The spanning ratios of β-Skeletons

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***Abstract*— In this paper we study the spanning ratio of the β-skeleton for $\beta \in [0,2]$. Both our upper-bounds and lower-bounds improve the previously best known results [10], [12].**

I. Introduction

Proximity graphs [1], [2], [3], [4] have been used extensively in various fields including pattern recognition, GIS (Geographic Information System), computer vision, and neural network [5], [3]. The spanning ratios of the proximity graphs are of great interest to many applications. For example, several results showed that Delaunay Triangulation has a constant bounded spanning ratio, which is at least $\frac{\pi}{2}$ [6] and at most $2\pi/(3\cos(\pi/6)) \simeq 2.42$ [2].

As one of the proximity graphs, β-skeletons have been studied extensively in [8], [9], [10], [11]. Our main concern in this paper is about the spanning ratio (or dilation) of the β-skeleton. Given a set S of n points in a two dimensional plane, two points u and v are β-*neighbors* in S if $N(u,v,\beta)$ contains no point of S, other than u or v, in its interior [1]. The most common definition of $N(u,v,\beta)$ is so-called *Lune-Based Neighborhood*, which is defined as follows.

Case 1: $\beta \geq 1$. $N(u,v,\beta)$ is the intersection of the two circles of radius $\frac{\beta\|uv\|}{2}$ centered at the points $p_2 = (1-\frac{\beta}{2})u + \frac{\beta}{2}v$ and $p_1 = \frac{\beta}{2}u + (1-\frac{\beta}{2})v$, respectively.

Case 2: $0 \leq \beta \leq 1$. $N(u,v,\beta)$ is the intersection of the two circles of radius $\frac{D(uv)}{2\beta}$ passing through both u and v.

Here $\|uv\|$ is the Euclidian distance between u and v. The β-skeleton of a point set S, denoted by $G_\beta(S)$, is the set of edges joining β-neighbors in S. When $\beta = 1$, the closed $N(u,v,\beta)$ corresponds exactly to the Gabriel neighborhood of u and v. When $\beta = 2$, the open $N(u,v,\beta)$ is the relative neighborhood of u and v. As β approaches ∞, the neighborhood of u and v approximates the infinite strip formed by translating the line segment (u,v) normal to itself. Notice when $\beta > 2$ the β-skeleton graph can be disconnected, so we restrict our attention to the case that $0 \leq \beta \leq 2$. As β approaches 0, $N(u,v,\beta)$ approximates the line segment connecting u and v. Thus, except in degenerate situations (three or more points co-linear), all point pairs are β-neighbors under this scheme for β sufficiently small, which means that we can find a β to make the β-skeleton of S a complete graph.

For $\beta \in [0,1]$, the spanning ratio of β-skeleton is at most $O(n^{c_2})$ [10], where $c_2 = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$ and at least $\Omega(n^{c_1})$ [12], where $c_1 = 1 - \log_5(3 + \sqrt{2 + 2\sqrt{1-\beta^2}})$. For the special β-skeleton such as Gabriel graph (GG) [1], [13], [10] ($\beta = 1$) and the relative neighborhood graph (RNG) [4], [14], [3], [15] ($\beta = 2$), Bose *et al.* [10] gave a bound which is $\Theta(\sqrt{n})$ and $\Theta(n)$ respectively. Since the spanning ratio increases over β for $\beta \in [1,2]$, the spanning ratio of the β-skeleton for $\beta \in [1,2]$ is at least $\Omega(n^{\frac{1}{2}})$ and at most $O(n)$, which is also the best known result till now.

The contribution of this paper is follows. We first prove that, for $\beta \in [1,2]$, the β-skeleton has spanning ratio at most $(n-1)^\gamma$, where $\gamma = 1 - \frac{1}{2}\log_2(\mu_2 + 1)$, $\mu_2 = \frac{2-\beta}{\beta}$. We then show that the Gabriel graph has *exact* spanning ratio $\sqrt{n-1}$ and the relative neighborhood graph has exact spanning ratio $n-1$. The spanning ratio of β-skeleton for $\beta \in [0,1]$ is at most $(n-1)^\gamma$, where $\gamma = \frac{1}{2} - \frac{1}{2}\log_2(\mu_1+1)$, $\mu_1 = \sqrt{1-\beta^2}$. Finally, we construct a point set whose β-skeleton, for $\beta \in [0,1]$, has spanning ratio n^{c_3}, where $c_3 = \frac{1}{2} - \frac{1}{2}\log_2(1 + \sqrt{\frac{\mu_1+1}{2}})$, which improves the previously best known lower bound [12].

II. Upper Bound of Spanning Ratios

Consider a geometry graph $G = (V,E)$ over a set V of n points. For each pair of points (u,v), the length of the shortest path connecting u and v measured by Euclidean distance is denoted by $D_G(u,v)$, while the direct Euclidean distance is $\|uv\|$. The spanning ratio (also dilation ratio or length stretch factor) of the graph G is defined by $\psi(G) = \max_{u,v\in G}\frac{D_G(u,v)}{\|uv\|}$. If the graph G is not connected, then $\psi(G)$ is infinity, so it is reasonable to focus on connected graphs only.

A. Fade Factor of β-skeleton

Our analysis of the upper bound of the spanning ratio of a β-skeleton relies on our definition of *fade factor* of a β-skeleton, which is defined as follows. Given a 2-dimensional point set S and its β-skeleton $G_\beta(S)$, choose any pair of points $u,v \in S$. If $uv \notin G_\beta(S)$, there must exist some point w other than u,v in $N(u,v,\beta)$. We say that the point w breaks edge (u,v) and define $x_1 = \frac{\|uw\|}{\|uv\|}$, $x_2 = \frac{\|vw\|}{\|uv\|}$ as the two *fade factors* of uv by w. We then study the property of fade factors of an edge uv not in the β-skeleton, illustrated by Figure 1.

Case 1: $\beta \in [1,2]$**.** In this case, w must lie in the shaded area $N(u,v,\beta)$. For symmetry, we assume that $\|wu\| \geq \|wv\|$, [2] In triangles $\triangle wup_1$ and $\triangle wvp_1$, we have $\|uw\|^2 = \|up_1\|^2 + \|wp_1\|^2 - 2\|up_1\|\|wp_1\|\cos\alpha$ and $\|vw\|^2 = \|vp_1\|^2 + \|wp_1\|^2 - 2\|vp_1\|\|wp_1\|\cos(\pi-\alpha)$. Consequently,

$$\frac{\|uw\|^2 - \|up_1\|^2 - \|wp_1\|^2}{\|up_1\|} + \frac{\|uw\|^2 - \|vp_1\|^2 - \|wp_1\|^2}{\|vp_1\|} = 0$$

$$\Rightarrow \quad \frac{x_1^2}{2-\beta} + \frac{x_2^2}{\beta} = \frac{1}{2} + \frac{\|wp_1\|^2}{\|uv\|^2}\frac{2}{\beta(2-\beta)} \leq \frac{1}{2-\beta}.$$

Suppose that $0 \leq \mu_2 = \frac{2-\beta}{\beta} \leq 1$. The fade factors, when $\beta \in [1,2]$ and $x_1 \geq x_2$, satisfy

$$x_1^2 + \mu_2 x_2^2 \leq 1 \tag{1}$$

Case 2. $\beta \in [0,1]$**.** In this case, we have $1 = x_1^2 + x_2^2 - 2x_1x_2\cos\theta$. Let $\cos\alpha = \sqrt{1-\beta^2}$. From $\theta + \alpha \geq \pi$, we have

$$1 \geq x_1^2 + x_2^2 - 2x_1x_2\cos(\pi-\alpha) = x_1^2 + x_2^2 + 2x_1x_2\cos\alpha \tag{2}$$

B. Construction of the fade factor tree

Our analysis of the spanning ratio is based on a concept called *fade factor tree*, which intuitively records the edge-breaking sequence for an edge uv not in β-skeleton. The exact definition is given along the following construction algorithm.

Algorithm 1: Constructing the Fade Factor Tree

1) Construct the root node corresponding to uv.

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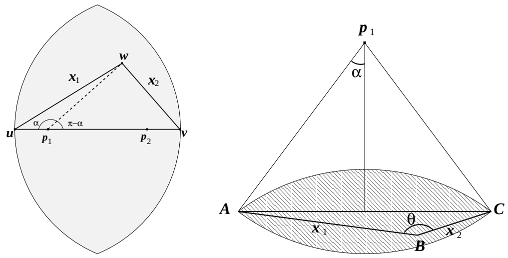
[1]There are two possible interpretations of the interior: one includes the boundary which is called *Closed Region* and the other excludes the boundary which is called *Open Region*. We always consider closed region here.

[2]This assumption implies that $\|wp_2\| \leq \|wp_1\| \leq \|uv\|$ and $x_1 \geq x_2$.



(a) $\beta \in [1,2]$ (b) $\beta \in [0,1]$

Fig. 1. The relations between fade factors of β-skeleton.

2) If there is no point inside $N(u,v,\beta)$ then stop. Otherwise, assume a point $w \in N(u,v,\beta)$. We put edge uw as uv's left child and edge wv as uv's right child and label these two branches with their fade factor $x_1 = \|uw\|/\|uv\|$ and $x_2 = \|wv\|/\|uv\|$ respectively. The leaf nodes uw, wv form path uwv.
3) If we already have a binary tree with k leaf nodes p_0p_1, p_1p_2, $\cdots$, $p_{k-1}p_k$, where $p_0 = u$, $v = p_k$. Let $S_1 = \{p_0, p_1, \cdots, p_k\}$. For every point $p \in S$, we test if p breaks some edge p_ip_{i+1}. We consider five cases here.
   a) If p doesn't break any p_ip_{i+1} then continue to try another point from S.
   b) If $p \in S - S_1$ and p breaks a single edge p_ip_{i+1} then similar to step (2), attach p_ip as the left child and pp_{i+1} as the right child of edge p_ip_{i+1}.
   c) If $p \in S - S_1$ and it breaks multiple edges, choose such broken edge p_rp_{r+1} with the minimum index r and p_sp_{s+1} with the maximum index s. Attach node p_rp to node p_rp_{r+1} and node pp_{s+1} to node p_sp_{s+1} in the tree. Mark all leaf nodes between p_rp and pp_{s+1}. If all descendant leaf nodes of an internal node have marks, then also mark it. Delete all nodes with marks.
   d) If $p \in S_1$, say $p = p_j$, and it breaks single edge p_ip_{i+1}. If $j > i+1$ then attach p_ip_j to node p_ip_{i+1}, and mark all leaf nodes p_mp_{m+1} for $i+1 \le m \le j-1$. If $j < i$ then attach p_jp_{i+1} to node p_ip_{i+1} and mark all leaf nodes p_mp_{m+1} for $j+1 \le m \le i$. If all descendant leaf nodes of an internal node have marks, then also mark it. Delete all nodes with mark.
   e) If $p \in S_1$, say $p = p_j$, and it breaks multiple edges, choose the edge with the minimum index p_rp_{r+1} and the maximum index p_sp_{s+1}. If $j < r$ then attach p_jp_{s+1} to p_jp_{j+1} and mark all leaf nodes between p_jp_{s+1} and p_sp_{s+1}. If $j > s+1$ then attach p_sp_j to p_sp_{s+1} and mark all leaf nodes between p_sp_{s+1} and $p_{j-1}p_j$. If $r+1 < j < s$ then attach p_rp_j to p_rp_{r+1} and attach p_jp_{s+1} to p_sp_{s+1}, then mark all nodes between p_rp_{r+1} and p_sp_{s+1}. If an internal node's all descendant leaf nodes have marks, then also mark it. Delete all nodes with mark.
4) When there is no updating to the tree, conduct the following reduction process: for every internal node, if it has only one child then remove its only child. Visiting all leaf nodes from left to right, we get a sequence of edges uE_0,$B_1E_1, \cdots, B_{l-1}E_{l-1}$,$B_lv$.

Observations of fade factor tree:

1) For every $0 \le i \le l-1$, we have $E_i = B_{i+1}$, so the sequence can be written as $u_0u_1, u_1u_2, \cdots, u_{l-1}u_l$. ($u_0 = u, u_l = v$).
2) $l \le n-1$, where n is the number of total points in S.
3) $u_0u_1, u_1u_2, \cdots, u_{l-1}u_l$ corresponds to a *simple path* connecting u and v in β-skeleton.

We can show that the above algorithm terminates. For detail of the proof, see the full version of the paper.

C. *Upper bound when $\beta \in [1,2]$*

Previously, Bose *et al.* [10] gave an upper bound $O(n)$ for β-skeleton when $\beta \in [1,2]$ from the fact that, for a point set S, the β_1-skeleton belongs to the β_2-skeleton when $\beta_1 \ge \beta_2$. They use the upper bound of the RNG ($\beta = 2$) as upper bound for $\beta \in [1,2]$. We improve it to

$$U(\beta, n) = (n-1)^\gamma,$$

where $\gamma = \max\{1 - \frac{1}{2}\log_2(\mu_2+1), g(\mu_2)\} = 1 - \frac{1}{2}\log_2(\mu_2+1)$, $\mu_2 = \frac{2-\beta}{\beta}$, and $g(\mu_2)$ is the solution to the equation $(\mu_2^{\frac{1}{g(\mu_2)}} + 1)^{2g(\mu_2)} = 1 + \mu_2$ (See appendix about the details of $g(\mu_2)$ and γ).

Before presenting our proof, we list some simple results. If $x_1 \ge x_2$ and subject to the constraint of (1), then for $a, b \ge 0$,

$$\max_{x_1,x_2}\{ax_1 + bx_2\} = \sqrt{a^2 + b^2/\mu_2}\,; \quad \text{if } b \le \mu_2 a \qquad (3)$$

$$\max_{x_1,x_2}\{ax_1 + bx_2\} = (b+a)/\sqrt{\mu_2+1}\,; \quad \text{if } b \ge \mu_2 a \qquad (4)$$

Now we prove our upper bound by induction on n. When $n = 3$, there are only three points u, v and w, and suppose that uv is the longest edge. If w doesn't break uv, then $\psi(G) = 1 \le U(\beta)$. Otherwise, the relation of the fade factors from (1) implies

$$\psi(G) = x_1 + x_2 \le 2^{1-\frac{1}{2}\log_2(1+\mu_2)} = U(\beta, 3).$$

Suppose for all $k < n$ we have $\psi(G) \le U(\beta, k)$. Then for $k = n$ and any pair of points u and v, we construct their fade factor tree T, and suppose the fade factors of the root are x_1 and x_2. Suppose there are n_l leaf nodes in root's left subtree and n_r leaf nodes in root's right subtree. Clearly, $n_l + n_r \le n-1$ and we have $\psi(G) \le U(\beta, n_l+1)x_1 + U(\beta, n_r+1)x_2$. By induction, we have $U(\beta, n_l+1) \le n_l^\gamma$ and $U(\beta, n_r+1) \le n_r^\gamma$. We consider two different cases here.

1) If $U(\beta, n_r+1) \ge \mu_2 U(\beta, n_l+1)$, we have

$$\begin{aligned}\psi(G) &\le (U(\beta, n_l+1) + U(\beta, n_r+1))/\sqrt{1+\mu_2} \\ &\le (n_l+n_r)^\gamma \cdot 2^{1-1+\frac{1}{2}\log_2(1+\mu_2)}/\sqrt{1+\mu_2} \\ &\le (n-1)^\gamma = U(\beta, n)\end{aligned}$$

2) If $U(\beta, n_r+1) \le \mu_2 U(\beta, n_l+1)$, we have $\psi(G) \le \sqrt{U(\beta, n_l+1)^2 + \frac{1}{\mu_2}U(\beta, n_r+1)^2}$. Let $f(x) = x^{2\gamma} + \frac{1}{\mu_2}(n-x-1)^{2\gamma}$. Differentiating $f(x)$, we get $f'(x) = 2\gamma[x^{2\gamma-1} - \frac{1}{\mu_2}(n-x-1)^{2\gamma-1}]$. Since $1/2 \le \gamma \le 1$, $f(x)$ reaches its minimum at a point $x_0 = (n-1)/(1+\mu_2^{\frac{1}{2\gamma-1}})$, increases when $x \ge x_0$, and decreases when $0 \le x \le x_0$. Notice that $U(\beta, n_r+1) \le \mu_2 U(\beta, n_l+1)$, which implies that the feasible region for x is $x_l = \frac{n-1}{\mu_2^{1/\gamma}+1} \le x \le n-2 = x_r$. It is easy to show that $x_l \le x_0$ when $\gamma \le 1$ and $\mu_2 \le 1$. Consequently, $U(\beta, n_l+1)x_1 + U(\beta, n_r+1)x_2$ reaches its maximum at point x_l or $x_r = n-2$. We can show that it reaches the maximum at point x_l. Thus,

$$\psi(G) \le \sqrt{1+\mu_2}\cdot(n-1)^\gamma/(\mu_2^{\frac{1}{\gamma}}+1)^\gamma.$$

Notice $(\mu_2^{\frac{1}{\gamma}}+1)^\gamma$ strictly increases over $[0,1]$ for γ. Thus

$$\begin{aligned}\psi(G) &\le \sqrt{1+\mu_2}\cdot(n-1)^\gamma/(\mu_2^{\frac{1}{\gamma}}+1)^\gamma \\ &\le \sqrt{1+\mu_2}\cdot(n-1)^\gamma/(\mu_2^{\frac{1}{g(\mu_2)}}+1)^{g(\mu_2)} = n^\gamma\end{aligned}$$

The upper bound we proved so far could be a loose bound, and usually the β-skeleton cannot reach this upper bound. At some extreme cases, we can show that the upper bounds are indeed tight.

When $\beta = 1$, the β-skeleton is the Gabriel Graph. Then $\mu_2 = \frac{2-\beta}{\beta} = 1$. It is easy to verify $g(1) = \frac{1}{2}$. Thus, $\gamma = \max\{1 - \frac{1}{2}\log_2(2), \frac{1}{2}\} = \frac{1}{2}$. In the following section, we construct an example such that GG has spanning ratio $(n-1)^{\frac{1}{2}}$. Consequently, we have

Theorem 1: The spanning ratio of Gabriel Graph is exactly $U(1,n) = (n-1)^{\frac{1}{2}}$.

When $\beta = 2$, the β-skeleton becomes the RNG. Notice $\mu_2 = 0$, and it is impossible that $U(\beta, n_r) \le \mu_2 U(\beta, n_l)$. Then we have $\gamma = 1$. In the following section we review an example in [10] such that RNG has spanning ratio $n-1$. Consequently, we have

Theorem 2: The spanning ratio of Relative Neighborhood Graph is exactly $U(2,n) = n-1$.

D. Upper bound when $\beta \in [0,1]$

The fade factors satisfy $x_1^2 + x_2^2 + 2x_1x_2\sqrt{1-\beta^2} \le 1$ when $\beta < 1$. Let $\mu_1 = \sqrt{1-\beta^2}$ here. For symmetry, assume that $x_1 \ge x_2$. Thus, $0 \le x_2 \le \sqrt{\frac{1}{2+2\sqrt{1-\beta^2}}}$. If $x_1 \ge x_2$ and subject to the constraint (2), then for $a > 0$, $b > 0$,

$$\max_{x_1,x_2}\{ax_1 + bx_2\} = \frac{\sqrt{a^2+b^2-2ab\mu_1}}{\sqrt{1-\mu_1^2}} \quad \text{if } b \ge a\mu_1 \qquad (5)$$

$$\max_{x_1,x_2}\{ax_1 + bx_2\} = a\mu_1 \quad \text{if } b < a\mu_1 \qquad (6)$$

When $\beta \in [0,1]$, we prove that the spanning ratio is at most

$$U(\beta, n) = (n-1)^{\frac{1-\log_2(1+\mu_1)}{2}},$$

We prove this bound similar to the case $\beta \in [1,2]$. When $k = 3$, it is easy to verify the correctness of the bound. Suppose when $k < n$ this bound holds. For $k = n$ we also construct the fade factor tree T, and assume the fade factors of the root are x_1 and x_2. Assume there are n_l leaf nodes in root's left subtree and n_r leaf nodes in its right subtree, where $n_l + n_r \le n-1$. We have $\psi(G) \le U(\beta, n_l+1)x_1 + U(\beta, n_r+1)x_2$. Let $a = U(\beta, n_l+1)$ and $b = U(\beta, n_r+1)$. We also prove it by cases:

Case 1: $b < a\mu_1$. In this case,we have $\psi(G) \le \mu_1 U(\beta, n_l+1) \le U(\beta, n)$.

Case 2: $b \ge a\mu_1$. In this case we have $\psi(G) \le \frac{\sqrt{a^2+b^2-2ab\mu_1}}{\sqrt{1-\mu_1^2}}$, and it reaches the maximum when $a = b$. Thus $\psi(G) \le U(\beta, \frac{n+1}{2})\sqrt{\frac{2}{1+\mu_1}} = U(\beta, n)$.

III. Lower bound of β-skeleton

A. Gabriel Graph ($\beta = 1$)

Gabriel Graph is a special case of β-skeleton with $\beta = 1$. We construct a set of n points whose Gabriel graph has spanning ratio exactly $\sqrt{n-1}$ as follows.

1) Let A_1A_0 be the diameter of a unit circle C_1.
2) We then generate a point A_k from A_{k-1} and A_{k-2} for $k \ge 2$. Draw a circle C_{k-1} using A_{k-1} and A_{k-2} as diameter, and let $\sin\angle A_kA_{k-1}A_{k-2} = \sin\angle\alpha_{k-1} = \frac{1}{\sqrt{n-k+1}}$.

Figure 2 (a) illustrates such construction. We notice that the graph is divided into two parts, all points with the odd index and all points with the even index. It is not difficult to prove the following properties of the constructed point set.

1) $A_kA_{k+2} = \frac{1}{\sqrt{n-1}}$, for $0 \le k \le n-2$.

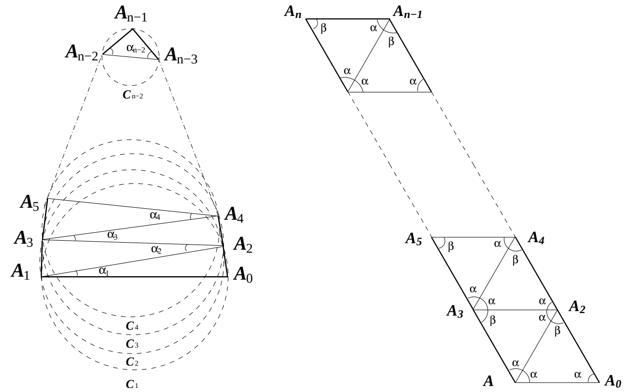


Fig. 2. Point sets that achieve the upper bounds of the spanning ratios.

2) Let $\alpha_k = \angle A_{k-1}A_kA_{k+1}$. Then $\sin\alpha_k = \frac{1}{\sqrt{n-k}}$ and $\angle\alpha_k \le \angle\alpha_{k+1}$. For every $1 \le k \le n-2$, $\angle A_{k-2}A_kA_{k+2} = \frac{\pi}{2} + \angle\alpha_k + \frac{\pi}{2} - \angle\alpha_{k+1}$. Thus, $\angle A_{k-1}A_kA_{k+1} < \frac{\pi}{2}$.
3) For every A_iA_j, if $|i-j| \ne 2$ then A_iA_j is not in the Gabriel Graph. Thus, the Gabriel graph are formed by these edges A_iA_{i+2}, $0 \le i \le n-3$, and $A_{n-2}A_{n-1}$.

Obviously, the spanning ratio of this graph is $\frac{D_G(A_0A_1)}{\|A_0A_1\|} = \frac{n-1}{\sqrt{n-1}} = \sqrt{n-1}$.

B. Relative Neighborhood Graph ($\beta = 2$)

For Relative Neighborhood Graph, the lower bound of the spanning ratio is $n-\epsilon$. We review the example used in [10], illustrated by Figure 2 (b).

Here, $\alpha = 60^\circ - \delta$ and $\beta = 60^\circ + 2\delta$. Notice that all triangles are similar. Assume that $\gamma = \frac{\sin\alpha}{\sin\beta}$. Then in triangle $A_{k-1}A_kA_{k+1}$, $1 \le k \le n-1$, we have $A_{k-1}A_k = \gamma^{k-1}$, $A_{k-1}A_{k+1} = A_kA_{k+1} = \gamma^k$. Thus, $D_G(A_0A_1) = \gamma^{n-1} + \sum_{i=1}^{n-1}\gamma^i$. When γ is sufficiently close to 1, we have $D_G(A_0A_1)$ is sufficiently close to $n-1$. Thus, the spanning ratio of the relative neighborhood graph is sufficiently close to $n-1$.

C. $1 > \beta > 0$ case

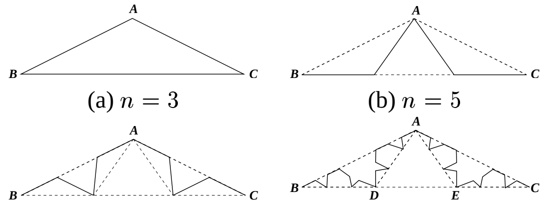
When $\beta \in [0,1]$, Eppstein [12] presents a fractal construction that provides a non-constant lower bound on the spanning ratio, and his result is summarized below:

Theorem 3: For any $n = 5^k + 1$, there exists a set of n points in the plane whose β-skeleton with $\beta \in (0,1]$ has the spanning ratio $\Omega(n^c)$, where $c = \log_5 \frac{5}{3+\sqrt{2+2\mu}}$ and $\mu = \sqrt{1-\beta^2}$.

In this paper, we give a different construction that achieves a better lower bound. Suppose that $\alpha = \arccos(\sqrt{1-\beta^2})$, and $\theta = \pi - \alpha$. Then for any $n = 2^k + 1$, let $P(\beta, k)$ be a path of 2^k segment (defined along our construction). Figure 3 illustrates our construction of β-skeleton for n points, which is described as follows.

1) If $k = 1$, construct a triangle $\triangle ABC$ such that $\angle ABC = \angle ACB = \frac{\pi-\theta}{4}$, so $\angle BAC = \frac{\pi+\theta}{2}$. Then $P(\beta, 1)$ is segments BAC. Call segment BC the *supporting segment* of $P(\beta, 1)$.
2) If $k \ge 1$, first construct $P(\beta, 1) = BAC$. Then construct two $P(\beta, k-1)$, scale the supporting segments to length $\|AB\|$, and align their supporting segments to AB and BC respectively. Notice there are two possible ways to place $P(\beta, k-1)$, we should choose the way such that $P(\beta, k-1)$ lies inside the triangle ΔABC.

Lemma 1: If $\angle BAC \ge \frac{\pi+\theta}{2}$, then $P(\beta, k)$ is a β-skeleton of its points, where $\theta = \pi - \arccos(\sqrt{1-\beta^2})$.



(a) $n = 3$ (b) $n = 5$ (c) $n = 9$ (d) $n = 2^k + 1$

Fig. 3. Constructing β-skeleton with large spanning ratio for $\beta \in [0, 1]$.

TABLE I

LOWER AND UPPER BOUNDS FOR SPANNING RATIONS OF β-SKELETONS. HERE THE CONSTANTS USED ARE $c_1 = 1 - \log_5(3 + \sqrt{2 + 2\mu_1})$, $c_2 = \frac{1}{2} - \frac{1}{2}\log_2(1 + \mu_1)$, $c_3 = \frac{1}{2} - \frac{1}{2}\log_2(1 + \sqrt{\frac{\mu_1+1}{2}})$, AND $c_4 = 1 - \frac{1}{2}\log_2(\mu_2 + 1)$. AND $\mu_1 = \sqrt{1 - \beta^2}$ AND $\mu_2 = (2 - \beta)/\beta$.

	$\beta \in (0,1)$	$\beta = 1$	$\beta \in (1,2)$	$\beta = 2$
OldLower	$\Omega(n^{c_1})$	$\Omega(\sqrt{n})$	$\Omega(\sqrt{n})$	$\Omega(n)$
OldUpper	$O(n^{c_2})$	$O(\sqrt{n})$	$O(n)$	$O(n)$
OurLower	$\Omega(n^{c_3})$	$\sqrt{n-1}$	$\Omega(\sqrt{n})$	$n - 1$
OurUpper	$O(n^{c_2})$	$\sqrt{n-1}$	$O(n^{c_4})$	$n - 1$

PROOF. In order to show that $P(\beta, k)$ is the β-skeleton, we prove that for any pair of no-adjacent points u and v, they do not belong to the β-skeleton. Obviously there must exist some $i < k$ such that u and v belong to the different copy of adjacent $P(\beta, i)$, assume that A is the common point of these two copies, then $\angle uAv \geq \angle DAE = \frac{\pi+\theta}{2} - 2 \cdot \frac{\pi-\theta}{4} = \theta$, which finishes our proof. Notice that the β-skeleton is still a connected graph. Thus, all line segments constructed belong to β-skeleton. ◻

Obviously, if the length of the supporting segment is normalized to 1, the spanning ratio is the total length of segments in $P(\beta, k)$.

Theorem 4: For any $\beta \in [0, 1]$, there exists a β-skeleton of $n = 2^k + 1$ points such that its spanning ratio is $\Omega((n - 1)^{\frac{1}{2} - \frac{1}{2}\log_2(1+\sqrt{\frac{\mu_1+1}{2}})})$, where $\mu_1 = \sqrt{1 - \beta^2}$.

This theorem can be enhanced such that we can construct examples for any integer n, but with a small constant degradation of the spanning ratio. For Gabriel Graph, from previous result by Eppstein [12], we get a spanning ratio of $\Omega(n^c)$ for $0.077 < c < 0.078$, and applying Theorem 4, we get a spanning ratio of $\Omega(n^{c_2})$ for $0.114 < c_2 < 0.115$, which is bigger than previous lower bound, but is still much smaller than the tight bound $\Theta(n^{\frac{1}{2}})$. In general, for $\beta \in [0, 1]$, our lower bound is always better than the previous one, which is discussed in the full version of the paper.

IV. CONCLUSION

We studied the spanning ratio of β-skeletons with β ranging from 0 to 2. This class of proximity graphs includes the Gabriel graph and the relative neighborhood graph. Table I summarizes our results compared with the previously best known results. For $\beta > 2$, β-skeletons are not guaranteed to be connected. Thus, the spanning ratios leap to infinity.

Several open problems remain for investigation. It would be interesting to close the gap between our lower bound and the upper bound for $\beta \in (0, 1)$ and $\beta \in (1, 2)$. We conjecture that our lower bound for $\beta \in (0, 1)$ is already tight.

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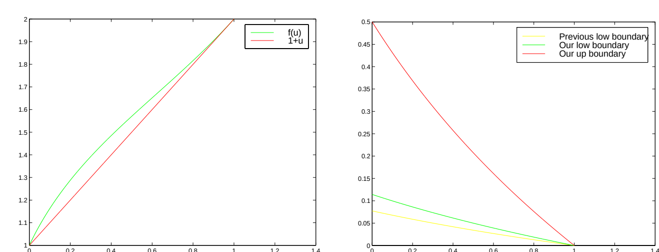
V. APPENDIX

In subsection II-C, we show that the β-skeleton, $\beta \in [1, 2]$, has spanning ration at most $(n - 1)^\gamma$, where $\gamma = \max\{1 - \frac{1}{2}\log_2(\mu_2 + 1), g(\mu_2)\}$, $\mu_2 = (2 - \beta)/\beta \in (0, 1]$. We then show that $1 - \frac{1}{2}\log_2(\mu_2 + 1) \geq g(\mu_2)$.

Let $f(x) = (\mu_2^{1/x} + 1)^{2x}$. For any $\mu_2 \in [0, 1]$, it is easy to verify that both $\mu_2^{1/x} + 1$ and $a^{2x} (a \geq 1)$ are increasing on $[0, 1]$, where a is a fixed constant. Thus, $f(x)$ increases over $[0, 1]$. With $f(0) = 1 \leq 1 + \mu_2$ and $f(1) = (1 + \mu_2)^2 \geq 1 + \mu_2$, the equation $(\mu_2^{\frac{1}{g(\mu_2)}} + 1)^{2g(\mu_2)} = 1 + \mu_2$ has exactly one solution $g(\mu_2)$ over $[0, 1]$. In fact, any solution to the inequality $(\mu_2^{\frac{1}{g(\mu_2)}} + 1)^{2g(\mu_2)} \geq 1 + \mu_2$ is an upper bound.

Now we compare the the value of $1 - \frac{1}{2}\log_2(\mu_2 + 1)$ and $g(\mu_2)$, which is equivalent to compare the value of $f(\mu_2) = (\mu_2^{\frac{1}{1-\frac{1}{2}\log_2(1+\mu_2)}} + 1)^{2-\log_2(1+\mu_2)}$ and $1 + \mu_2$ for $\mu_2 \in [0, 1]$. Figure 4(a) shows that $f(\mu_2) \geq 1 + \mu_2$, which means for $\mu_2 \in [0, 1]$ $\gamma = \max\{1 - \frac{1}{2}\log_2(\mu_2 + 1), g(\mu_2)\} = 1 - \frac{1}{2}\log_2(\mu_2 + 1)$. See full version of the paper for arithmetic proof.



(a) $g(\mu_2)$ is unnecessary. (b) lower bounds for $\beta \in [0, 1]$.

Fig. 4. (a) The upper bound for $\beta \in [1, 2]$ can be simplified. (b) Our lower bound for $\beta \in [0, 1]$ is strictly better than previous result.