

Constraint on the early Universe by relic gravitational waves: From pulsar timing observations

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Recent pulsar timing observations by the Parkers Pulsar Timing Array (PPTA) and European Pulsar Timing Array (EPTA) teams obtained the constraint on the relic gravitational waves at the frequency $f_* = 1/\text{yr}$, which provides the opportunity to constrain H_* , the Hubble parameter, when these waves crossed the horizon during inflation. In this paper, we investigate this constraint by considering the general scenario for the early Universe: we assume that the effective (average) equation-of-state w before the big bang nucleosynthesis stage is a free parameter. In the standard hot big-bang scenario with $w = 1/3$, we find that the current PPTA result follows a bound $H_* \leq 1.15 \times 10^{-1} m_{\text{pl}}$, and the EPTA result follows $H_* \leq 6.92 \times 10^{-2} m_{\text{pl}}$. We also find that these bounds become much tighter in the nonstandard scenarios with $w > 1/3$. When $w = 1$, the bounds become $H_* \leq 5.89 \times 10^{-3} m_{\text{pl}}$ for the current PPTA and $H_* \leq 3.39 \times 10^{-3} m_{\text{pl}}$ for the current EPTA. In contrast, in the nonstandard scenario with $w = 0$, the bound becomes $H_* \leq 7.76 m_{\text{pl}}$ for the current PPTA.

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I. INTRODUCTION

A stochastic background of relic gravitational waves, generated during the early inflationary stage, is a necessity dictated by general relativity and quantum mechanics [1–3]. The relic gravitational waves have a wide range spreading spectra, and their amplitudes depend only on the Hubble parameter in the inflationary stage, when the waves crossed the horizon, and the expansion history of Universe after the waves reentered the horizon. So their detection provides a direct way to study the physics in the early Universe in both stages, during and after the inflation.

Recently, there have been several experimental efforts to constrain the amplitude of relic gravitational waves in the different frequencies. The current observations of cosmic microwave background (CMB) radiation by the WMAP satellite place an interesting bound on the so-called tensor-to-scalar ratio $r \leq 0.20$ [4,5], which is equivalent to the constraint on the energy density $\Omega_{\text{gw}}(f)$ of relic gravitational waves at the lowest frequency range $f \sim 10^{-17}$ Hz. Among various direct observations, LIGO S5 has also experimentally obtained so far the most stringent bound $\Omega_{\text{gw}}(f) \leq 6.9 \times 10^{-6}$ around $f \sim 100$ Hz [6]. In addition, there are two bounds on the integration $\int \Omega_{\text{gw}}(f) d \ln f \leq 1.5 \times 10^{-5}$, obtained by the big bang nucleosynthesis (BBN) observation [7] and the CMB observation [8]. These bounds have been used to constrain the Hubble parameter (or the potential density of inflaton) in the inflationary stage, when the corresponding waves crossed the horizon [4,6,9].

The timing studies on the millisecond pulsars provide a unique way to constrain the amplitude of gravitational waves in the frequency range $f \in (10^{-9}, 10^{-7})$ Hz [10].

Recently, the Parkers Pulsar Timing Array (PPTA) team and the European Pulsar Timing Array (EPTA) team have reported their observational results on the stochastic background of gravitational waves and given the upper limit of $\Omega_{\text{gw}}(f)$ at the frequency $f = 1/\text{yr}$ [11,12]. In this paper, we shall infer from these bounds the constraint on H , the Hubble parameter at the waves' horizon-crossing time during inflation. In the calculation, we have considered a general early cosmological model, i.e., we assume the effective (average) equation-of-state w before the BBN stage can be of any value, which includes a wide range of cosmological scenarios. The derived bound of H would limit various inflation models.

II. RELIC GRAVITATIONAL WAVES IN THE STANDARD HOT BIG-BANG UNIVERSE

Incorporating the perturbation to the spatially flat Friedmann-Robertson-Walker (FRW) spacetime, the metric is

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij}dx^i dx^j)], \quad (1)$$

where a is the scale factor of the universe, and η is the conformal time, which relates to the cosmic time by $ad\eta = dt$. The perturbation of spacetime h_{ij} is a 3×3 symmetric matrix. The gravitational-wave field is the tensorial portion of h_{ij} , which is transverse-traceless $\partial_i h^{ij} = 0$, $\delta^{ij} h_{ij} = 0$.

Relic gravitational waves satisfy the linearized evolution equation [1]:

$$\partial_\mu(\sqrt{-g}\partial^\mu h_{ij}) = -16\pi G\pi_{ij}. \quad (2)$$

The anisotropic portion π_{ij} is the source term, which can be given by the relativistic free-streaming gas [13] and the scalar field in the preheating stage [14]. However, it has

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been deeply discussed that the relativistic free-streaming gas can only affect the relic gravitational waves at the frequency range $f \in (10^{-16}, 10^{-10})$ Hz, which could be detected by the future CMB observations [15]. The generation of a stochastic background of gravitational waves in the preheating stage has also been deeply studied (see, for instance, [14]), where the gravitational radiation was produced in interactions of classical waves created by resonant decay of a coherently oscillating field. However, it was found that the typical frequencies of this kind of gravitational waves are quite high, i.e., $f > 10^4$ Hz. Even if the model with low energy $H \sim 100$ GeV is considered, the gravitational waves are important only at the frequency range $f \sim 1$ Hz [14], which could be detected by the future laser interferometer detectors. So, both effects cannot obviously influence the relic gravitational waves at the frequency $f \in (10^{-9}, 10^{-7})$ Hz. For these reasons, in this paper we shall ignore the contribution of the external sources. So the evolution of gravitational waves is only dependent on the scale factor and its time derivative. It is convenient to Fourier transform the equation as follows:

$$h_{ij}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{s=\mp, \times} [h_k(\eta) \epsilon_{ij}^{(s)} c_k^{(s)} e^{i\vec{k}\cdot\vec{x}} + \text{c.c.}], \quad (3)$$

where c.c. stands for the complex conjugate term. The polarization tensors are symmetry, transverse-traceless $k^i \epsilon_{ij}^{(s)}(\vec{k}) = 0$, $\delta^{ij} \epsilon_{ij}^{(s)}(\vec{k}) = 0$, and satisfy the conditions $\epsilon^{(s)ij}(\vec{k}) \epsilon_{ij}^{(s')}(\vec{k}) = 2\delta_{ss'}$ and $\epsilon_{ij}^{(s)}(-\vec{k}) = \epsilon_{ij}^{(s)}(\vec{k})$. Since the relic gravitational waves we will consider are isotropy, and each polarization state is the same, we have denoted $h_k^{(s)}(\eta)$ by $h_k(\eta)$, where $k = |\vec{k}|$ is the wave number of the gravitational waves, which relates to the frequency by $k \equiv 2\pi f$. (The present scale factor is set $a_0 = 1$.) So Eq. (2) can be rewritten as

$$h_k'' + 2\frac{a'}{a} h_k' + k^2 h_k = 0, \quad (4)$$

where the *prime* indicates a conformal time derivative $d/d\eta$. For a given wave number k and a given time η , we can define the transfer function t_f as

$$t_f(\eta, k) \equiv h_k(\eta)/h_k(\eta_i), \quad (5)$$

where η_i is the initial conformal time. This transfer function can be obtained by solving the evolution equation (4).

The strength of the gravitational waves is characterized by the gravitational-wave energy spectrum,

$$\Omega_{\text{gw}} \equiv \rho_{\text{gw}}/\rho_0, \quad (6)$$

where $\rho_{\text{gw}} = \frac{1}{32\pi G} \langle \dot{h}_{ij} \dot{h}^{ij} \rangle$, the critical density is $\rho_0 = \frac{3H_0^2}{8\pi G}$, and $H_0 = 100h \cdot \text{km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}$ is the current Hubble constant. Using Eqs. (3) and (5), the energy density of gravitational waves can be written as [16]

$$\rho_{\text{gw}} = \int \frac{dk}{k} \frac{P_t(k) t_f^2(\eta_0, k)}{32\pi G}, \quad (7)$$

where $P_t(k) \equiv \frac{2k^3}{\pi^2} |h_k(\eta_i)|^2$ is the so-called primordial power spectrum of relic gravitational waves. Thus, we derive that the current energy density of relic gravitational waves

$$\Omega_{\text{gw}} \equiv \int d \ln k \Omega_{\text{gw}}(k), \quad \text{and} \quad \Omega_{\text{gw}}(k) = \frac{P_t(k)}{12H_0^2} t_f^2(\eta_0, k), \quad (8)$$

where the *dot* indicates a cosmic time derivative d/dt .

Now, let us discuss the terms $P_t(k)$ and $t_f(\eta_0, k)$ separately. The primordial power spectrum of relic gravitational waves is usually assumed to be power-law as follows:

$$P_t(k) = A_t(k_*) \left(\frac{k}{k_*} \right)^{n_t}. \quad (9)$$

This is a generic prediction of a wide range of scenarios of the early Universe, including the inflation models. Here, we should mention that there might be deviations from power-law if we consider the relic gravitational waves in a fairly large wave number span. *In this paper, as a conservative consideration, we assume this form is held only when k is very close to the pivot wave number k_* .* In the above expression, n_t is the spectral index when $k \rightarrow k_*$. ($n_t = 0$ corresponds to the scale-invariant power spectrum.) $A_t(k_*)$ is directly related the value of the Hubble parameter H at time when wavelengths corresponding to the wave number k_* crossed the horizon [1,3,17],

$$A_t^{1/2}(k_*) = \frac{4}{\sqrt{\pi}} \frac{H_*}{m_{\text{Pl}}} \Big|_{k_* = a_* H_*}, \quad (10)$$

where $m_{\text{Pl}} \equiv 1/\sqrt{G}$ is the Planck mass.

Now, let us turn to the transfer function t_f , defined in (5), which describes the evolution of gravitational waves in the expanding Universe. From Eq. (4), we find that this transfer function can be directly derived, so long as the scale factor as a function of time is given. Actually, the analytical or numerical forms of t_f have been discussed by a number of authors (see, for instance, [18–21]).

In this paper, we shall use the following analytical approximation for this transfer function. It has been known that, during the expansion of the Universe, the mode function $h_k(\eta)$ of the gravitational waves behaves differently in two regions [18]. When waves are far outside the horizon, i.e., $k \ll aH$, the amplitude of h_k keeps constant, and when inside the horizon, i.e., $k \gg aH$, the amplitude is damping with the expansion of Universe, i.e., $h_k \propto 1/a(\eta)$. In the standard hot big-bang cosmological model, we assume that the inflationary stage is followed by a radiation dominant stage, and then the matter dominant stage and the Λ dominant stage. In this scenario, by numerically

integrating Eq. (4), one finds that the damping function i_f can be approximately described by the following form [22–25]

$$i_f(\eta_0, k) = \frac{-3j_2(k\eta_0)}{k\eta_0} \frac{\Omega_m}{\Omega_\Lambda} \sqrt{1 + 1.36\left(\frac{k}{k_{\text{eq}}}\right) + 2.50\left(\frac{k}{k_{\text{eq}}}\right)^2}, \quad (11)$$

where $k_{\text{eq}} = 0.073\Omega_m h^2 \text{Mpc}^{-1}$ is the wave number corresponding the Hubble radius at the time that matter and radiation have equal energy density, and $\eta_0 = 1.41 \times 10^4 \text{Mpc}$ is the present conformal time. The factor Ω_m/Ω_Λ encodes the damping effect due to the recent accelerating expansion of the Universe [19,22]. In this damping factor, we have ignored the small effects of neutrino free-streaming [13] and various phase transition [21].

We can define a new function

$$\mathcal{T}(k) \equiv i_f(\eta_0, k)/\sqrt{12H_0^2}. \quad (12)$$

So, the current density of relic gravitational waves becomes $\Omega_{\text{gw}}(k) = P_r(k)\mathcal{T}^2(k)$. In this paper, we shall focus on the wave number $k \gg k_{\text{eq}}$. In this range, we have [22,24–26]

$$\mathcal{T}^2(k) = \frac{15}{16} \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^2 \frac{1}{H_0^2 \eta_0^4 k_{\text{eq}}^2}, \quad (13)$$

and a parametrized form for the current density of relic gravitational waves

$$\Omega_{\text{gw}}(k) = \left(\frac{H_*}{m_{\text{Pl}}}\right)^2 \left(\frac{k}{k_*}\right)^{n_r} \left(\frac{15}{\pi} \frac{1}{H_0^2 \eta_0^4 k_{\text{eq}}^2} \frac{\Omega_m^2}{\Omega_\Lambda^2}\right). \quad (14)$$

For the wave number $k = k_*$, the value of $\Omega_{\text{gw}}(k_*)$ depends only on the value of H_* . So, in this standard scenario, an observational bound on the $\Omega_{\text{gw}}(k_*)$ corresponds to a bound on the Hubble parameter H_* , which will be shown clearly in Sec. IV.

III. DAMPING FACTOR IN THE GENERAL MODEL OF THE EARLY UNIVERSE

Although, in the standard hot big-bang universe, a radiation dominant stage is always assumed after the inflationary stage, there is no observational evidence to show this is held before the BBN stage. Actually, this assumption can be violated in a number of cases, for example, the existence of the reheating stage [18] or the existence of the cosmic phase transition [21]. So, in general, before the BBN stage, one can assume that the average equation-of-state of the Universe is w , and the scale factor satisfies a simple power-law form

$$a \propto \eta^{1+\beta}. \quad (15)$$

The constant β relates to w by $\beta = (-3w + 1)/(3w + 1)$. Obviously, when $w = 1/3$, i.e., $\beta = 0$, it returns to the

standard model. However, if the Universe is dominated by the kinetic energy of inflaton, one has $w = 1$ and $\beta = -1/2$. On the other hand, for a matter-dominated era, one has $w = 0$ and $\beta = 1$.

Now, let us discuss the evolution of relic gravitational waves in this general cosmological model. In principle, it can be done by directly solving Eq. (4). In this paper, in order to avoid the complicated numerical calculation, we give an approximate method as below.

We consider the wave h_k with the wave number k , which crossed the horizon at $a = a_k$ and the corresponding Hubble parameter is H_k . So one has $k = a_k H_k / a_0$. One knows that, when the waves are in the horizon, $h_k \propto 1/a(\eta)$, damping with the expansion of the Universe, and when the waves are out of the horizon, $h_k = \text{constant}$, keeping its initial value. So one can define a ratio which accounts for the damping of the gravitational waves,

$$\frac{h_k(\eta_0)}{h_k(\eta_i)} = \frac{a_k}{a_0} = \frac{a_k}{a_b} \frac{a_b}{a_0}, \quad (16)$$

where a_b is the scale factor at the temperature of the Universe being 1 MeV, i.e., the BBN stage.

In the standard model, where $\beta = 0$ in (15) is assumed (i.e., $w = 1/3$, the radiation-dominant stage), we have

$$\frac{H_k}{H_b} = \left(\frac{a_b}{a_k}\right)^2, \quad (17)$$

where H_b is the Hubble parameter in the BBN stage. Taking into account the relation $k = a_k H_k / a_0$, we obtain that

$$\frac{h_k(\eta_0)}{h_k(\eta_i)} = \frac{a_b}{a_0} \left(\frac{a_b H_b}{a_0 k}\right). \quad (18)$$

However, in the general case with $\beta \neq 0$, we assume h_k crossed the horizon at $a = \tilde{a}_k$ and the corresponding Hubble parameter is \tilde{H}_k . (Note that, in general, $\tilde{a}_k \neq a_k$ and $\tilde{H}_k \neq H_k$, but $k = \tilde{a}_k \tilde{H}_k / a_0$ is still satisfied.) From the equation in (15), it follows that

$$\frac{\tilde{H}_k}{H_b} = \left(\frac{a_b}{\tilde{a}_k}\right)^{(2+\beta)/(1+\beta)}.$$

So, in this general case, we have

$$\frac{h_k(\eta_0)}{h_k(\eta_i)} = \frac{\tilde{a}_k}{a_0} = \frac{a_b}{a_0} \left(\frac{a_b H_b}{a_0 k}\right)^{1+\beta}. \quad (19)$$

Comparing Eqs. (19) and (18), we can define the damping factor $D(k)$ as follows:

$$D(k) \equiv \left(\frac{h_k(\eta_0)}{h_k(\eta_i)}\right)_{\text{general}} / \left(\frac{h_k(\eta_0)}{h_k(\eta_i)}\right)_{\text{standard}} \quad (20)$$

$$= \left(\frac{a_b H_b}{a_0 H_0}\right)^\beta \left(\frac{H_0}{k_*}\right)^\beta \left(\frac{k}{k_*}\right)^{-\beta}. \quad (21)$$

Thus, in this general scenario, the current density of relic gravitational waves becomes

$$\Omega_{\text{gw}}(k) = P_t(k) \mathcal{T}^2(k) D^2(k), \quad (22)$$

which satisfies $\Omega_{\text{gw}}(k) \propto k^{n_t - 2\beta}$ when k is close to k_* . Using the formulae in (9), (10), (13), and (21), and substituting the cosmological parameters ($h = 0.702$, $T_{\text{CMB}} = 0.276$ K, $\Omega_\Lambda = 0.725$, $\Omega_m = 0.275$, and $z_{\text{eq}} = 3454$) [4], we get the following simple result:

$$\log_{10} \Omega_{\text{gw}}(k_*) = 1.25 - \frac{13.48}{3w + 1} + 2 \log_{10} \left(\frac{H_*}{m_{\text{Pl}}} \right), \quad (23)$$

where $k_* \equiv 2\pi f_*$, and $f_* = 1/\text{yr}$ is used. In Sec. IV, we shall compare this with the observational results.

IV. CONSTRAINT BY THE PULSAR TIMING OBSERVATIONS

Pulsar timing observations provide a unique opportunity to study the gravitational waves at the frequency range $f \in (10^{-9}, 10^{-7})$ Hz. In 2006, Jenet *et al.* analyzed the PPTA data and archival Arecibo data for several millisecond pulsars. By focusing on the gravitational waves with the wave number k_* (where $k_* = 2\pi f_*$ and $f_* = 1/\text{yr}$), and assuming the density of gravitational waves satisfies $\Omega_{\text{gw}}(k) = k^{2+2\alpha}$ at around $k \sim k_*$, the authors obtained the 2σ upper limit on $\Omega_{\text{gw}}(k_*)$ as a function of α [11], which has been shown in Fig. 1 (solid line 1). This figure shows that $\Omega_{\text{gw}}(k_*) \leq 4.05 \times 10^{-8}$ when $\alpha = -1$. However, this upper bound increases to be 1.98×10^{-6} when $\alpha = 0$.

Recently, this upper limit has been updated. In [12], the authors have used the current data from the EPTA to determine an upper limit on the stochastic gravitational-wave background as a function of the spectral slope α . The 1σ and 2σ bounds are shown in Fig. 1 ([blue] lines 2 and 3), which are slightly lower than those in the PPTA case for any given α .

It is interesting that, in [11], the authors have also investigated the possible upper limit (or a definitive detection) of the stochastic background of gravitational waves by using the potential completed PPTA data-sets (20 pulsars with an rms timing residual of 100 ns over 5 years). We have also plotted this potential upper limit in Fig. 1 ([red] dotted line 4).

Now, let us compare these observations with the analytical formulae of relic gravitational waves in Sec. III. In order to compare, it is necessary to relate the parameter α with the theoretical models. In Sec. III, Eq. (22) shows that $\Omega_{\text{gw}}(k) \propto k^{n_t - 2\beta}$, where $\beta = (-3w + 1)/(3w + 1)$. Comparing this with the assumed form $\Omega_{\text{gw}}(k) \propto k^{2+2\alpha}$, we get the interesting relation

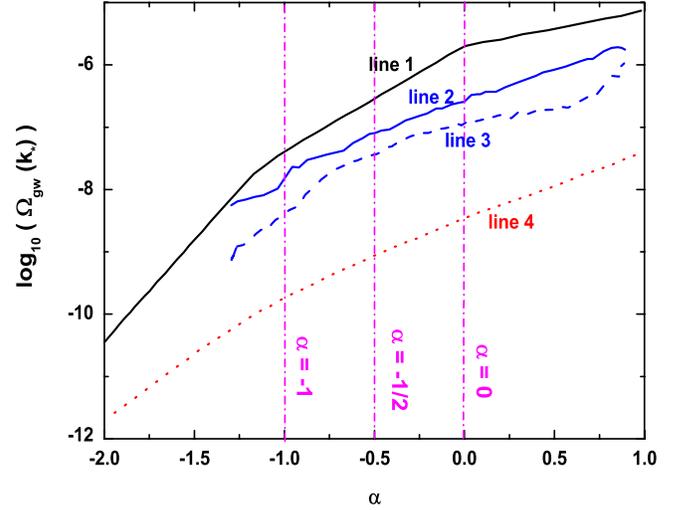


FIG. 1 (color online). The upper limit of $\Omega_{\text{gw}}(k_*)$ as a function of the parameter α . The black solid line line 1 is for the current PPTA 2σ result [11]; the [blue] solid line 2 is for the current EPTA 2σ result [12]; the [blue] dashed line 3 is for the current EPTA 1σ result [12]; and the [red] dotted line 4 is for the future PPTA 2σ result [11].

$$\alpha = \frac{n_t}{2} - \frac{2}{3w + 1}. \quad (24)$$

This relation shows that, in the standard hot big-bang scenario with $w = 1/3$, and the scale-invariant primordial power spectrum with $n_t = 0$, we have $\alpha = -1$. In this case, let us use the bounds of gravitational waves to constrain the Hubble parameter H_* in the inflationary stage. Taking into account the formula in Eq. (23) and using $w = 1/3$, we obtain the 2σ upper limit of H_* , i.e., $H_* \leq 1.15 \times 10^{-1} m_{\text{Pl}}$ for the current PPTA case, $H_* \leq 6.92 \times 10^{-2} m_{\text{Pl}}$ for the current EPTA case, and the future PPTA is expected to give $H_* \leq 7.94 \times 10^{-3} m_{\text{Pl}}$. These results are listed in Table I.

Although the inflation models always predict the nearly same Hubble parameter throughout the inflationary stage, it is necessary to constrain H , the Hubble parameter, at quite different stages of inflation, which encodes the evolution information of inflaton. Here, let us compare the bound of H inferred from pulsar timing with those obtained in CMB observations and LIGO observations. The recent

TABLE I. The 2σ upper limit of the quantity H_*/m_{Pl} inferred from various pulsar timing observations. In this table, we have assumed $n_t = 0$.

	Current PPTA	Current EPTA	Future PPTA
$w = 0$	7.76		1.41
$w = 1/3$	1.15×10^{-1}	6.92×10^{-2}	7.94×10^{-3}
$w = 1$	5.89×10^{-3}	3.39×10^{-3}	3.55×10^{-4}
$w \rightarrow \infty$	3.16×10^{-4}	1.20×10^{-4}	1.41×10^{-5}

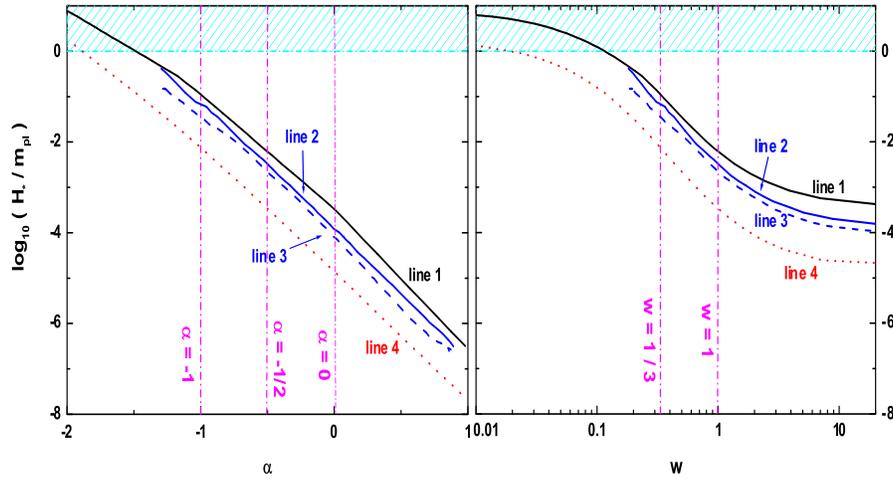


FIG. 2 (color online). The upper limit of the Hubble parameter H_* as a function of the parameter α (left panel) and the average equation-of-state w (right panel), where we have assumed $n_t = 0$. In both panels, the black solid line lines 1 are for the current PPTA 2σ result; the [blue] solid lines 2 are for the current EPTA 2σ result; the [blue] dashed lines 3 are for the current EPTA 1σ result, and the [red] dotted lines 4 are for the future PPTA 2σ result.

CMB observations by the WMAP satellite provide the constraint on the tensor-to-scalar ratio $r \leq 0.20$ [4], which is equivalent to the bound of $H/m_{\text{pl}} \leq 6.92 \times 10^{-6}$, where H is the Hubble parameter of inflation when the waves with frequency $f = 1.94 \times 10^{-17}$ Hz crossed the horizon. The recent LIGO 5S reported so far the tightest constraint $\Omega_{\text{gw}}(f) \leq 6.9 \times 10^{-6}$ on relic gravitational waves at the frequency $f \approx 100$ Hz [6], which corresponds to $H/m_{\text{pl}} \leq 1.46$. Comparing these results, we find that the current and the potential future pulsar timing constraints on H are quite tighter than that of LIGO, but much looser than the CMB constraint.

Now, let us relax the assumptions of the early Universe. We assume only $n_t = 0$, which is approximately held in a wide range of inflation models. So, we can constrain the Hubble parameter H_* in a wide range of w by the following inequality:

$$\log_{10}\left(\frac{H_*}{m_{\text{pl}}}\right) \leq \frac{1}{2}(U(\alpha) - 6.74\alpha - 1.25), \quad (25)$$

where $U(\alpha)$ is the upper limit of $\Omega_{\text{gw}}(k_*)$ based on the pulsar timing observations, which is a function of the parameter α . [In this case, α relates to w by the relation $\alpha = -2/(3w + 1)$.]. The bounds of H_* as functions of α (left panel) and w (right panel) are shown in Fig. 2. These bounds in three special cases with $w = 0$ (i.e., $\alpha = -2$), $w = 1$ (i.e., $\alpha = -1/2$) and $w \rightarrow \infty$ (i.e., $\alpha = 0$) are also listed in Table I. Clearly, we find that a larger w corresponds to a tighter bound of H_* . Especially in the limit case with $w \rightarrow \infty$, the current EPTA gives the constraint $H_* \leq 1.20 \times 10^{-4} m_{\text{pl}}$, and the future PPTA is expected to give a bound of $H_* \leq 1.41 \times 10^{-5} m_{\text{pl}}$.

In the end, let us discuss the most general case with free parameters n_t and w . In this case, the inequality (25) becomes the constraint on the physical parameters n_t and H_* as follows:

$$\log_{10}\left(\frac{H_*}{m_{\text{pl}}}\right) - 1.69n_t \leq \frac{1}{2}(U(\alpha) - 6.74\alpha - 1.25). \quad (26)$$

Here, we should remember that α relates to the physical parameters by Eq. (24). The spectra index n_t influences the bound of H_* mainly by slightly changing the corresponding relation between α and w . In Fig. 3, we calculate the

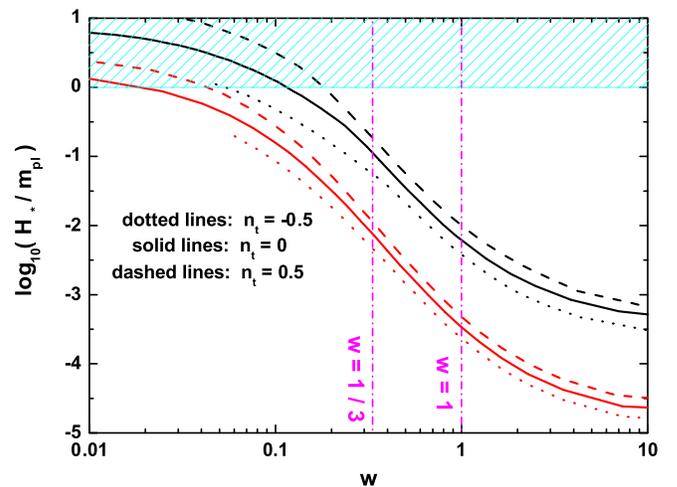


FIG. 3 (color online). The upper limit of the Hubble parameter H_* as a function of the average equation-of-state w , where we have considered three cases with $n_t = -0.5, 0, 0.5$. The three upper black lines are for the current PPTA 2σ result, and the three lower [red] lines are for the future PPTA 2σ result.

upper bound of H_* in two special cases with $n_t = 0.5$ and $n_t = -0.5$, and compare them with those in the case of $n_t = 0$. This figure shows that the parameter n_t only slightly affects the bound of H_* , and a larger n_t follows a looser bound of H_* . For example, the current PPTA observations follow $H_* \leq 1.82 \times 10^{-1} m_{\text{Pl}}$ at the case with $n_t = 0.5$ and $w = 0$, which is only 1.6 times larger than the

bound $H_* \leq 1.14 \times 10^{-1} m_{\text{Pl}}$ at the case with $n_t = 0$ and $w = 0$.

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