Generating Well-Shaped Delaunay Meshes in 3D

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Abstract

A triangular mesh in 3D is a decomposition of a given geometric domain into tetrahedra. The mesh is well-shaped if the aspect ratio of every of its tetrahedra is bounded from above by a constant. It is Delaunay if the interior of the circum-sphere of each of its tetrahedra does not contain any other mesh vertices. Generating a well-shaped Delaunay mesh for any 3D domain has been a long term outstanding problem. In this paper, we present an efficient 3D Delaunay meshing algorithm that mathematically guarantees the well-shape quality of the mesh, if the domain does not have acute angles. The main ingredient of our algorithm is a novel refinement technique which systematically forbids the formation of slivers, a family of bad elements that none of the previous known algorithms can cleanly remove, especially near the domain boundary — needless to say, that our algorithm ensure that there is no sliver near the boundary of the domain.

Keywords: Mesh generation, Delaunay triangulations, slivers, computational geometry, algorithms.

1 Introduction

Mesh generation is the process of breaking a domain into a collection of primitive elements. In this paper we exclusively consider three-dimensional Delaunay meshes whose elements are tetrahedra. A mesh is Delaunay if the circumsphere of any tetrahedron element does not contain any mesh vertices inside. We assume that the spatial domain is given in terms of its piecewise linear complex boundary (*PLC*) [16].

The size and shape of the tetrahedra is important because it influences the convergence and stability of numerical algorithms such as the finite element method; see Strang and Fix [15]. Generating meshes with small aspect ratio is one of the most important steps in numerical simulations. The *aspect ratio* of a tetrahedron is usually defined as its circumradius divided by its inradius. The aspect ratio of a mesh is the largest aspect ratio of all of its tetrahedral elements. A mesh is *well-shaped* if it has small aspect ratio. Unfortunately, currently there is no method that guarantees to generate well-shaped 3D Delaunay meshes. An alternative but weaker quality measurement is to use the *radius-edge ratio* [10]. It is the circumradius divided by the shortest edge length of the tetrahedron. The radius-edge ratio of a mesh is the maximum radius-edge ratio among all of its elements. A mesh is *almost good* if it has small radius-edge ratio. Numerous methods [3, 7, 9, 11, 13, 14] guarantee to generate 3D almost-good Delaunay meshes.

Slivers are the only elements that have small radius-edge ratio but have large aspect ratio. Talmor [16] notes that even well-spaced vertex set does not prevent slivers from its Delaunay triangulation. Thus the main difficulty of three-dimensional mesh generation comes from the existence of slivers. Mitchell and Vavasis [12] use oct-trees to generate a well-shaped tetrahedral mesh for a domain bounded by a specified polyhedral boundary. But the final mesh is not Delaunay. The Delaunay refinement or sphere packing based method fail to address the problem of slivers.

Chew [4] sketched an algorithm that eliminates slivers by adding points in a randomized manner. For each tetrahedron with circumradius larger than the unit length, it adds a random point within a half unit of the circumcenter to the point set. Chew showed that there exists a point that will not introduce new slivers with circumradius less than one unit. The Delaunay triangulation is then updated if no new sliver is introduced by the new point. However, his algorithm generates constant density meshes. In addition, his algorithm does not address the slivers completely.

Recently, Cheng *et al.* [2] developed an algorithm that, given an almost good Delaunay triangulation, constructs an assignment of weights so the weighted Delaunay triangulation is free of slivers inside. We refer the reader to [2] for a description of weighted Delaunay

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triangulations. Then Edelsbrunner *et al.* [5] developed a new algorithm that perturbs the vertices of an almost-good mesh such that the Delaunay triangulation of perturbed vertices is free of slivers inside. Notice that there are no any boundary treatments by these two algorithms presented in [2] and [5].

The main result of this paper is a refinement-based technique that generates meshes with small aspect ratio. It first eliminates all tetrahedra with large radius-edge ratio. It then adds points around the circumcenter of any sliver τ so that it avoids creating new small slivers at the same time. Here a created sliver is *small* if its circumradius is less than a constant b factor of the circumradius of this sliver τ. It keeps adding points until the mesh has small radius-edge ratio and does not have slivers. The tetrahedra with large radius-edge ratio have priority over slivers to be refined. We prove that for any element τ of an almost-good mesh, there is a point p around its circumcenter such that the insertion of p will not introduce new *small* slivers. When circumcenter of bad tetrahedron encroaches boundary triangles or segments, we split these boundary triangles or segments instead of refining that bad tetrahedron. We prove the termination guarantee of our algorithm by showing that the distance between the closest mesh vertices is just decreased by a constant factor compared with that of the input mesh. Our algorithm differs from Chew's algorithm in that we generate a non-uniform mesh and our algorithm eliminates all original slivers without introducing any slivers in final mesh.

The remainder of the paper is structured as follows. Section 2 introduces the basic concept such as Delaunay triangulation, sliver, picking regions. Our refinement-based algorithm is presented in Section 3. It specifies how to avoid creating new small slivers, how to eliminate existed slivers, and how to remove elements with large radius-edge ratio. The termination guarantee of the algorithm is presented in Section 4. In Section 5, we show that the generated well-shaped mesh has size within a constant factor of the size of any almost-good meshes for the same domain. Section 6 concludes the paper with discussions.

2 Preliminaries

2.1 Delaunay Triangulation Due to their numerous desirable properties, and the abundance of the well studied algorithms to construct them [1, 6], Delaunay triangulations are widely used in generating tetrahedral meshes. The following properties about Delaunay triangulations are extensively used in this paper. After inserting a new vertex p, all new tetrahedra created in the Delaunay triangulation of the new vertex set are incident on p. And the new Delaunay triangulation can be obtained by efficient operations local the new vertex. The nearest neighbor graph defined by a vertex set is contained in the Delaunay triangulation of the vertex set. In other words, the shortest edge length of the Delaunay triangulation is the closest distance among mesh vertices. This fact is used in proving the termination guarantee of our algorithm.

2.2 Parameterizing Slivers For later convenience, we use R_τ, L_τ and $\rho(\tau)$ to denote the circumradius, the shortest edge length and the radius-edge ratio of an element τ. Let $pqrs$ be a tetrahedron with volume V and the shortest edge length L. As [2, 5], we define $\sigma = \sigma(pqrs) = V/L^3$ as a measure of its quality. Call $pqrs$ a *sliver* if $\rho(pqrs) \le \varrho_0$ and $\sigma(pqrs) < \sigma_0$, where ϱ_0, σ_0 are constants that we specify later.

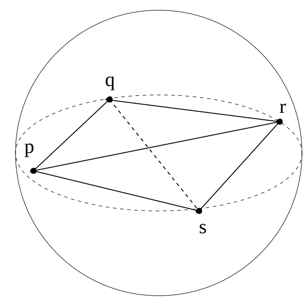

Figure 1: A sliver example.

The following lemma verifies our definition of $\sigma(\tau)$ for tetrahedron τ.

LEMMA 2.1. *The aspect ratio of τ is at most $\frac{\sqrt{3}\rho(\tau)^3}{\sigma(\tau)}$.*

The proof is omitted here. Hereafter, we use (c, R) to denote the sphere centered at a point c with radius R.

For any base triangle qrs, there is a set of points p such that $pqrs$ is a sliver. We call them the *forbidden region* F_{qrs} of base triangle qrs. Let Y be the circumradius of triangle qrs. The following lemmas proved in [8] are essential for analyzing our algorithm.

LEMMA 2.2. [FORBIDDEN VOLUME] *For any base triangle qrs, the volume of the forbidden region F_{qrs} is at most c_3Y^3, where $c_3 = 2\pi^2(48\varrho_0\sigma_0)^2$.*

LEMMA 2.3. [FORBIDDEN AREA] *For triangle qrs and a plane $\mathcal{H}$, the intersection of the forbidden region F_{qrs} with $\mathcal{H}$ has area at most c_4Y^2, where $c_4 = 192\pi\varrho_0\sigma_0$.*

LEMMA 2.4. [FORBIDDEN LENGTH] *For triangle qrs and a line $\mathcal{L}$, the intersection of the forbidden region F_{qrs} with $\mathcal{L}$ has length at most c_5Y, where $c_5 = 16\sqrt{3\varrho_0\sigma_0}$.*

2.3 Picking Region A tetrahedron is called *bad* if it has large radius-edge ratio or is a sliver. The algorithm removes each bad tetrahedron by inserting a point inside its circumsphere. We then discuss where to select such point. Let's consider a tetrahedron τ. We only pick a point from the interior of sphere $(c_\tau, \delta R_\tau)$, where $\delta < 1$ is a constant to be specified later. We call the solid ball $(c_\tau, \delta R_\tau)$ the *picking region* of τ.

A PLC domain satisfies the projection condition [14] if for any vertex p encroaching a boundary triangle, there is a boundary triangle that contains the projection of p inside. We will always assume that the input domain satisfies the projection condition. We only split the boundary triangle containing the projection of the encroaching point p. We always assume that the PLC domain does not have acute angles similar to [14]. This will guarantee that the boundary protections terminate in finite number of steps.

For a triangle qrs, its *equatorial sphere* is the smallest sphere containing points q, r, and s. For a segment qr, its *diametric sphere* is the smallest sphere containing points q and r. Without confusion, sometimes we will just use circumsphere to denote the smallest sphere containing a triangle or segment. A point *encroaches* boundary if it is contained in the circumsphere of any boundary triangle or segment. Here a triangle is boundary triangle if it belongs to a two-dimensional Delaunay triangulation of a boundary face; a segment is boundary segment if defines the polygonal boundary faces. We call the element, whose circumcenter encroaches a boundary triangle or segment, the *encroaching element*; that boundary triangle or segment is called the *encroached element*. Shewchuk [14] proved the following radii relation between the encroaching element and encroached element.

Lemma 2.5. [Encroachment Relations] *The circumradius of the encroached element is at least $\frac{1}{\sqrt{2}}$ factor of the circumradius of the encroaching element.*

We are now in the position to study how to split the encroached boundary triangles or boundary segments. Consider a boundary triangle qrs. Let (c_{qrs}, R_{qrs}) be its circumcircle. We call the disk $(c_{qrs}, \delta R_{qrs})$ the *picking region* of qrs. A point from the picking region is inserted if triangle qrs is encroached. Then consider an encroached boundary segment qr. Let c be its middle point. Then a point from segment qr with distance no more than $\delta||qr||$ to c is selected to split qr. In other words, the *picking region* of qr is a subsegment on qr centered at c with length $2\delta||qr||$. Without confusion, we will use $(c_\tau, \delta R_\tau)$ to denote the picking region of an element τ. Here τ can be a tetrahedron, a triangle or a segment. See Fig. 2 for illustrations.

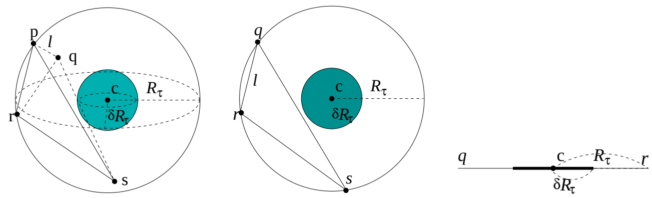

Figure 2: The picking region of a tetrahedron, a boundary triangle and a boundary segment respectively.

3 Refinement Algorithms

In this section, we present the refinement-based algorithm to generate well-shaped Delaunay meshes.

3.1 Algorithm Outline For any bad element in the mesh, we add a point p inside its picking region. The insertion of point p removes this bad element, but may create many new bad tetrahedra. It is possible that inserting any point in the picking region will create a new sliver [8]. The algorithm has the following four components.

Algorithm: Remove-Sliver-by-Refinement (ϱ_0, σ_0, δ, b)

Enforce Empty Encroachment: For any encroached boundary segment, add its midpoint and update the Delaunay triangulation. For any encroached boundary triangle, add its circumcenter c and update the Delaunay triangulation. If c encroaches any boundary segment, we split the encroached boundary segment instead of adding c.

Clean Bad Elements: For any bad tetrahedron τ, find a point p in its picking region whose insertion avoids creating small slivers. If such p does not exist, then add the circumcenter c_τ of τ.

Here, tetrahedra with a large radius-edge ratio have priority over slivers to be split.

If the circumcenter c_τ encroaches boundary, we apply the following rules instead.

Encroach Equatorial Sphere: For any encroached boundary triangle μ, add a point p, whose insertion avoids creating new small slivers, from the picking region of μ. If such p does not exist, then add the circumcenter c_μ of μ. Update the Delaunay triangulation. However, if c_μ encroaches any boundary segment, apply the following rule instead of finding p from $(c_\mu, \delta R_\mu)$.

Encroach Diametric Sphere: For any diametric sphere of boundary segment, if it contains point c_τ

or c_μ inside, then add a point p, whose insertion avoids creating new small slivers, from the segment's picking region. If such p does not exist, we split the segment by adding its midpoint. Update the Delaunay triangulation.

3.2 Select Point in Picking Region The key part of the algorithm is to find a point p whose insertion avoids creating new small slivers. One approach is based on a randomized selection as Chew did [4]. We randomly select a point p from the picking region and construct a local mesh whose elements are all incident on p. If there is a small sliver in the local mesh, we discard p and reselect a new point from the picking region randomly. The above procedure is repeated for constant rounds. By defining slivers and small slivers properly, we can show that the above procedure is expected to find point p if such a point exists.

4 Termination Guarantee

In this section, we prove that the algorithm will terminate if we define what is bad element properly. After the algorithm terminates, the generated mesh elements have small aspect ratio.

4.1 Existence Notice that the tetrahedra with large radius-edge ratio have priority to be refined than the slivers. Miller *et al.* [10] proved that, given an almost-good mesh, the lengths of edges sharing a common vertex are within a constant factor of each other, where the constant depending on the radius-edge ratio of the mesh. We call it the *length variation bound* ν_0.

LEMMA 4.1. [CONSTANT SMALL SLIVERS] *There are at most constant number of triangles that can form small slivers with points from the picking region of an element.*

PROOF. Let's consider an element τ of an almost-good mesh. Let T be set of all triangles that can form small slivers with points from $(c_\tau, \delta R_\tau)$ in the new Delaunay triangulation. Assume that triangle qrs forms a sliver $pqrs$ together with a point p from the picking region. Then edges of $pqrs$ have lengths at most $2R_{pqrs}$, which is at most $2bR_\tau$ from $R_{pqrs} \le bR_\tau$. Then the edges incident on vertex q before point p is introduced have length at least $2bR_\tau/\nu_0$, where ν_0 is the length variation bound of meshes with radius-edge ratio ϱ_0. It implies that the closest distance among all vertices of all triangles of T is at least $2bR_\tau/\nu_0$.

It is simple to show that the vertices of all triangles in T are inside the sphere $(c_\tau, (\delta + 2b)R_\tau)$. Then by a volume argument, we know that the number of vertices of T is bounded from above by a constant. For convenience, we will use W to denote such constant. $\square$

It remains to show that, given an almost-good mesh, there is a point in the picking region of any element to avoid creating new small slivers. The following conditions are sufficient: (1) $W \cdot c_3(bR_\tau)^3 < (\delta R_\tau)^3$; (2) $W \cdot c_4(bR_\tau)^2 < (\delta R_\tau)^2$; and (3) $W \cdot c_5 bR_\tau < \delta R_\tau$, where W is a constant depending on ϱ_0, δ and b. Recall that $c_3 = 2\pi^2(48\varrho_0\sigma_0)^2$, $c_4 = 192\pi\varrho_0\sigma_0$, and $c_5 = 16\sqrt{3\varrho_0\sigma_0}$. In other words, the σ_0 used to define sliver has to satisfy all three inequalities.

$$\begin{aligned}
\textbf{Volume}: &\quad \sigma_0 \le \frac{\delta\sqrt{\delta}}{48\sqrt{2Wb\pi\varrho_0 b}} \le \frac{(\varrho_0-2)\sqrt{\varrho_0-2}}{192\pi\varrho_0^4\sqrt{W}};\\
\textbf{Area}: &\quad \sigma_0 \le \frac{\delta^2}{192\pi W\varrho_0 b^2} \le \frac{(\varrho_0-2)^2}{768\pi W\varrho_0^5};\\
\textbf{Length}: &\quad \sigma_0 \le \frac{\delta^2}{768W^2\varrho_0 b^2} \le \frac{(\varrho_0-2)^2}{3072W^2\varrho_0^5}.
\end{aligned}$$

We summarize the above discussions by the following Existence Theorem.

THEOREM 4.1. [EXISTENCE] *Given an almost-good mesh, there is a point in the picking region of each element whose insertion will not introduce small slivers.*

4.2 Termination We then show that the algorithm terminates. We first classify the bad elements to three classes: *original slivers* in the mesh, *created slivers* by inserting some points, *tetrahedra with large radius-edge ratio*. See Figure 3 for data flow illustration of

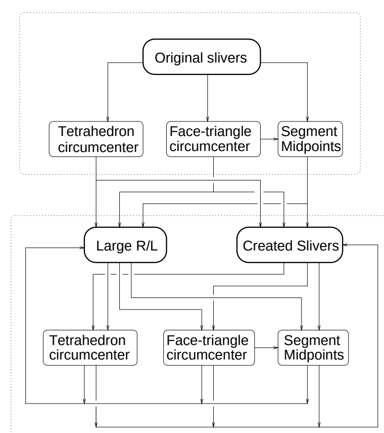

Figure 3: Dataflow diagram illustrating the bad elements evolution.

these three bad tetrahedra. Then the operations of the algorithm are categorized as eliminating original slivers, eliminating created slivers, and eliminating tetrahedra with large radius-edge ratio. Observe that the distance among mesh vertices generated by this algorithm will possibly decrease along the insertion of new points. For example, the insertion of a point p in the picking region of a sliver τ could possibly decrease the shortest distance by a constant factor. However, on the other hand, we will show that the shortest distance for any intermediate mesh is at least a constant factor of that of the original mesh. Then by a volume argument, we know that the algorithm will terminate. For convenient, we use l_{org} to denote the shortest edge length of the original mesh after enforcing that all equatorial spheres of boundary triangles and diametric spheres of boundary segments are empty.

4.3 Eliminate Original Slivers For simplicity, assume that the input mesh is almost-good. Let's first study the case of eliminating original slivers of the input mesh. Notice that in the algorithm, we do not distinguish the original slivers from the created slivers by point insertion. For the sake of convenience of analysis, we assume that all original slivers are removed first.

LEMMA 4.2. [ORIGINAL SLIVERS] *After eliminating all original slivers, the length of the shortest edge of the mesh is at least* $(1-\delta)/4$ *of that of the original mesh.*

PROOF. We consider an original sliver τ and assume that point p is inserted to eliminate τ. If p is selected from the picking region of τ, then for a mesh vertex v

$$||v-p|| \geq (1-\delta)R_\tau \geq (1-\delta)L_\tau/2 \geq (1-\delta)l_{org}/2.$$

If p is selected from the picking region of an encroached boundary triangle or boundary segment μ, then $R_\mu \geq R_\tau/2$. And $R_\tau \geq L_\tau/2 \geq l_{org}/2$ implies

$$||v-p|| \geq (1-\delta)R_\mu \geq (1-\delta)l_{org}/4.$$

Then the lemma follows. ◻

4.4 Eliminate Large Radius-Edge Ratio Then let's study the scenario when we insert points to remove tetrahedra with large radius-edge ratio. These tetrahedra could be original or be created by inserting points. Consider a tetrahedron τ with $R_\tau \geq \varrho_0 L_\tau$. The following lemma bounds the shortest edge length after a point p is inserted to process τ.

LEMMA 4.3. [LARGE RADIUS-EDGE RATIO] *The length of the shortest introduced edge after eliminating* τ *with* $\frac{R_\tau}{L_\tau} > \varrho_0$ *is at least* $\frac{(1-\delta)\varrho_0}{2}L_\tau$.

PROOF. There are two cases when eliminating a tetrahedron τ with large radius-edge ratio.

The first case is that a point p from the picking region of τ is inserted to remove τ. The length of the shortest edge introduced after inserting p is at least $(1-\delta)R_\tau \geq (1-\delta)\varrho_0 L_\tau$.

The second case is that the circumcenter c_τ encroaches domain boundary. Assume that c_τ encroaches the circumsphere (v, R_v) of a boundary triangle or segment. Recall that $R_v \geq R_\tau/2$. Thus the shortest edge introduced after selecting p from $(v, \delta R_v)$ has length at least $\frac{1-\delta}{2}R_\tau \geq \frac{1-\delta}{2}\varrho_0 L_\tau$. Notice that it may need splitting the domain boundary several times till tetrahedron τ is eliminated. The above argument is true for each boundary splitting. ◻

Thus, if $(1-\delta)\varrho_0 \geq 2$, inserting points to eliminate any tetrahedron with radius-edge ratio larger than ϱ_0 will not introduce shorter edges to the mesh.

4.5 Eliminate New Created Slivers It remains to show that the shortest edge length will not decrease when points are inserted to eliminate created slivers. Let's consider a sliver τ created by inserting a point from the picking region of an element; say $f(\tau)$. In other words, $f(\tau)$ is responsible for creating the new sliver τ. Element $f(\tau)$ is called the *parent* of sliver τ. There are three cases: $f(\tau)$ is a sliver; $f(\tau)$ is a tetrahedron with large radius-edge ratio; $f(\tau)$ is a boundary triangle or segment encroached by a bad element directly or indirectly.

Let us first consider the case that $f(\tau)$ is a sliver. Then we have $R_\tau \geq bR_{f(\tau)}$ because the insertion of point in the picking region of sliver $f(\tau)$ will always avoid creating small slivers.

LEMMA 4.4. [LARGE SLIVER- SLIVER] *The length of the shortest introduced edge after eliminating sliver* τ *is at least* $\frac{1-\delta}{4}b \cdot L_{f(\tau)}$, *where* $L_{f(\tau)}$ *is the shortest edge length of the parent sliver element* $f(\tau)$.

PROOF. First consider the case that the circumcenter c_τ of τ does not encroach the boundary. The shortest edge introduced after inserting a point from the picking region of τ has length at least $(1-\delta)R_\tau$, which is at least $(1-\delta)bR_{f(\tau)} \geq (1-\delta)bL_{f(\tau)}/2$.

Then consider the case that c_τ encroaches the circumsphere (v, R_v) of a boundary triangle or segment directly or indirectly. Notice that $R_v \geq R_\tau/2$. Thus the length of the shortest edge introduced after inserting a point from $(v, \delta R_v)$ is at least $\frac{1-\delta}{2}R_\tau$, which is at least $\frac{(1-\delta)b}{4}L_{f(\tau)}$. Then the lemma follows. ◻

Then we study the second case that the parent $f(\tau)$ is a tetrahedron with large radius-edge ratio.

LEMMA 4.5. [LARGE SLIVER -LARGE R/L] *Assume sliver* τ *is created by eliminating tetrahedron* $f(\tau)$ *with* $\frac{R_\tau}{L_\tau} > \varrho_0$. *Then the length of the shortest edge introduced by eliminating sliver* τ *is at least* $\frac{(1-\delta)^2\varrho_0}{4} \cdot L_{f(\tau)}$, *where* $L_{f(\tau)}$ *is the shortest edge length of* $f(\tau)$.

PROOF. We first consider the case that the circumcenter c_τ of τ does not encroach the boundary. The shortest edge e introduced by eliminating τ has length at least $(1-\delta)R_\tau$. Assume that $\tau = pqrs$ is created by the insertion of point p from the picking region of $f(\tau)$. Then the shortest edge connected to p has length at least $(1-\delta)R_{f(\tau)}$. From $R_\tau \geq \frac{\|p-q\|}{2}$, we have $R_\tau \geq \frac{(1-\delta)R_{f(\tau)}}{2}$. Then the length of e is at least

$$\frac{(1-\delta)^2 R_{f(\tau)}}{2}.$$

We then consider the case that c_τ encroaches the circumsphere (v, R_v) of a boundary triangle or segment directly or indirectly. Notice that $R_v \geq R_\tau/2$. Thus the length of the shortest edge introduced after selecting a point from $(v, \delta R_v)$ is at least $\frac{1-\delta}{2}R_\tau$, which is at least

$$\frac{(1-\delta)^2 R_{f(\tau)}}{4}.$$

Then $R_{f(\tau)} \geq \varrho_0 L_{f(\tau)}$ completes the proof. ◻

It remains to show that splitting a sliver τ created by splitting a boundary triangle or segment μ will not introduce shorter edges.

LEMMA 4.6. [LARGE SLIVER- BOUNDARY] *Assume sliver τ is created by splitting a boundary triangle or segment μ. Then the length of the shortest edge introduced by eliminating τ is at least*

- $|e| \geq \frac{(1-\delta)^2 \varrho_0 L_{p(\mu)}}{8}$, *where $L_{p(\mu)}$ is the shortest edge length of $p(\mu)$ and $p(\mu)$ has large radius-edge ratio.*
- $|e| \geq \frac{(1-\delta)bL_\mu}{4}$, *where L_μ is the shortest edge length of μ and parent element $p(\mu)$ is a sliver.*

PROOF. Recall that we split μ because there is a bad tetrahedron whose circumcenter encroaches the domain boundary. Let $p(\mu)$ be that bad tetrahedron. There are two cases about $p(\mu)$: it is a sliver or it is a tetrahedron with large radius-edge ratio.

Let us first study the case that $p(\mu)$ has radius-edge ratio larger than ϱ_0. Then as proved by previous lemmas, the length of the shortest edge e introduced by splitting τ is at least $\frac{(1-\delta)^2 R_\mu}{4}$, where R_μ is the circumradius of μ. Notice that here, we may need split boundary triangles or segments if the circumcenter of sliver τ encroaches the domain boundary. It is always true that $R_\mu \geq R_{p(\mu)}/2$. From $R_{p(\mu)} \geq \varrho_0 L_{p(\mu)}$, we have $R_\mu \geq \varrho_0 L_{p(\mu)}/2$, where $R_{p(\mu)}$ and $L_{p(\mu)}$ is the circumradius and the shortest edge length of $p(\mu)$ respectively. Consequently, we have

$$|e| \geq \frac{(1-\delta)^2 \varrho_0 L_{p(\mu)}}{8}.$$

We then study the case that $p(\mu)$ is a sliver. Recall that the mesh should be almost-good when we split sliver $p(\mu)$. Then the Existence Theorem 4.1 guarantees that τ is not a small sliver, i.e., $R_\tau \geq bR_\mu$. Notice that the length of the shortest edge e introduced by splitting τ is at least $\frac{(1-\delta)R_\tau}{2}$, which is at least $\frac{(1-\delta)bR_\mu}{2}$. The fact that $R_\mu \geq L_\mu/2$ implies that $|e| \geq \frac{(1-\delta)bL_\mu}{4}$. ◻

The above lemma implies that, if $(1-\delta)b \geq 4$ and $(1-\delta)^2\varrho_0 \geq 8$, eliminating a created sliver τ will not introduce shorter edges.

4.6 Main Theorem Combining all the above analysis, we have the following theorem.

THEOREM 4.2. [SHORTEST EDGE LENGTH] *The length of the shortest edge introduced by eliminating all original slivers is at least $(1-\delta)/4$ factor of that of the original mesh. If we select b, δ and ϱ_0 such that $(1-\delta)b \geq 4$ and $(1-\delta)^2\varrho_0 \geq 8$, the shortest edge length of the mesh will never decrease after all the original slivers are eliminated.*

Consequently, the shortest distance between all mesh vertices is at least $\frac{1-\delta}{4}$ factor of that of the original mesh. It is straightforward to show that the above algorithm is guaranteed to terminate by a volume argument. Then refinement algorithm generates well-shaped three-dimensional Delaunay meshes.

5 Good Grading Guarantee

This section is devoted to study the mesh size of the generated mesh, or more specifically, the relation between the nearest neighbor function $N()$ defined by the final mesh and the local feature size function *lfs*() defined by the input domain. Here $N(v)$ is the distance from v to the second nearest mesh vertex; and $mathitlfs(x)$ is the radius of the smallest sphere centered at x intersects two non-incident input segments or input vertices.

We study the spacing relations among intermediate meshes by using similar idea as Ruppert and Shewchuk did. With each vertex v, associate an *insertion edge length* e_v equal to the length of the shortest edge connected to v immediately after v is introduced into the Delaunay mesh. Notice that v may not have to be inserted into the mesh actually. For the sake of convenience of analyzing, we also define a *parent vertex* $p(v)$ for each vertex v, unless v is an input vertex. Intuitively, for any noninput vertex v, $p(v)$ is the vertex "responsible" for the insertion of v. We discuss in detail what means by responsible here. If v is inserted inside the picking region of a tetrahedron τ with $\rho(\tau) \geq \varrho_0$, then $p(v)$ is the most recently inserted end point of the shortest edge of τ. If v is inserted inside the picking region of an original sliver τ, then $p(v)$ is an end point of the shortest edge of τ. If v is inserted inside the picking region of a created sliver τ, then $p(v)$ is the vertex of τ that is responsible for creating τ, i.e., the most recently inserted vertex of τ. If v is inserted inside the picking region of an encroached boundary triangle or segment, then $p(v)$ is the encroaching vertex. For the sake of simplicity, always assume that the encroaching vertex is not an input vertex, because Ruppert [13] and Shewchuk [14] showed that the nearest neighbor function $N()$ defined on the Delaunay mesh after enforcing the domain boundary is within a constant factor of the local feature size function, i.e., $N(v) \sim lfs(v)$.

Notice that the parent vertex $p(v)$ of v does not need to be inserted into the mesh actually. We then show that the insertion edge length e_v for any introduced mesh vertex v is related to that of its parent vertex $p(v)$. Notice that here v may not be inserted due to encroaching also.

LEMMA 5.1. [INSERTION EDGE LENGTH] *Let v be a vertex of the final mesh generated and let $p = p(v)$ be the vertex responsible for the insertion of v. Then we have $e_v \geq lfs(v)$ for an input vertex v; and $e_v \geq C \cdot e_p$ for Steiner point v, where*

1. *$C = (1-\delta)\varrho_0$ if v is selected from the picking region of a tetrahedron with $\frac{R_\tau}{L_\tau} > \varrho_0$;*
2. *$C = \frac{1-\delta}{\sqrt{3}}$ if v is selected from the picking region of an original sliver;*
3. *$C = \frac{1-\delta}{1+\delta} b$ if v is selected from the picking region of a created sliver and the parent $f(\tau)$ is also a sliver;*
4. *$C = \frac{1-\delta}{2}$ if v is selected from the picking region of a created sliver and the parent element $f(\tau)$ has radius-edge ratio more than ϱ_0;*
5. *$C = \frac{1-\delta}{1+\delta} b$ if v is selected from the picking region of a created sliver and the parent $f(\tau)$ is a boundary triangle or segment encroached by a sliver;*
6. *$C = \frac{1-\delta}{2}$ if v is selected from the picking region of a created sliver and the parent element $f(\tau)$ is a boundary triangle or segment encroached by a tetrahedron with large radius-edge ratio;*
7. *$C = \frac{1-\delta}{\sqrt{2}(1+\delta)}$ if v is selected from the picking region of an encroached boundary triangle or segment.*

PROOF. If v is an original input vertex, then the length e_v of the shortest edge connected to v is at least $lfs(v)$ from the definition of $lfs(v)$. Thus $e_v \geq lfs(v)$ and the theorem holds.

Then consider non-input vertex v. Assume that v is selected from the picking region of an element τ. It is always true that $e_v \geq (1-\delta)R_\tau$, where R_τ is the circumradius of τ.

If τ is a tetrahedron with radius-edge ratio at least ϱ_0, then parent p is one of the end points of the shortest edge of τ. Here p could be the most recently inserted Steiner vertex or an original vertex of τ. Let L_τ be the length of the shortest edge pq of τ. Then q is original vertex or is inserted before p. In both cases, we have $e_p \leq \|p-q\|$. [1] Then $e_p \leq \|p-q\| = L_\tau \leq \frac{R_\tau}{\varrho_0}$. Thus

$$e_v \geq (1-\delta)R_\tau \geq (1-\delta)\varrho_0 \cdot e_p.$$

If τ is an original sliver, then parent p is one of the end points of the shortest edge of τ. Assume τ has four vertices p, q, r, s. Let L_τ be the length of the shortest edge pq of τ. Then $R_\tau \geq Y \geq L_\tau/\sqrt{3}$, where Y is the circumradius of triangle pqr. Similar to previous case, we have $e_p \leq \|p-q\| \leq \sqrt{3}R_\tau$. Thus

$$e_v \geq (1-\delta)R_\tau \geq \frac{1-\delta}{\sqrt{3}} \cdot e_p.$$

Then consider that τ is a created sliver. There are three cases about the parent element $f(\tau)$ of τ: $f(\tau)$ is a sliver; $f(\tau)$ has large radius-edge ratio; $f(\tau)$ is an encroached boundary triangle or segment. Recall that the parent vertex $p = p(v)$ is the most recently inserted vertex of τ.

We first study that the parent element $f(\tau)$ is a sliver. Recall that the insertion of $p = p(v)$ from the picking region of sliver $f(\tau)$ will always avoid creating small slivers. Thus, we have $R_\tau \geq bR_{f(\tau)}$, where $R_{f(\tau)}$ is the circumradius of element $f(\tau)$. The length of the shortest edge connected to p is $e_p \leq (1+\delta)R_{f(\tau)}$. Thus, we have

$$e_v \geq (1-\delta)bR_{f(\tau)} \geq \frac{1-\delta}{1+\delta} b \cdot e_p.$$

Then we study that parent element $f(\tau)$ has radius-edge ratio at least ϱ_0. Assume that tetrahedron τ has four vertices p, q, r, s. Notice that e_p is no more than $\|p-q\|$. We also have $R_\tau \geq \|p-q\|/2$. Thus, we have

$$e_v \geq (1-\delta)R_\tau \geq \frac{1-\delta}{2} \cdot e_p.$$

The final subcase is that the parent element $f(\tau)$ is an encroached boundary triangle or segment. We first

[1] It can not guarantee that $e_p = \|p-q\|$, because the shortest edge connected to p after p is introduced could be in other tetrahedron incident on p.

consider the scenario that $f(\tau)$ is encroached by a sliver directly or indirectly. We then know that the insertion of the parent vertex p of $f(\tau)$ will always avoid creating small slivers. Thus we have $R_\tau \geq bR_{f(\tau)}$, where $R_{f(\tau)}$ is the circumradius of element $f(\tau)$. The length of the shortest edge connected to p is $e_p \leq (1+\delta)R_{f(\tau)}$. Thus, we have

$$e_v \geq (1-\delta)bR_{f(\tau)} \geq \frac{1-\delta}{1+\delta} b \cdot e_p.$$

We then consider the scenario that $f(\tau)$ is encroached by a tetrahedron with large radius-edge ratio directly or indirectly. Here parent p is selected from the picking region of $f(\tau)$. Assume that tetrahedron τ has four vertices p, q, r, s. Notice that e_p is no more than $\|p-q\|$. We also have $R_\tau \geq \|p-q\|/2$. Thus, we have

$$e_v \geq (1-\delta)R_\tau \geq \frac{1-\delta}{2} \cdot e_p.$$

Finally, we study the situation that τ is a boundary triangle or boundary segment. Here p is always an encroaching circumcenter of a bad tetrahedron or a boundary triangle μ. Notice that here p was considered for insertion but was rejected due to encroaching. It is simple to show that $R_\mu \leq \sqrt{2}R_\tau$, where R_μ is the circumradius of μ. Recall that we split boundary triangle τ only if it contains the projection of the encroaching circumcenter inside. The length e_p of the shortest edge connected to point p (if it was inserted) is at most $(1+\delta)R_\mu$. Thus we have

$$e_v \geq (1-\delta)R_\mu \geq \frac{1-\delta}{\sqrt{2}(1+\delta)} \cdot e_p.$$

□

The previous lemma 5.1 is concerned about the relationship between the insertion edge length of a child and its parent, if there is any. For a vertex v, as [14], we define $D_v = \frac{lfs(v)}{e_v}$. We call D_v the *density ratio* at point v. Clearly, initially D_v is at most one for an input vertex v, and after inserting new vertices, D_v tends to become larger. Notice that D_v is defined just immediately after v is introduced to the mesh; it is not defined based on the final mesh. The next lemma will discuss the relationship between D_v and D_p of parent vertex $p = p(v)$.

LEMMA 5.2. [DENSITY RATIO RELATIONS] *Let v be a vertex with parent $p = p(v)$ if there is any. Assume that $e_v \geq C \cdot e_p$. Then $D_v \leq \frac{1+\delta}{1-\delta} + \frac{D_p}{C}$.*

PROOF. If v is inserted inside the picking region of a bad tetrahedron τ, p is then on the circumsphere of τ. Thus $e_v \geq (1-\delta)R_\tau$, and $\|v-p\| \leq (1+\delta)R_\tau$, where R_τ is the circumradius of τ. If v is inserted inside the picking region of an encroached boundary triangle or segment μ, then p is inside the circumsphere of μ. Thus $e_v > (1-\delta)R_\mu$, and $\|v-p\| < (1+\delta)R_\mu$, where R_μ is the circumradius of μ. In both cases, we have $\|v-p\| \leq \frac{1+\delta}{1-\delta}e_v$.

From the 1-Lipschitz condition of the local feature size function $lfs()$, we have

$$\begin{aligned} lfs(v) &\leq lfs(p) + \|v-p\| \\ &\leq D_p \cdot e_p + \frac{1+\delta}{1-\delta}e_v \\ &\leq \frac{D_p}{C}e_v + \frac{1+\delta}{1-\delta}e_v. \end{aligned}$$

The lemma follows by dividing both side by $e_v > 0$. □

THEOREM 5.1. [BOUNDED DENSITY] *There are fixed constants $D_1 \geq 1$, $D_2 \geq 1$ and $D_3 \geq 1$ such that for any vertex v inserted or rejected at the picking region of a bad tetrahedron, $D_v \leq D_3$; for any vertex v inserted or rejected at the picking region of an encroached boundary triangle, $D_v \leq D_2$; for any vertex v inserted or rejected at the picking region of an encroached boundary segment, $D_v \leq D_1$. Hence, there is a constant $D = \max\{D_1, D_2, D_3\}$ such that $D_v \leq D$ for all mesh vertex v.*

PROOF. We prove the theorem by induction. First consider any original input vertex p, the length e_p of the shortest edge connected to p is at least $lfs(p)$ from the definition of $lfs(p)$. Thus $D_p = \frac{lfs(p)}{e_p} \leq 1$. Then assume that the lemma is true for the parent vertex p of vertex v. Hereafter, let $\alpha = \frac{1+\delta}{1-\delta}$.

If v is selected from the picking region of a tetrahedron τ with radius-edge ratio at least ϱ_0, then $e_v \geq (1-\delta)\varrho_0 \cdot e_p$. Therefore, by above Lemma 5.1, we have

$$D_v \leq \frac{1+\delta}{1-\delta} + \frac{D_p}{(1-\delta)\varrho_0} \leq \alpha + \frac{\alpha D_p}{\varrho_0}.$$

Notice that here point p could be an original input vertex, or a Steiner vertex. In other words, we have $D_p \leq \max\{D_1, D_2, D_3\}$. Thus a sufficient condition that one can prove that $D_v \leq D_3$ is

$$\alpha + \frac{\max\{D_1, D_2, D_3\}}{(1-\delta)\varrho_0} \leq D_3 \tag{5.1}$$

If v is selected from the picking region of an original sliver τ, then $e_v \geq \frac{1-\delta}{\sqrt{3}} \cdot e_p$. Here p is an original input vertex, i.e., $D_p \leq 1$. Apply Lemma 5.1 to v. Thus a sufficient condition that one can prove that $D_v \leq D_3$ is

$$\alpha + \frac{\sqrt{3}}{1-\delta} \leq D_3 \tag{5.2}$$

Then consider that v is selected from the picking region of a created sliver τ. There are three cases about the parent element $f(\tau)$ of τ: $f(\tau)$ is a sliver; $f(\tau)$ has large radius-edge ratio; $f(\tau)$ is an encroached boundary triangle or segment.

If parent element $f(\tau)$ is a sliver, then $e_v \geq \frac{1-\delta}{1+\delta} b \cdot e_p$. Here, we have $D_p \leq D_3$. Thus a sufficient condition that one can prove that $D_v \leq D_3$ is

$$\alpha + \alpha \frac{D_3}{b} \leq D_3 \tag{5.3}$$

If parent element $f(\tau)$ is a tetrahedron with large radius-edge ratio, then $e_v \geq \frac{1-\delta}{2} \cdot e_p$. The fact that p is inserted from $f(\tau)$ with large radius-edge ratio implies that $e_p \geq (1-\delta)\varrho_0 \cdot e_{p(p)}$, where $p(p)$ is the parent element of vertex p. Apply Lemma 5.2 to vertices v and $p(v)$. Notice that $D_{p(p(v))} \leq \max\{D_1, D_2, D_3\}$. Thus a sufficient condition to prove $D_v \leq D_3$ is

$$\alpha + 2\alpha^2 + 2\alpha^2 \frac{\max\{D_1, D_2, D_3\}}{\varrho_0} \leq D_3 \tag{5.4}$$

We then study that the parent element $f(\tau)$ is an encroached boundary triangle or segment. We first consider the scenario that $f(\tau)$ is encroached by a sliver directly or indirectly. Thus we have $e_v \geq \frac{1-\delta}{1+\delta} b \cdot e_p$. Notice that parent vertex p is from a boundary face or segment. Then a sufficient condition to prove $D_v \leq D_3$ is

$$\alpha + \alpha \frac{\max\{D_1, D_2\}}{b} \leq D_3 \tag{5.5}$$

We then consider the scenario that $f(\tau)$ is encroached by a tetrahedron μ with large radius-edge ratio directly or indirectly. Thus we have $e_v \geq \frac{1-\delta}{2} \cdot e_p$. If $f(\tau)$ is encroached directly by μ, then $e_p \geq \frac{1-\delta}{\sqrt{2}(1+\delta)} \cdot e_{c_\mu}$, where c_μ is the circumcenter of tetrahedron μ, i.e., the parent vertex of p. If $f(\tau)$ is encroached by the circumcenter $p(p)$ of a triangle that is encroached by μ, then we have $e_{p(p)} \geq \frac{1-\delta}{\sqrt{2}(1+\delta)} \cdot e_{c_\mu}$. Because μ has large radius-edge ratio, we have $e_{c_\mu} \geq (1-\delta)\varrho_0 e_{p(c_\mu)}$. Similarly, we apply Lemma 5.2 to vertex v, p, $p(p)$ and c_μ. Notice that parent vertex $p(c_\mu)$ of c_μ could be from interior or on boundary. Then a sufficient condition to prove $D_v \leq D_3$ is

$$\alpha + 2\alpha^2 + 2\sqrt{2}\alpha^3 + 4\alpha^4 + 4\alpha^4 \frac{\max\{D_1, D_2, D_3\}}{\varrho_0} \leq D_3 \tag{5.6}$$

If v is selected from the picking region of a boundary triangle μ then $e_v \geq \frac{1-\delta}{\sqrt{2}(1+\delta)} \cdot e_p$. Here parent p could be the circumcenter of a tetrahedron with large radius-edge ratio or a sliver. In other words, $D_p \leq D_3$. Thus a sufficient condition that one can prove that $D_v \leq D_2$ is

$$\alpha + \sqrt{2}\alpha D_3 \leq D_2 \tag{5.7}$$

If v is selected from the picking region of a boundary segment μ, then $e_v \geq \frac{1-\delta}{\sqrt{2}(1+\delta)} \cdot e_p$. Here parent p could be the circumcenter of a tetrahedron or a boundary triangle. In other words, $D_p \leq \max\{D_2, D_3\}$. Thus a sufficient condition that one can prove that $D_v \leq D_1$ is

$$\alpha + \sqrt{2}\alpha \max\{D_2, D_3\} \leq D_1 \tag{5.8}$$

Notice that some inequalities are satisfied if other inequalities were satisfied. One can show that above inequalities are simultaneously satisfied if we choose

$$\begin{aligned}
D_3 &\geq \frac{\varrho_0(\alpha + 2\alpha^2 + 2\sqrt{2}\alpha^3 + 4\alpha^4) + 4\alpha^5 + 4\sqrt{2}\alpha^6}{\varrho_0 - 8\alpha^6} \\
D_3 &\geq \frac{\alpha b + \alpha^2 + \sqrt{2}\alpha^3}{b - 2\alpha^3}, \\
D_3 &\geq \alpha + \sqrt{3}\alpha, \\
D_2 &= \alpha + \sqrt{2}\alpha D_3, \\
D_1 &= \alpha + \sqrt{2}\alpha^2 + 2\alpha^2 D_3.
\end{aligned}$$

□

Thus, to guarantee a good grading on the final mesh generated, we need that $\varrho_0 > 8\alpha^6$ and $b > 2\alpha^3$. The following theorem concludes that the generated mesh has good grading, i.e., for any mesh vertex v, $N(v)$ is at least some constant factor of *lfs*(v). Ruppert and Shewchuk had similar theorems for classic Delaunay refinement methods; see [13, 14].

THEOREM 5.2. [GOOD GRADING] *For any mesh vertex v generated by refinement algorithm, the distance connected to its nearest neighbor vertex u is at least $\frac{lfs(v)}{D+1}$.*

The proof is omitted here. Ruppert showed that the nearest neighbor value $N(v)$ of a mesh vertex v of any almost-good mesh is at most a constant factor of *lfs*(v), where the constant depends on the radius-edge ratio. The above Theorem 5.2 shows that the nearest neighbor $N(v)$ for the sliver-free Delaunay mesh is at least some constant factor of *lfs*(v). Then we have the following theorem.

THEOREM 5.3. [LINEAR SIZE] *The size of the generated sliver-free Delaunay mesh is within a small constant factor of any almost-good mesh for the same domain, where the constant depends on the radius-edge ratio of the meshes.*

This theorem also implies that given an almost-good mesh with n vertices, our refinement algorithm will remove the slivers by introducing at most $O(n)$ new mesh vertices. Thus the time complexity of the algorithm is $O(n \log n)$ given an almost-good mesh with n vertices.

6 Discussions

In this paper, we present a refinement based method that guarantees to remove all slivers in the mesh. In other words, any tetrahedron generated in the mesh has radius-edge ratio no more than ϱ_0 and the volume is at least σ_0 times the cube of its shortest edge length.

Notice that the σ_0 derived from all the proofs may be too small for any practical use (even it is better than previous results [2, 5]). We would like to conduct some experiments to see what σ_0 can guarantee that there is no small slivers created. Recall that the termination guarantee does not depend on the definition of sliver. Only the existence of point p, which will not introduce small slivers, in the picking region depends on the sliver definition. Based on this observation, we can have a variation of this algorithm. We remove a tetrahedron τ with small value $\sigma(\tau)$ only if we find a point p in the picking region of τ such that the new tetrahedra with circumradius less than bR_τ is better (with larger V/L^3 value than $\sigma(\tau)$). In other words, there is a point p inside the picking region to improve the local mesh quality. Moreover, we could have different definitions about slivers depending upon the location of the tetrahedron: inside or near the domain boundary. Then using the same proofs as before, we can prove that it will generate a sliver-free three-dimensional mesh with termination and quality guarantees.

References

[1] Cavendish, J. C., Field, D. A., and Frey, W. H. An approach to automatic three-dimensional finite element mesh generation. *Internat. J. Numer. Methods Engrg. 21* (1985).

[2] Cheng, S. W., Dey, T. K., Edelsbrunner, H., Facello, M. A., and Teng, S. H. Silver exudation. In *Proc. 15th ACM Symposium on Computational Geometry* (1999), pp. 1–13.

[3] Chew, L. P. Constrained Delaunay triangulations. *Algorithmica 4* (1989), 97–108.

[4] Chew, L. P. Guaranteed-quality delaunay meshing in 3d (short version). In *13th ACM Sym. on Comp. Geometry* (1997), pp. 391–393.

[5] Edelsbrunner, H., Li, X. Y., Miller, G., Stathopoulos, A., Talmor, D., Teng, S. H., ÜngÖr", A., and Walkington, N. Smoothing and cleaning up slivers. In *ACM Symposium on Theory of Computing (STOC00)* (2000).

[6] Edelsbrunner, H., and Shah, N. R. Incremental topological flipping works for regular triangulations. *Algorithmica 15* (1996).

[7] Li, X. Y. Functional delaunay refinement. In *7th International Conference on Numerical Grid Generation in Computational Field Simulations* (2000).

[8] Li, X. Y. *Sliver-free Three Dimensional Delaunay Mesh Generation.* PhD thesis, University of Illinois at Urbana-Champaign, 2000.

[9] Li, X. Y., Teng, S. H., and Üngör, A. Biting: advancing front meets sphere packing. *Int. Jour. for Numerical Methods in Eng* (2000).

[10] Miller, G. L., Talmor, D., Teng, S. H., and Walkington, N. A delaunay based numerical method for three dimensions: generation, formulation, and partition. In *Proc. 27^{th} Annu. ACM Sympos. Theory Comput.* (1995), pp. 683–692.

[11] Miller, G. L., Talmor, D., Teng, S. H., Walkington, N., and Wang, H. Control volume meshes using sphere packing: generation, refinement, and coarsening. In *5th International Meshing Roundtable* (1996), Sandia National Laboratories, pp. 47–61.

[12] Mitchell, S. A., and Vavasis, S. A. Quality mesh generation in three dimensions. In *ACM Symposium on Computational Geometry* (1992), pp. 212–221.

[13] Ruppert, J. A new and simple algorithm for quality 2-dimensional mesh generation. In *Third Annual ACM-SIAM Symposium on Discrete Algorithms* (1992), pp. 83–92.

[14] Shewchuk, J. R. Tetrahedral mesh generation by delaunay refinement. In *14th Annual ACM Symposium on Computational Geometry* (1998), pp. 86–95.

[15] Strang, G., and Fix, G. J. *An Analysis of the Finite Element Method.* Prentice-Hall, 1973.

[16] Talmor, D. *Well-Spaced Points for Numerical Methods.* PhD thesis, Carnegie Mellon University, 1997.