}h(k)k^2\tau$ is the major contributor, it is a linear function of the time τ, as in the case of density perturbation The induced polarization $\beta_T^{(1)}$ can be given directly,

$$\beta_T^{(1)} = -CF_T^{(1)}(\tau_d)\Delta\tau_d = -\frac{3}{2}\frac{h(k)}{\zeta(k)}\beta_S , \tag{29}$$

in which β_S is the result caused by the density perturbation. It should be mentioned that the present calculation can not determine directly the relative magnitude of the density perturbation and gravitational waves $\frac{h(k)}{\zeta(k)}$. Various inflation models have different theoretical predictions on this ratio And one of the major points of the study of the CMB polarization is that by comparing the theory with the observation we can make a distinction between the contributions from density perturbations and gravitational waves[27–30].

The second term of the perturbation source $F_T^{(2)} = \frac{1}{140}h(k)k^4\tau^3$ is the minor contributor, and, different from the first term, it has the form $\propto \tau^3$ We first substitute $F_T^{(2)}$ into Eq.(11) and obtain

$$\xi_T^{(2)}(\tau) = \frac{1}{140}h(k)k^4e^{\kappa(\tau)}\int_0^\tau \tau'^3e^{-\kappa(\tau')}\mathrm{d}\tau' . \tag{30}$$

The integration on the right side of this formula is similar to the integration $A(\tau)$ in the calculation of density perturbations, when we have made the following alternations in $A(\tau)$, i.e., $b^{-1/c} \to b^{-2/c}$, and $\Gamma(-\frac{1}{c}, bx) \to \Gamma(-\frac{2}{c}, bx)$ Then, by expressing τ_0 with τ_d and $\Delta\tau$, respectively, we obtain

$$\xi_T^{(2)}(\tau) \simeq B_T^{(2)}(k)e^{bx}\Gamma(-\frac{2}{c}, bx) , \tag{31}$$

in which

$$B_T^{(2)}(k) \equiv F_T^{(2)}(\tau_d)\Delta\tau_d(\frac{10^3}{145\ 8}\frac{1}{cb^{2/c}}). \tag{32}$$

Substituting $\xi_T^{(2)}$ into Eq.(12) and changing the variable of integration to x, then, from $q(\tau')d\tau' = -bdx$ we have

$$\beta_T^{(2)}(\tau) = \frac{1}{10}bB_T^{(2)}(k)\int_\infty^x \Gamma(-\frac{2}{c}, bx)e^{\frac{7}{10}bx}\mathrm{d}x \tag{33}$$

By Eq (22), we can obtain the polarization caused by the second term of the gravitational wave perturbation

$$\beta_T^{(2)} \equiv \beta_T^{(2)}(\tau_0) = -DF_T^{(2)}(\tau_d)\Delta\tau_d , \tag{34}$$

in which the coefficient D is

$$D \equiv \left(\frac{10^2}{145.8}\frac{1}{c\,b^{2/c}}\right)\Gamma(1-\frac{2}{c})\,F(1-\frac{2}{c}, 1; 2, \frac{7}{10}) . \tag{35}$$

Similar to C, it decreases with increasing c, as does the amplitude of $\beta_T^{(2)}$. For the Jones-Wyse model, $D = 0\ 22$. For the Hu-Sugiyama model, if we take $\Omega_b = 0.045$, then $D = 0.32$. In summary, under the long-wave approximation the polarization caused by the perturbation of remnant gravitational waves is

$$\beta_T \simeq -[CF_T^{(1)}(\tau_d) + DF_T^{(2)}(\tau_d)]\Delta\tau_d \,. \tag{36}$$

6. ELECTRIC-TYPE AND MAGNETIC-TYPE POLARIZATIONS AND CURRENT OBSERVATIONAL RESULTS

In general, we can use the spin-weighted spherical harmonics to construct the electric-type polarization Δ_E and the magnetic-type polarization Δ_B with the given Stokes parameters of polarization. It helps us to distinguish the scalar perturbation from the tensorial perturbation Density perturbation induces only electric-type polarization Δ_E and no magnetic-type polarization Δ_B, while tensorial perturbation results in both Δ_E and Δ_B[31].

For the scalar perturbation, during the recombination process and within the long-wave limits, the polarization $\Delta_P(k,\mu)$, which depends on the angle $\mu = cos\theta$, can be obtained directly from β_S:

$$\Delta_P(k,\mu) = (1-\mu^2)e^{ik\mu(\tau_d-\tau_0)}\beta_S. \tag{37}$$

Then, by operating continuously on $\Delta_P(k,\mu)$ with the spin up operator for two times, the so-called electric-type polarization can be obtained:

$$\Delta_E = \partial_\mu^2\left[-(1-\mu^2)\Delta_P(k,\mu)\right] = \partial_\mu^2\left[-(1-\mu^2)^2 e^{ik\mu(\tau_d-\tau_0)}\beta_S\right]$$

Some direct calculations within the long-wave limits will yield the following simple linear relation between β_S and Δ_E:

$$\Delta_E \simeq 4(1-3\mu^2)e^{ik\mu(\tau_D-\tau_0)}\beta_S \,.$$

The Hu-Sujiyama model and Harari-Zaldarriaga model differ from ours only in the coefficient of β_S, and the amplitude of Δ_E given by these two models is 1 5~2.3 times of our result.

According to the common definition, the correlation function of the electric-type polarization $< \Delta_E\Delta_E >$ is proportional to the square amplitude of Δ_E. However, the WMAP observation can not provide this correlation of electric-type polarization, so a comparison with the model calculation can not be made at present What is given by the WMAP observations is the power spectrum corresponding to the cross correlation $< \Delta_T\Delta_E >$ between the temperature and the electric-type polarization.

$$C_{TE,l} = (4\pi)^2\int k^2 \mathrm{d}k\, P(k)\Delta_{Tl}\Delta_{El}\,,$$

in which

$$\Delta_{El} = \frac{1}{\sqrt{4\pi(2l+1)}}\sqrt{\frac{(l+2)!}{(l-2)!}}\sum_{m=-l}^{l}\int \mathrm{d}\Omega Y_{lm}^*\Delta_E$$

It is equivalent to the projection of Δ_E in the function space on bases of spherical harmonics. Similarly, Δ_{Tl} is the projection of the temperature anisotropy ΔT, and $P(k)$ is the spectrum of the primordial perturbation of density fields, which is usually provided by the inflation model It should be noticed that although certain differences in $P(k)$ exist among the different models, their functional forms are all close to a scale-invariant spectrum. The greatest uncertainty is that, up to now, the theoretical amplitudes of $P(k)$ given by all inflation models are greater than the actual limit by several orders of magnitude So, a common practice is to make an arbitrary adjustment of the calculated amplitude. As $C_{TE,l}$ contains simultaneously $P(k)$ and β_S, so a 1~2-fold difference in the coefficient of β_S can hardly be perceived Besides, in the observed $l < 20$ large scale, $C_{TE,l}$ is quite diffuse. This indicates that the reionization process happened at redshift $z =\sim 17\pm 3$[7]. In order to make an accurate analysis, it is necessary to take the reionization process into consideration. In this respect, the study of physical models is just beginning, and a lot of unknown factors remain

Tensorial perturbation will cause magnetic-type polarization, this situation is more complicated. WMAP has not yet observed magnetic-type polarization. As ΔT and Δ_E have the positive parity, while Δ_B has the negative parity, so for the cross correlations related to the magnetic-type polarization, we have $< \Delta T \Delta_B >= 0$ and $< \Delta_E \Delta_B >= 0$. Only if its correlation $< \Delta_B \Delta_B > \neq 0$, will the magnetic-type polarization be proportional to the square of β_T. The concrete formulae can be found in Reference [31]. To directly detect remnant gravitational wave in the universe is rather difficult[32] At present, we expect that future CMB detection can observe directly magnetic-type polarization, but first we must receive signals of remnant gravitational waves And this is an important problem in the study of the CMB and remnant gravitational waves.

7. DISCUSSION

In this paper, we have studied the CMB polarization produced by the Thompson scattering of photons in the recombination stage of the early universe. Starting from the Boltzmann equation, the polarizations induced by primordial density perturbation and remnant gravitational waves are calculated separately for the general recombination process. In the case of density perturbation, by giving the integration a careful treatment, an approximate analytic expression of polarization valid for the general recombination process is obtained For the Jones-Wyse model, our amplitude of polarization is less than that given by Hararri-Zaldarriaga's simple approximate formula by a factor of 2.3. As for the Hu-Sugiyama model, if the baryon density is taken to be $\Omega_b = 0.045$, then our amplitude of polarization is less than that of Hararri-Zaldarriaga's formula by a factor of 1.5 times For the case of remnant gravitational waves, we have made a discussion in the long-wave approximation By expanding the primordial perturbation into a power series in the wave number, keeping the first two terms, then inegrating separately, an approximate analytic formula for the polarization is obtained, and it depends as well on the recombination model. The first term of the result is similar to the result for the density perturbation, and the coefficient D of the second term is different from the coefficient C of the first term. Our approximate analytic solutions indicate clearly that for either case of density or gravitational wave perturbation, as the parameter c of the optical-depth function decreases, the peak of the visibility function curve of the recombination process becomes narrower, the decoupling time is shortened, so decreasing the amplitude of the polarization induced by the Thompson scattering. Finally, the present status of the observations of the electric-type and magnetic-type polarizations, as well as their relations with β, are briefly mentioned.

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