Gauge Theories with Spontaneous Symmetry Breaking

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The Higgs Mechanism

Realization of Symmetries we met:

- Global symmetry $\rightarrow$ conserved current.
- SSB of Global Symmetry: interactions are constraint, coupling constants related; for continuous symm, current is still Conserved, massless Goldstone.
- Gauge symmetry: local symmetry, require massless vector bosons, interactions are highly restricted.
- What happens if we combine SSB and local gauge symmetry? — gauge boson acquire masses, and we can choose a gauge s.t. the goldstone degree of freedom becomes the third d.o.f of the massvie gauge boson.
**The Higgs Mechanism**

An Abelian example:

\[ \mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi), \quad D_\mu = \partial_\mu + ieA_\mu. \]

Gauge symmetry: \( \phi(x) \to e^{i\alpha} \phi(x), \quad A_\mu(x) \to A_\mu - \frac{1}{e} \partial_\mu \alpha(x). \)

Choose Potential: \( V(\phi) = -\mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2, \quad \mu^2 > 0 \)

- Minimum of the potential: \( |\langle \phi \rangle| = \phi_0 = \frac{\mu}{\sqrt{\lambda}}. \) We choose the real vev: \( \langle \phi \rangle = \phi_0 = \frac{\mu}{\sqrt{\lambda}}. \)
- Expand about \( \phi_0, \phi = \phi_0 + \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)), \) potential, quadratic terms: \( V(\phi) \sim \frac{1}{2} (2\mu^2) \phi_1^2 + O(\phi_3^2), \) \( m_1 = \sqrt{2}\mu, \) \( m_2 = 0. \) \( \phi_2 \) goldstone.
- Kinetic terms:

\[
|D_\mu \phi|^2 = \partial^\mu \phi^* \partial_\mu \phi - ieA_\mu ( - \partial^\mu \phi^* \phi + \phi^* \partial^\mu \phi ) + e^2 \phi^* \phi A_\mu A^\mu = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 + \sqrt{2} e \phi_0 A_\mu \partial^\mu \phi_2 + e^2 \phi_0^2 A_\mu A^\mu + \ldots
\]

- \( A_\mu \) acquire mass: \( m_A^2 = 2e^2 \phi_0^2, \) \( \frac{1}{2} m_A^2 A_\mu A_\mu = \frac{1}{2} m_A^2 (A^0 A^0 - A^i A^i), \)

physical space-like \( A^i \) have the correct sign of mass.
**The Higgs Mechanism**

- There is an interaction term: $\mathcal{L} \sim \sqrt{2}e\phi_0 A_\mu \partial^\mu \phi_2$, mixing term between $A_\mu$ gauge boson and $\phi_2$ goldstone. No $\sqrt{2}e\phi_0 A_\mu \partial^\mu \phi_1$ term, cancelled.

- Recall in QED, photon does not acquire mass, photon selfenergy $1\Pi \sim (k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2)$, $\Pi(k^2)$ does not have $1/k^2$ pole. photon propagator: $\sim -i\frac{(g^{\mu\nu} - k^\mu k^\nu / k^2)}{k^2 (1 - \Pi(k^2))} - i\frac{\xi k^\mu k^\nu}{k^4}$. If $\Pi(k^2)$ has a $1/k^2$ pole, photon acquire mass.

- Goldstone provides the right pole to give mass to gauge boson and make the vacuum polarization amplitude properly transverse.
  
  $i\mathcal{L} \sim \sqrt{2}e\phi_0 A_\mu (-ik^\mu \phi_2(k)) = m_A k^\mu \phi_2(k)$
  
  Also treat mass term as perturbation:

  
  \[
  \begin{align*}
  &\quad = i\sqrt{2}e\phi_0 (-ik^\mu) = m_A k^\mu.
  \end{align*}
  \]

Also treat mass term as perturbation:

\[
\begin{align*}
&\quad = i m_A^2 g^{\mu\nu} + (m_A k^\mu) \frac{i}{k^2} (-m_A k^\nu) \\
&\quad = i m_A^2 \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right).
\end{align*}
\]
The Higgs Mechanism

- Massless vector: two physical transverse polarizations.
- Massive vector boson, three physical d.o.f.: \( \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0, \partial_\mu A^\mu \sim \partial_\mu \partial_\nu F^{\mu\nu} = 0. \) \( k \cdot \epsilon = 0. \) \( k^2 = m^2. \) Three space-like polarization vectors. \( k^\mu = (E, 0, 0, k), \epsilon^{T(1)} = (0, 1, 0, 0), \) \( \epsilon^{T(2)} = (0, 0, 1, 0), \epsilon^L = \frac{1}{m}(k, 0, 0, E), \) longitudinal polarization. Roughly speaking, Goldstone provides the third d.o.f of the massive gauge boson.
- Unitary gauge: the Goldstone does not appear as an independent physical particle. Decompose \( \phi(x) = e^{-i\alpha(x)} \phi'(x), \) \( \phi' \) a real field. \( \phi'(x) = \frac{1}{\sqrt{2}} \sigma(x) + \phi_0. \) \( \phi_0 = \frac{\mu}{\sqrt{\lambda}} \) is the v.e.v. \( \alpha(x) \) can be gauged away. \( A_\mu = A'_\mu + \frac{i}{e} \partial_\mu \alpha. \)

\[
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)
\]

rename \( \phi' \rightarrow \phi \)

\[
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (\partial_\mu - i e A_\mu)\phi(\partial^\mu + i e A^\mu)\phi - V(\phi)
\]

\[
= -\frac{1}{4}(F_{\mu\nu})^2 + (\partial_\mu \phi)^2 + e^2 \phi^2 A_\mu A^\mu - V(\phi)
\]

\[
= -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu \sigma)^2 + e^2 \left( \frac{1}{\sqrt{2}} \sigma + \phi_0 \right)^2 A_\mu A^\mu - V \left( \frac{1}{\sqrt{2}} \sigma + \phi_0 \right)
\]

\[
= \cdots + e^2 \phi_0^2 A_\mu A^\mu + \frac{1}{2} e^2 \sigma^2 A_\mu A^\mu + \sqrt{2} e^2 \phi_0 \sigma A_\mu A^\mu - V \left( \frac{1}{\sqrt{2}} \sigma + \phi_0 \right)
\]

\[
m_A^2 = 2 e^2 \phi_0^2 = 2 e^2 \frac{\mu^2}{\lambda}, \quad m_\sigma^2 = 2 \mu^2. \) No goldstone modes in the Lagrangian.
SSB with Non-Abelian gauge symmetry: $\phi_i$ real. (complex
$\phi \to \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$).

Symmetric transformation: $\phi \to (1 + i\alpha^a t^a)_{ij} \phi_j = (1 - \alpha^a T^a)_{ij} \phi_j$, $t^a$: pure imaginary, hermitian. $T^a = -it^a$ real anti-symmetric.

Covariant derivative $D_\mu \phi = (\partial_\mu - igA_\mu t^a) \phi = (\partial_\mu + gA_\mu T^a) \phi$

Kinetic term:

$$\frac{1}{2} \sum_i D_\mu \phi_i D^\mu \phi_i = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + gA_\mu \partial^\mu \phi_i T^a_{ij} \phi_j + \frac{1}{2} g^2 A_\mu A^{b\mu} (T^a \phi)_i (T^b \phi)_i$$

SSB: $\phi_i = (\phi_0)_i + \tilde{\phi}_i (x)$, $\langle \Omega | \phi_i | \Omega \rangle = (\phi_0)_i$

- mass matrix: $(m^2)_{ab} = g^2 \sum_i (T^a \phi_0)_i (T^b \phi_0)_i$.
  - $(m^2)_{ab}$ positive semi-definite: $v^a m^2_{ab} v_b = g^2 \sum_i [v^a (T^a \phi_0)_i]^2 \geq 0$.
  - Positive eigenvalues: massive gauge bosons.
  - Zero-eigenvalues: massless gauge bosons.
- Unbroken generators: $T^a_{ij} \phi_0, j = 0$, $(m^2)_{ab} = 0$, for $\forall b$.
  - Broken generators: $T^a_{ij} \phi_0, j \neq 0$, also $T^b \Rightarrow (m^2)_{ab} \neq 0$.
- Linearly independent broken generators, $(m^2)_{ab}$ nondegenerate, nonvanishing eigenvalues, massive gauge boson.
  - Broken generators $\leftrightarrow$ massive gauge bosons, also correspond to goldstones.
- Unbroken generators: massless gauge bosons.
The Higgs Mechanism

- Interaction vertex: \( gA_\mu^a \partial^\mu \phi_i (T^a_{ij} \phi_0;j) \);
  \((T^a_{ij} \phi_0;j) \neq 0\) broken generator \(T^a\), \((T^a_{ij} \phi_0;j)\) Goldstone excitation direction, \(\partial_\mu \phi_i\) is projected onto the goldstone direction.

- Gauge boson propagator receives a contribution from the Goldstone boson.

\[
\begin{align*}
\mu_a & \quad \bullet \quad \bullet \quad \nu & \quad b \quad = \sum_j (g k^\mu (T^a_{0j})_j) \frac{i}{k^2} (-g k^\nu (T^b_{0j})_j) \\
\end{align*}
\]

1PI+ goldstone:

\[
\text{Propagator} = -\frac{i}{k^2} \delta_{ab} g_{\mu \nu} + \left( -\frac{i}{k^2} \delta_{aa'} g_{\mu \mu'} \right) (g_{\mu' \nu'} - \frac{k_{\mu'} k_{\nu'}}{k^2}) i m_{a'b'}^2 \left( -\frac{i}{k^2} \delta_{b' \nu} g_{\nu' \nu} \right) + \ldots
\]

\[
= -i (g_{\mu \nu} - \frac{k_{\mu} k_{\nu}}{k^2}) [k^2 - m^2]^{-1} - \frac{i}{k^2} \frac{k_{\mu} k_{\nu}}{k^2} \delta_{ab}
\]
A Non-Abelian example: \( \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \), \( \phi_{(1,2)} \) doublet complex fields in SU(2) fundamental rep..

Covariant derivative: \( D_\mu \phi = (\partial_\mu - igA^a_\mu \tau^a) \phi \). \( \tau^a = \frac{\sigma^a}{2} \).

SSB: \( \langle \phi \rangle \neq 0 \), we can always choose unitary transformation s.t.

\[
\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} = \phi_0. \text{ Expand } \phi = \phi_0 + \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^1 + i\pi^2}{\pi^3 + i\pi^4} \end{pmatrix}
\]

- Gauge boson mass from:
  \[
  |D_\mu \phi|^2 \sim g^2 \phi_0^\dagger A_\mu A^\mu \phi_0 = \frac{1}{2} g^2 \begin{pmatrix} 0 & v \end{pmatrix} \tau^a \tau^b \begin{pmatrix} 0 \\ v \end{pmatrix} A^a_\mu A^{b,\mu} = \frac{g^2 v^2}{8} A^a_\mu A^{a,\mu}
  \]

- \( A^{1,2,3} \) all are massive. \( m_A^2 = \frac{1}{2} gv \). All the SU(2) symmetry are broken:

\[
 i\tau^1 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} iv \\ 0 \end{pmatrix} \sim \pi^2, \quad i\tau^2 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v \\ 0 \end{pmatrix} \sim \pi^1, \quad i\tau^3 \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ -iv \end{pmatrix} \sim \pi^4
\]

Goldstone: \( \pi^2, \pi^1, \pi^4 \).
**The Higgs Mechanism**

SU(2) adjoint rep.: $\phi_a$, real fields, $a = 1, \ldots 3$.

Covariant derivative: $D_\mu \phi_a = \partial_\mu \phi_a + g \epsilon_{abc} A_\mu^b \phi_c$.

\[
\mathcal{L} = \frac{1}{2} D_\mu \phi^a D^\mu \phi^a + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} (\phi^2)^2
\]

vev: $\phi^2_0 = \sum_{i=1}^3 (\phi^i_0)^2 = \frac{\mu^2}{\lambda} = v^2$, 2-sphere, choose $\phi_0 = (0, 0, v)$.

\[
T^1 = i t^1 = \epsilon^{a1b} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
T^2 = i t^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
T^3 = i t^3 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}
\]

\[
T^1 \phi_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = - \begin{pmatrix} 0 \\ v \end{pmatrix},
T^2 \phi_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix},
T^3 \phi_0 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0
\]

- $T^1, T^2$ generators are broken generators. $T^3$ is not broken. We would expect the $A^1_\mu, A^2_\mu$ are massive, and $A^3_\mu$ is massless.

- mass terms from kinetic energy part: (using $\epsilon_{abc} \epsilon_{ab'c'} = \delta_{bb'} \delta_{cc'} - \delta_{bc'} \delta_{cb'}$)

\[
\frac{1}{2} D_\mu \phi^a D^\mu \phi^a \sim \frac{1}{2} g^2 \sum_a (\epsilon_{abc} A_\mu^b \phi^c_0) (\epsilon_{ab'c'} A^{\mu, b'} \phi^c_0)
\]

\[
= \frac{1}{2} g^2 (A^a_\mu A^a_{\mu} \phi^2_0 - A^b_\mu A^c_{\mu} \phi^b_0 \phi^c_0) = \frac{1}{2} g^2 v^2 (\langle A_1^1 \rangle^2 + \langle A_\mu^2 \rangle^2)
\]

Only $A^1_\mu, A^2_\mu$ acquire masses: $m_1^2 = m_2^2 = gv$. $A^3$ remains massless.
The Higgs Mechanism

Unitarity gauge: \( \phi = e^{i(t^1 \varphi_1 + t^2 \varphi_2)} \varphi \rightarrow \varphi, \quad \varphi = \begin{pmatrix} 0 \\ 0 \\ v + \sigma \end{pmatrix} \).

Kinetic part:

\[
\frac{1}{2} (D_\mu \phi)^2 \rightarrow \frac{1}{2} (D_\mu \varphi)^2 = \frac{1}{2} \partial^\mu \sigma \partial_\mu \sigma + \frac{1}{2} g^2 ((A^1_\mu)^2 + (A^2_\mu)^2) (v + \sigma)^2.
\]

\[
V[\phi] \rightarrow V[\varphi] = -\frac{1}{2} \mu^2 (\sigma + v)^2 + \frac{\lambda}{4} (v + \sigma)^4
\]

Two of the gauge bosons acquire the same mass and one remain massless.
Another example: SU(3) gauge theory: $\phi^a$ in adjoint rep., $a = 1, \ldots, 8$
t, $3 \times 3$ matrix in fundamental rep. $tr(t^a t^b) = \frac{1}{2} \delta^{ab}$,
Define $\Phi = \phi^a t^a$, $A_\mu = A_\mu^a t^a$.
$\Phi$ transforms as: $\Phi \rightarrow e^{i \alpha^a t^a} \Phi e^{-i \alpha^a t^a} = \Phi + i \alpha^a [t^a, \Phi] + \ldots$,
- Infinitesimal: $\delta \Phi = i \alpha^a [t^a, \Phi]$.
  If $i [t^a, \Phi] = 0$, $t^a$ unbroken generators. If $i [t^a, \Phi] \neq 0$, $t^a$ broken generators.
- Covariant derivative: $(D_\mu \phi)^a = \partial_\mu \phi^a + g f^{abc} A_\mu^b \phi^c$,
in $\Phi$, $D_\mu \Phi = \partial_\mu \Phi - ig [A_\mu, \Phi]$
- $A_\mu$ mass term from kinetic part: $\frac{1}{2} [(D_\mu \phi)^a]^2 = tr [(D_\mu \Phi)^2] \sim$
$- g^2 tr ([A_\mu, \Phi] [A_\mu, \Phi]) = - g^2 tr ([t^a, \Phi] [t^b, \Phi]) A_\mu^a A_\mu^b$.$m_{ab} = 2 g^2 tr ([t^a, \Phi_0] [t^b, \Phi_0])$.
- Vev of $\Phi$, $\langle \Phi \rangle = \Phi_0$, hermitian and traceless, can be diagonalized by
unitary transformaion, eigenvalues are real.
- Different vev can have different breaking pattern. Diagonal traceless real
matrix: 2 independent, $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & \end{pmatrix}$.
SU(3) generators:
$t^a = \begin{pmatrix} \tau^a & 0 \\ 0 & 0 \end{pmatrix}$, for $a = 1, 2, 3$;
$t^8 = \frac{1}{2 \sqrt{3}} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$
$t^4 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $t^5 = \frac{1}{2} \begin{pmatrix} 1 & \end{pmatrix}$; $t^6 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $t^7 = \frac{1}{2} \begin{pmatrix} 0 & \end{pmatrix}$
The Higgs Mechanism

\[ \Phi_0 = |\phi_0| \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = 2\sqrt{3}|\phi_0|t^8 \]

- \( \Phi_0 \) commutes with \( t^1, t^2, t^3, t^8; t^{1,2,3,8} \), unbroken generators, \( SU(2) \times U(1) \) unbroken. \( t^{4,5,6,7} \) broken generators.
- \( A^{4,5,6,7} \) massive gauge bosons. Mass matrix \( m_{ab}^2 = -2g^2 tr([t^a, \Phi_0][t^b, \Phi_0]) \)

\[ [t^4, \Phi] = 2\sqrt{3}[t^4, t^8]|\phi_0| = -3it^5|\phi_0|, \quad [t^5, \Phi] = 2\sqrt{3}[t^5, t^8]|\phi_0| = 3it^4|\phi_0|, \]

\[ [t^6, \Phi] = 2\sqrt{3}[t^6, t^8] = -3it^7|\phi_0|, \quad [t^7, \Phi] = 2\sqrt{3}[t^7, t^8] = 3it^6|\phi_0|, \]

mass term:

\[ m_{ab}^2 = -2g^2 tr([t^a, \Phi_0][t^b, \Phi_0]) A^a_\mu A^{\mu, b} = 9g^2|\phi_0|^2 \sum_{a=4,5,6,7} A^a_\mu A^{\mu, a}, \]

- \( m_{4,5,6,7}^2 = 9g^2|\phi_0|^2 = (3g|\phi_0|)^2 \cdot m_{1,2,3,8}^2 = 0. \)
Another: $\Phi_0 = |\phi_0| \left( \begin{array}{cc} 1 & -1 \\ 0 & 0 \end{array} \right) = 2|\phi_0|t^3$.

Only $t^3, t^8$ commute with $\Phi_0$, unbroken, $U(1) \times U(1)$ unbroken. $t^{1,2,4,5,6,7}$ are broken.

$m_{ab}^2 = -2g^2 \text{tr}([t^a, \Phi_0][t^b, \Phi_0])$

$$[t^1, \Phi] = 2[t^1, t^3]|\phi_0| = -2it^2|\phi_0|, \quad [t^2, \Phi] = 2[t^2, t^3]|\phi_0| = 2it^1|\phi_0|,$$
$$[t^4, \Phi] = 2[t^4, t^3]|\phi_0| = -it^5|\phi_0| \quad [t^5, \Phi] = 2[t^5, t^3]|\phi_0| = it^4|\phi_0|,$$
$$[t^6, \Phi] = 2[t^6, t^3]|\phi_0| = it^7|\phi_0|, \quad [t^7, \Phi] = 2[t^7, t^3]|\phi_0| = it^6|\phi_0|$$

mass term:

$$m_{ab}^2 A_\mu^a A^{b,\mu} = \sum_{a=4,5,6,7} g^2 |\phi_0|^2 A_\mu^a A^{a,\mu} + \sum_{a=1,2} 4g^2 |\phi_0|^2 A_\mu^a A^{a,\mu}$$

$m_{1,2}^2 = 4g^2|\phi_0|^2, \quad m_{4,5,6,7}^2 = g^2|\phi_0|^2, \quad m_{3,8}^2 = 0$.

Gauge bosons acquire masses: gauge bosons corresponding to broken generators split to different masses.
Formal description of the Higgs Mechanism:

- Global symmetry of $L_0 \rightarrow$ conserved current $J^\mu$.
- If we require the global symmetry parameter to depend on $x$, continuous and differentiable and nonvanishing in a small region: $\delta L = (\partial_\mu \alpha^a) J^{\mu,a}$.

For on-shell fields $\int d^4 x \delta L \big|_{\text{on shell fields}} = 0$,
\[
\int d^4 x \partial_\mu \alpha J^{\mu,a} = -\int \alpha \partial_\mu J^{\mu,a} = 0 \Rightarrow \partial_\mu J^{\mu,a} = 0. \quad J^{\mu,a} \text{ is the conserved Noether current.}
\]
- If we require the local invariance: introducing $A_\mu$

\[
L = L_0 - g A^a_\mu J^{\mu,a} + O(A^2) ; \quad A^a_\mu \rightarrow A^a_\mu + \frac{1}{g} \partial_\mu \alpha + O(A)
\]

The gauge field is always coupled to the global current at $O(g)$.

- Global SSB: for broken generators, $J^{\mu,a}$ destroy or create the Goldstone $\pi_k$, $\langle \Omega | J^{\mu,a} | \pi_k(p) \rangle = -ip^\mu F^a_k e^{-ip \cdot x}$, in general $F^a_k$ depends only on Lorentz invariant $p^2$ of on-shell $\pi_k$. $F^a_k$ constants. $F^a_k \neq 0$ for broken generators $t^a$, and $F^a_k = 0$ for unbroken generators.
- $J^{\mu,a}$ conserved: $0 = \partial_\mu \langle \Omega | J^{\mu,a} | \pi_k(p) \rangle = -p^2 F^a_k e^{-ip \cdot x}$, so $p^2 = 0$.
- example: For weakly coupled scalar theory, Noether current

\[
J^{\mu,a} = \partial_\mu \phi_i T^a_{ij} \phi_j, \quad \langle \Omega | \partial_\mu \phi_i T^a_{ij} \phi_j | \phi_i(p) \rangle = -ip_\mu (T^a_0)_{ij} e^{-ip \cdot x}, \quad \text{so} \quad F^a_{ij} = T^a_{ij} \phi_0, \text{ nonzero only for broken generator — direction of Goldstone exitation.}
\]
**Formal description of the Higgs Mechanism:**

Local symmetry: $A_\mu$ couples to the Neother current. The corresponding global symmetry is spontaneous broken.

Vertex $-iA_\mu^a J^{\mu,a}$, Amplitude for a gauge boson to convert to a Goldstone:

$$\int d^4x \langle k; a | -iA_\mu^b(x) J^{\mu,b}(x) | \pi_j(p) \rangle = \epsilon^{a*}_\mu (-g k^\mu F^a_j) \delta(4)(k - p)$$

The vacuum polarization diagram of gauge boson: still transverse, (current conservation, ward id.)

$$a \sim b = i \left( g^{\mu \nu} - \frac{k^\mu k^\nu}{k^2} \right) \cdot (m_{ab}^2 + O(k^2))$$

The $1/k^2$ pole term will give the mass to gauge bosons in the propagator—contribution from Goldstone.

$$= (g k^\mu F^a_j) \frac{i}{k^2} (-g k^\nu F^b_j).$$

so $m_{ab}^2 = g^2 F^a_j F^b_j$

This can be used in any SSB theory, not only in broken by a scalar field v.e.v, but also by nonperturbative effects.

Gauge bosons coupled to the SSB currents $\rightarrow$ massive.
The Glashow-Weinberg-Salam Theory of Weak Interactions

Gauge theory with SSB —An experimentally correct unified description of electro-weak Interactions, GWS theory.

(1) Gauge and Scalar sector:

- **Gauge Symmetry**: $SU(2) \times U(1)_Y$. Gauge boson $A^a_\mu$ for $SU(2)$, and $B_\mu$ for $U(1)_Y$.

- **Scalar field**: $\phi = (\phi_1, \phi_2)^T$ in fund. rep of $SU(2)$. Charge $1/2$ under $U(1)$.
  Transformation of $\phi$: $\phi \rightarrow e^{i\alpha^a \tau^a} e^{i\beta/2} \phi$. ($\tau^a = \frac{\sigma^a}{2}$, $\tau^0 = \frac{I}{2}$).

- **If the scalar field get vev**: $\langle \phi \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v \end{array} \right)$:

  Unbroken generator: $\tau^1 + \tau^0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$, Symmetry para: $\alpha^3 = \beta$, $\alpha^{1,2} = 0$.
  Broken generator (can be a little arbitrary): $\tau^1, \tau^2, \tau^3 - \tau^0 = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right)$.
  We expect that there are three massive gauge boson and one massless gauge boson.

- **Define**: $\phi = \left( \begin{array}{c} \varphi_1 + i\varphi_2 \\ v + \varphi_3 + i\varphi_4 \end{array} \right)$,

Goldstone:

- $i\tau^1 \left( \begin{array}{c} 0 \\ v \end{array} \right) = \left( \begin{array}{c} i v \\ 0 \end{array} \right) \sim \varphi_2$,
- $i\tau^2 \left( \begin{array}{c} 0 \\ v \end{array} \right) = \left( \begin{array}{c} v \\ 0 \end{array} \right) \sim \varphi_1$,
- $i(\tau^1 - \tau^0) \left( \begin{array}{c} 0 \\ v \end{array} \right) = \left( \begin{array}{c} 0 \\ iv \end{array} \right) \sim \varphi_4$.
Covariant derivative for $\phi$: $D_\mu \phi = (\partial_\mu - igA^a_\mu \tau^a - ig'B_\mu \tau^0) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

For direct product group, each simple group factor can have a separate gauge coupling constant.

- In general, $G_1 \times G_2$, transformation in matrix rep
  
  $U_1 \otimes U_2 = e^{it^a_\alpha \alpha^a} \otimes e^{iT^b_\beta \beta^b}$, we always use $t^a$ to represent $t^a \otimes I$, $T^b$ represents $I \otimes T^b$, $U_1 \otimes U_2 = e^{it^a_\alpha \alpha^a} \cdot e^{iT^b_\beta \beta^b}$

  
  $-igA^a_\mu t^a - ig'B^b_\mu T^b \rightarrow U_1 U^\dagger - igU_A^a_\mu t^a U^\dagger - ig'UB'^b_\mu T^b U^\dagger$

  
  $= U_1 \partial_\mu U^\dagger_1 + U_2 \partial_\mu U^\dagger_2 - igU_1 A^a_\mu t^a U^\dagger_1 - ig'U_2 B'^b_\mu T^b U^\dagger_2$

  
  $U_1 A^a_\mu t^a U^\dagger_1$ only transforms to combinations of $t^a$, not to mix with $T^a$, and similar for $T^b$. So the two kinds of gauge fields transform separately,

  
  $A^a_\mu t^a \rightarrow \frac{i}{g} U_1 \partial_\mu U^\dagger_1 + U_1 A^a_\mu t^a U^\dagger_1$, $B^a_\mu T^a \rightarrow \frac{i}{g'} U_2 \partial_\mu U^\dagger_2 + U_2 A^a_\mu T^a U^\dagger_2$.

  
  $g$ and $g'$ can be different.
The Glashow-Weinberg-Salam Theory of Weak Interactions

Masses of gauge bosons: from kinematic term \((D_\mu \phi)^\dagger D^\mu \phi\):

\[
\begin{align*}
\frac{1}{\sqrt{2}} (0, v) (igA^a_\mu \tau^a + i\frac{g'}{2} B_\mu) (-igA^{\mu, b} \tau^b - i\frac{g'}{2} B^\mu) = \frac{1}{\sqrt{2}} (0, v)
\end{align*}
\]

\[
\begin{align*}
= & \frac{1}{2} v^2 \left( \frac{g^2}{4} A^a_\mu A^{a\mu} + \frac{g'^2}{4} B_\mu B^\mu - \frac{1}{2} gg' B^\mu A^3_\mu \right) \\
= & \frac{1}{2} \left( \frac{g^2}{4} v^2 (A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}) + \frac{v^2}{4} (gA^3_\mu - g' B^\mu)^2 \right) \\
= & \frac{1}{4} g^2 v^2 W^+ W^- + \frac{1}{2} \left( \frac{1}{4} v^2 (g^2 + g'^2) \right) Z^0_\mu Z^0,\mu
\end{align*}
\]

We have defined

\[
W^\pm = \frac{1}{\sqrt{2}} (A^1_\mu \mp iA^2_\mu), \quad m_W = \frac{1}{2} gv,
\]

\[
Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (gA^3_\mu - g' B^\mu), \quad m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}
\]

We can also define another gauge boson orthogonal to \(Z^0_\mu\), which is massless

\[
A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} (g' A^3_\mu + gB_\mu), \quad m_A = 0
\]

\(A_\mu\) massless gauge boson — electromagnetic vector potential.
Covariant derivative in terms of $W^\pm, Z^0$ and $A_\mu$.

$SU(2)$ Generator $T^a$ in some rep; $Y$: $U(1)$ generator. Define $T^\pm = T^1 \pm iT^2$,

$$D_\mu = \partial_\mu - igA_\mu^a T^a - ig' YB_\mu$$

$$= \partial_\mu - ig \frac{g}{\sqrt{2}}[W^+ T^+ + W^- T^-] - \frac{ig}{\sqrt{g^2 + g'^2}}(g^2 T^3 - g'^2 Y)Z_\mu^0$$

$$- \frac{g'}{\sqrt{g^2 + g'^2}}(T^3 + Y)A_\mu$$

$A_\mu$ massless, correspond to the unbroken generator $T^3 + Y$.

Electric charge $e = \frac{gg'}{\sqrt{g^2 + g'^2}}$, electric charge quantum number $Q = T^3 + Y$.

We will see later for electron: $Q = -1$.

Define $\theta_w$ weak mixing angle, $\sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}$, $\cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}}$.

Then, $e = g \sin \theta_w = g' \cos \theta_w$, $(Z^0_A) = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} (A^3_B)$

$$- \frac{i}{\sqrt{g^2 + g'^2}}(g^2 T^3 - g'^2 Y) = - \frac{i}{\sqrt{g^2 + g'^2}}((g^2 + g'^2) T^3 - g'^2 Q) = -i \frac{g}{\cos \theta_w} (T^3 - \sin^2 \theta_w Q).$$

Covariant derivative:

$$D_\mu = \partial_\mu - ig \frac{g}{\sqrt{2}}[W^+ T^+ + W^- T^-] - \frac{i g}{\cos \theta_w} (T^3 - Q \sin^2 \theta_w) Z_\mu - i e QA_\mu,$$ $g = \frac{e}{\sin \theta_w}$.

Original gauge coupling: $g, g' \rightarrow e, \theta_w; \text{ vev: } v \rightarrow m_W = \frac{1}{2} gv$, $m_W = m_Z \cos \theta_w$.

$e, \theta_w, m_W$ independent paras.
The Glashow-Weinberg-Salam Theory of Weak Interactions

Scalar Lagrangian: \( \mathcal{L}_\phi = |D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \),

In Unitarity gauge: \( \phi = U(x) \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right) \rightarrow \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right) \).

\( h(x) \): Higgs field, chargeless.

For \( \phi \), \( Q = T^3 + Y = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \), \( T^+ = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \), \( T^- = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \)

\( D_\mu \phi = U(x) \frac{1}{\sqrt{2}} \left( \partial_\mu - i \frac{g}{\sqrt{2}} [W^+ T^+ + W^- T^-] \right) \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right) \right) \)

\( = U(x) \frac{1}{\sqrt{2}} \left( \partial_\mu h(x) + \frac{ig}{2 \cos \theta_w} Z_\mu (v + h(x)) \right) \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right) \right) \)

\( |D_\mu \phi|^2 = \frac{1}{2} (\partial_\mu h(x))^2 + \frac{g^2}{8 \cos^2 \theta_w} Z_\mu^2 (v + h(x))^2 + \frac{g^2 v^2}{4} W^+_\mu W^-_{\mu} \left( 1 + \frac{h(x)}{v} \right)^2 \)

\( = \frac{1}{2} (\partial_\mu h(x))^2 + \left( m_W^2 W^+_{\mu} W^-_{\mu} + \frac{1}{2} m_Z^2 Z_\mu Z_\mu \right) \left( 1 + \frac{h(x)}{v} \right)^2 \)

Potential term in Lagrangian: \( v^2 = \mu^2 / \lambda \), \( m_h = \sqrt{2} \mu \).

\( \mu^2 |\phi|^2 - \lambda |\phi|^4 = \frac{1}{2} \mu^2 (v + h(x))^2 - \frac{1}{4} \lambda (v + h(x))^4 \)

\( = -\frac{1}{2} m_h^2 h^2 - \frac{1}{\sqrt{2}} \sqrt{\lambda} m_h h^3 - \frac{1}{4} \lambda h^4 + \text{const.} \)
The Glashow-Weinberg-Salam Theory of Weak Interactions

$$|D_\mu \phi|^2 = \frac{1}{2} (\partial_\mu h(x))^2 + \left( m_W^2 W^\mu_\mu W^\mu - \frac{1}{2} m_Z^2 Z^\mu_\mu Z^\mu \right) \left( 1 + \frac{h(x)}{v} \right)^2$$

$$\mu^2 |\phi|^2 - \lambda |\phi|^4 = - \frac{1}{2} m_h^2 h^2 - \frac{1}{\sqrt{2}} \sqrt{\lambda} m_h^3 h^3 - \frac{1}{4} \lambda h^4 + \text{const.}$$

- $$m_h = \sqrt{2\mu} = \sqrt{2\lambda} v$$, proportional to $$v$$, $$\sqrt{\lambda}$$.  
- Interaction with gauge boson: 3-point coupling $\propto \text{mass}^2$ of the gauge boson.  
- Self-3-point interaction: $\propto m_h^2$.

$$W^+ \gamma^\mu \quad = \quad 2i \frac{m_W^2}{v} g^{\mu\nu}$$

$$W^- \gamma^\nu \quad = \quad 2i \frac{m_Z^2}{v} g^{\mu\nu}$$

$$Z^0 \gamma^\mu \quad = \quad -3i \sqrt{\frac{\lambda}{2}} m_h = -3i \frac{m_h^2}{v}$$
The Glashow-Weinberg-Salam Theory of Weak Interactions

Gauge boson kinetic term:

\[ L \sim -\frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a - \frac{1}{4} B_{\mu \nu} B_{\mu \nu} \]

\[ F_{\mu \nu}^1 = \partial_\mu A_\nu - \partial_\nu A_\mu + g (A_\mu^2 A_\nu^1 - A_\mu^1 A_\nu^2), \]

\[ F_{\mu \nu}^2 = \partial_\mu A_\nu - \partial_\nu A_\mu + g (A_\mu^1 A_\nu^3 - A_\mu^3 A_\nu^1), \]

\[ F_{\mu \nu}^3 = \partial_\mu A_\nu - \partial_\nu A_\mu - g (A_\mu^3 A_\nu^2 - A_\mu^2 A_\nu^3), \]

\[ B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \]

\[
\frac{1}{\sqrt{2}} (F_{\mu \nu}^1 - iF_{\mu \nu}^2) = D_\mu W^+_{\nu} - D_\nu W^+_\mu, \quad \frac{1}{\sqrt{2}} (F_{\mu \nu}^1 + iF_{\mu \nu}^2) = D^\dagger_\mu W^-_{\nu} - D^\dagger_\nu W^-_\mu
\]

where

\[ D_\mu = \partial_\mu - ig A_\mu^3 = \partial_\mu - ig (\sin \theta_w A_\mu + \cos \theta_w Z_\mu) = \partial_\mu - ie (A_\mu + \cot \theta_w Z_\mu) \]

\[ W^+: \text{ charge } +1, \quad W^-: \text{ charge } -1. \]

\[
F_{\mu \nu}^2 = \partial_\mu A_\nu - \partial_\nu A_\mu - ig \left( W^+_{\mu} W^-_{\nu} - W^-_{\mu} W^+_{\nu} \right)
\]

\[
= \sin \theta_w F_{\mu \nu} + \cos \theta Z_{\mu \nu} - ig \left( W^+_{\mu} W^-_{\nu} - W^-_{\mu} W^+_{\nu} \right)
\]

\[ B_{\mu \nu} = \cos \theta_w F_{\mu \nu} - \sin \theta_w Z_{\mu \nu} \]

where \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\), and \(Z_{\mu \nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu\)

\[ L \sim -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} - \frac{1}{4} Z_{\mu \nu} Z_{\mu \nu} - \frac{1}{2} (D^\dagger_\mu W^-_{-\nu} - D^\dagger_\nu W^-_{-\mu}) (D_\mu W^+_{\nu} - D_\nu W^+_{\mu}) \]

\[ + ie (F_{\mu \nu} + \cot \theta_w Z_{\mu \nu}) W^+_{\mu} W^-_{\nu} - \frac{e^2}{\sin^2 \theta_w} (W^{+\mu} W^-_{\mu} W^{+\nu} W^-_{\nu} - W^{+\mu} W^+_{\mu} W^-_{-\nu} W^-_{\nu}) \]
The Glashow-Weinberg-Salam Theory of Weak Interactions

Coupling to fermions:

• If the quantum number of the fermion fields are fixed, the covariant derivatives are uniquely determined — fixing the coupling between gauge bosons and fermions.

• Massless fermions: left-handed and right-handed fermions decouple in kinetic terms:

\[
\bar{\psi} i \partial \psi = \bar{\psi}_L i \partial \psi_L + \bar{\psi}_R i \partial \psi_R, \quad \psi_{L,R} = \frac{(1 \mp \gamma_5)}{2} \psi, \quad \bar{\psi}_{L,R} = \bar{\psi} \frac{(1 \pm \gamma_5)}{2}
\]

( massive term \( m \bar{\psi} \psi = m (\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \))

So for massless fermions, \( \psi_L \) and \( \psi_R \) could be in different representations of the gauge group.

• In early days, in experiment only left-handed neutrino and no right-handed neutrino were found.

Left-handed neutrino presents in the beta decay, coming with lepton, \( e^- \), or \( \mu^- \). \( W^\pm \) only couples with left-handed states of quark and lepton.

We assign the left-handed fermion fields eg. \((\nu_L, e^-_L)\) to \( SU(2) \) doublets, and right-handed fermion fields to singlet. (We still suppose there is no right-handed neutralinos.)

• We have then \( T^3 \) for fermions, and from experiment we have their \( Q \) charge quantum numbers. From \( Q = T^3 + Y \), we can find out \( Y \) quantum numbers.
Consider only one family of fermions:

<table>
<thead>
<tr>
<th>left-handed</th>
<th>$Q$</th>
<th>$T^3$</th>
<th>$Y$</th>
<th>right-handed</th>
<th>$Q$</th>
<th>$T^3$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_L = \begin{pmatrix} u_L \ d_L \end{pmatrix}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{6}$</td>
<td>$u_R$</td>
<td>$\frac{2}{3}$</td>
<td>$0$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>$E_L = \begin{pmatrix} \nu_L \ e_L \end{pmatrix}$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$(\nu_R)$ not consider $e_R$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

fermion fields: fermion and antifermion particle.

The first generation: $Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $E_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, $u_R$, $d_R$, $e_R$.

The second generation: $Q_L = \begin{pmatrix} c \\ s \end{pmatrix}_L$, $E_L = \begin{pmatrix} \nu_\mu \\ \mu_\nu \end{pmatrix}_L$, $c_R$, $s_R$, $\mu_R$.

The third generation: $Q_L = \begin{pmatrix} t \\ b \end{pmatrix}_L$, $E_L = \begin{pmatrix} \nu_\tau \\ \tau_\nu \end{pmatrix}_L$, $t_R$, $b_R$, $\tau_R$. 
Fermion mass terms:

- $\psi_L$: SU(2) fundamental rep; $\psi_R$ SU(2) singlet. Mass term: $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$ not singlet, forbidden by $SU(2)$ symmetry and $U(1)_Y$.

- Scalar field $\phi$ also in $SU(2)$ fundamental rep. We can construct interaction term:

$$\Delta L_e = -\lambda_e \bar{E}_L \cdot \phi e_R + h.c. = -\lambda_e(\bar{\nu}_e, \bar{e}_L) \left( \frac{\phi_1}{\sqrt{2}} \right) e_R + h.c.$$  

$\lambda_e$: Yukawa coupling constant. Y charges: $\bar{E}, \frac{1}{2}; \phi, \frac{1}{2}; e_R: -1$.

- Lepton mass term from scalar vev:

$$\Delta L_e \sim -\lambda_e(\bar{\nu}_e, \bar{e}_L) \left( \frac{0}{\sqrt{2}} \right) e_R + h.c. = -\lambda_e v \left( \bar{e}_L e_R + \bar{e}_R e_L \right) = -\frac{\lambda_e v}{\sqrt{2}} \bar{e}e.$$  

$m_e = \frac{\lambda_e v}{\sqrt{2}}$. mass $\propto$ Yukawa coupling $\lambda_e$, and vev.

$\therefore m_w = \frac{1}{2} g v$, $m_Z = \frac{1}{2} g v / \cos \theta_w$, $m_e = 0.5\text{MeV}$, $m_w = 80\text{GeV}$.

$\Rightarrow \frac{m_e}{m_w} = \frac{\sqrt{2}\lambda_e}{g} \sim 6 \times 10^{-6}$. $\lambda_e$ is put by hand very small.
Quack mass: We have right-handed down quark

\[ \Delta L_Q = -\lambda_d \bar{Q}_L \cdot \phi d_R - \lambda_u \epsilon^{ab} (\bar{Q}_L)_a \phi^*_b u_R + h.c. \]

\( \lambda_u, d \) Yukawa couplings.

- \( Y \) charge: \( \bar{Q}_L \cdot \phi d_R, (\frac{1}{6} + \frac{1}{2} - \frac{1}{3}) = 0; \) \( (\bar{Q}_L)_a \phi^*_b u_R, (\frac{1}{6} - \frac{1}{2} + \frac{2}{3}) = 0 \)

- \( SU(2) \) fundamental rep. is pseudoreal: \( \bar{Q}_{La} \rightarrow M^*_a \bar{Q}_{Lc}, \phi^*_b \rightarrow M^*_b \phi^*_b', \)

\[ \epsilon^{ab} (\bar{Q}_L)_a \phi^*_b u_R \rightarrow \epsilon^{ab} M^*_a \bar{Q}_{Lc} \phi^*_b = \epsilon^{cb'} \det M^* \bar{Q}_{Lc} \phi^*_b = \epsilon^{cb'} \bar{Q}_{Lc} \phi^*_b. \]

- Mass term

\[ \Delta L_Q \sim -\frac{1}{\sqrt{2}} \lambda_d v (\bar{d}_L d_R + \bar{d}_R d_L) - \frac{1}{\sqrt{2}} \lambda_u v (\bar{u}_L u_R + \bar{u}_R u_L) \]

\[ = -\frac{1}{\sqrt{2}} \lambda_d v \bar{d} d - \frac{1}{\sqrt{2}} \lambda_u v \bar{u} u \]

\[ m_u = \frac{1}{\sqrt{2}} \lambda_u v; \quad m_d = \frac{1}{\sqrt{2}} \lambda_d v \]

- Interaction with higgs: Unitary gauge, \( \phi \rightarrow \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} (v + h) \end{pmatrix} \)

\[ \Delta L_Q + \Delta L_e = -m_e (1 + \frac{h}{v}) \bar{e} e - m_d (1 + \frac{h}{v}) \bar{d} d - m_u (1 + \frac{h}{v}) \bar{u} u \]

Coupling \( \propto \) mass
The Glashow-Weinberg-Salam Theory of Weak Interactions

Fermions and Gauge boson Interaction: one generation of fermion

\[ L = \bar{E}_L i\partial E_L + \bar{e}_R i\partial e_R + \bar{Q}_L i\partial Q_L + \bar{u}_R i\partial u_R + \bar{d}_R i\partial d_R \]

Covariant derivative:

\[ iD_{\mu} = i\partial_{\mu} + \frac{g}{\sqrt{2}} [ W^+ T^+ + W^- T^- ] + \frac{g}{\cos\theta_w} ( T^3 - Q \sin^2\theta_w ) Z_{\mu} + eQA_{\mu} \]

Doublet:

\[ \begin{align*}
    \frac{1}{\cos\theta_w} ( \frac{1}{2} - Q \sin^2\theta_w ) Z_{\mu} + eQA_{\mu} \\
    \frac{g}{\sqrt{2}} W_{\mu} \\
    \frac{g}{\cos\theta_w} \left( -\frac{1}{2} - Q \sin^2\theta_w \right) Z_{\mu} + eQA_{\mu}
\end{align*} \]

Singlet:

\[ \begin{align*}
    \frac{gQ \sin^2\theta_w}{\cos\theta_w} Z_{\mu} + eQA_{\mu} \\
    \frac{gQ \sin^2\theta_w}{\cos\theta_w} \left( \frac{1}{2} - Q \sin^2\theta_w \right) Z_{\mu} + eQA_{\mu}
\end{align*} \]

\[ \mathcal{L} = \bar{E}_L i\partial E_L + \bar{e}_R i\partial e_R + \bar{Q}_L i\partial Q_L + \bar{u}_R i\partial u_R + \bar{d}_R i\partial d_R \]

\[ + g( W_{\mu}^+ J_{W}^{\mu \mu} + W_{\mu}^- J_{W}^{\mu -\mu} + Z_0 J_{Z}^{\mu \mu}) + eA_{\mu} J_{EM}^{\mu} \]

\[ J_{W}^{\mu +} = \frac{1}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L) ; \]

\[ J_{W}^{\mu -} = \frac{1}{\sqrt{2}} (\bar{\nu}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L) ; \]

\[ J_{Z}^{\mu} = \frac{1}{\cos\theta_w} \left[ \bar{\nu}_L \gamma^\mu \left( \frac{1}{2} \right) \nu_L + \bar{e}_L \gamma^\mu \left( -\frac{1}{2} + \sin^2\theta_w \right) e_L + \bar{e}_R \gamma^\mu \left( \sin^2\theta_w \right) e_R \right. \]
\[ \left. + \bar{u}_L \gamma^\mu \left( \frac{1}{2} - \frac{2}{3} \sin^2\theta_w \right) u_L + \bar{u}_R \gamma^\mu \left( -\frac{2}{3} \sin^2\theta_w \right) u_R \right. \]
\[ \left. + \bar{d}_L \gamma^\mu \left( -\frac{1}{2} + \frac{1}{3} \sin^2\theta_w \right) d_L + \bar{d}_R \gamma^\mu \left( \frac{1}{3} \sin^2\theta_w \right) d_R \right] ; \]

\[ J_{EM}^{\mu} = \bar{\nu} \gamma^\mu (-1)e + \bar{u} \gamma^\mu \left( \frac{2}{3} \right) u + \bar{d} \gamma^\mu \left( -\frac{1}{3} \right) d. \]
The Glashow-Weinberg-Salam Theory of Weak Interactions

• The second and third generation: $e \rightarrow \mu, \tau, \nu_e \rightarrow \nu_\mu, \nu_\tau$, $u \rightarrow c, t$, $d \rightarrow s, b$.

• Interaction with $W^\pm_\mu$, only left-handed fermions, charged current weak interaction.

• Interaction with $Z_\mu$, neutral current weak interaction, both left-handed and right-handed fermions, coupling different. No flavor changing neutral current at tree level.

• Interaction with $A_\mu$, electromagnetic current. vector coupling — left right-handed couplings are the same.
The Glashow-Weinberg-Salam Theory of Weak Interactions

We have three generation of quarks and leptons

\[ Q^i_L = \left( \begin{array}{c} u^i \\ d^i \end{array} \right)_L = \left( \begin{array}{c} (u) \\ (d) \end{array} \right)_L, \left( \begin{array}{c} (c) \\ (s) \end{array} \right)_L, \left( \begin{array}{c} (t) \\ (b) \end{array} \right)_L \right), u^i_R = (u_R, c_R, t_R), d^i_R = (d_R, s_R, b_R). \]

Gauge coupling: quarks of the same quantum number have the same gauge coupling.
However, coupling with higgs will in general mix flavors. The most general renormalizable gauge invariant coupling

\[ \Delta L_Q = -\lambda^{ij}_d \overline{Q}^i_L \cdot \phi d^j_R - \lambda^{ij}_u \epsilon^{ab} \left( \overline{Q}^i_L \right)_a \phi^*_b u^j_R + h.c. \]

\( \lambda^{ij}_d, \lambda^{ij}_u \): most general, not necessarily symmetric or Hermitian, complex-valued matrices.
We could diagonalize the matrices by unitary transformations different on left- and right-handed fermion fields.

\[ \lambda^{ij}_u = (U_u D_u W_u^\dagger)^{ij}; \quad D_u = \begin{pmatrix} D^1_u \\ D^2_u \\ D^3_u \end{pmatrix}, \quad D^i_u: \text{real} \]

\[ \lambda^{ij}_d = (U_d D_d W_d^\dagger)^{ij}; \quad D_d = \begin{pmatrix} D^1_d \\ D^2_d \\ D^3_d \end{pmatrix}, \quad D^i_d: \text{real} \]

change variables: \( u^i_R \rightarrow W^{ij}_{u} u^j_R, \quad d^i_R \rightarrow W^{ij}_{d} d^j_R, \quad u^i_L \rightarrow U^{ij}_u u^j_L, \quad d^i_L \rightarrow U^{ij}_d d^j_L. \)
change variables to mass eigenstates: $u_R^i \rightarrow W^i u_R^i$, $d_R^i \rightarrow W^i d_R^i$, $u_L^i \rightarrow U^i u_L^i$, $d_L^i \rightarrow U^i d_L^i$.

\[
\Delta L_Q = -\frac{1}{\sqrt{2}}D_d^i \bar{d}_L^i (v + h(x)) d_R^i - \frac{1}{\sqrt{2}}D_u^i \bar{u}_L^i (v + h(x)) u_R^i + h.c.
\]

- **masses:** $m_d^i = \frac{1}{\sqrt{2}} D_d^i v$, $m_u^i = \frac{1}{\sqrt{2}} D_u^i v$
- **Interaction with higgs:** coupling $m_d^i \bar{d}_L^i (h(x)) d_R^i$, proportional to fermion masses
- **$\bar{u}_L^i \gamma^\mu u_L^i \rightarrow (\bar{u}_L U_u^i)^i \gamma^\mu (U_u u_L)^i = \bar{u}_L^i \gamma^\mu u_L^i$ invariant. $\bar{\psi}_{L,R} \gamma^\mu \psi_{L,R}$ invariant under the transformations. The $J^{\mu}_{EM}$, $J^{\mu}_{Z}$ are of this form. QCD coupling are also invariant— like $J^{\mu}_{EM}$ vector coupling.
- **$J^\pm_W$ has interaction between $U_L$ and $d_L$**

\[
J^\pm_W \sim \frac{1}{\sqrt{2}} \bar{u}_L^i \gamma^\mu d_L^i \rightarrow \frac{1}{\sqrt{2}} (\bar{u}_L U_u^i)^i \gamma^\mu (U_d d_L)^i = \frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu (U_u^i U_d) d_L
\]

\[= \frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu V^{ij} d_L^j \]

$V^{ij}$ : Cabibbo-Kobayashi-Maskawa matrix.
We can still have transformation freedom: \( q_i^L \rightarrow e^{i\alpha_i} q_i^L, \) \( q_i^R \rightarrow e^{i\alpha_i} q_i^R, \) does not change mass and kinetic terms, and other diagonal interaction terms.

Mass term:
\[
D_i d^i_L (v + h) d^i_R \rightarrow D_i e^{-i\beta_i} d^i_L (v + h) e^{i\beta_i} d^i_R, \quad \text{invariant;}
\]

Interaction term
\[
\bar{u}_L^i \gamma^\mu u^i_L \rightarrow \bar{u}_L^i e^{-i\alpha_i} \gamma^\mu e^{i\alpha_i} u^i_L, \quad \text{invariant.}
\]

CKM:
\[
\bar{u}_L^i V_{ij} d^j_L \rightarrow \bar{u}_L^i e^{-i\alpha_i} V_{ij} e^{i\beta_j} d^j_L
\]

Example: for two generations: \( V_{ij} \in U(2), \) \( i, j = 1, 2, \) four parameters, one real, three phases:
\[
(\bar{u}_L^1, \bar{u}_L^2) V \begin{pmatrix} d^1_L \\ d^2_L \end{pmatrix} \rightarrow (\bar{u}_L^1 e^{-i\alpha_1}, \bar{u}_L^2 e^{-i\alpha_2}) V \begin{pmatrix} e^{i\beta_1} d^1_L \\ e^{i\beta_2} d^2_L \end{pmatrix}
\]
\[
= (\bar{u}_L^1, \bar{u}_L^2 e^{-i(\alpha_2-\alpha_1)}) V \begin{pmatrix} e^{i(\beta_1-\alpha_1)} d^1_L \\ e^{i(\beta_2-\alpha_1)} d^2_L \end{pmatrix}
\]

Eliminate three independent phases of \( V. \) \( V \) has only one real freedom, Cabbibo angle.
\[
V = \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix}
\]

\( V^{ij} \) a unitary 3 \( \times \) 3 matrix, 9 free para: 3 parameterize real orthogonal matrix, 6 phases; redefining the 6 fermion phases, five independent phase freedom can be used to eliminate the CKM phases. So, we can absorb 5 phases into fermions. Only three real parameters and one phase parameter is left in \( V. \) One matrix element is complex.

Complex coupling constant \( \rightarrow \) CP violation. CP violation only presents in the charged current weak interaction with \( W^\pm. \)
The Glashow-Weinberg-Salam Theory of Weak Interactions

CKM matrix, parameterization:

\[
V = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
e^{-i\delta/2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\delta/2}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} \\
0 & 1 & 0 \\
-s_{13} & 0 & c_{13}
\end{pmatrix}
\times
\begin{pmatrix}
e^{i\delta/2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\delta/2}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
-s_{12} c_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix}
\]

\[c_{ij} = \cos \theta_{ij}, \quad s_{ij} = \sin \theta_{ij}\]
Leptonic sector: if there is no right neutrino, no neutrino mass, there will be no CKM like matrix.

$$\Delta L_l = -\lambda_l^{ij} \bar{E}_L^i \cdot \phi e_R^j + h.c.$$ 

just need to diagonalize $\lambda_l^{ij} = U_l D_l W^\dagger$, transform $e_L^i \rightarrow U_l e_L$, $\nu_L \rightarrow U_l \nu_L$, $e_R \rightarrow W_l e_R$.

in $J^+_W$: $\bar{\nu}_L^i \gamma^\mu e_L^i$ invariant.