NON-ABELIAN GAUGE INVARIANCE

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Local Gauge Invariance

Rev: Abelian gauge theory, QED:

\[ \mathcal{L} = -\frac{1}{4} (F^{\mu\nu})^2 + i \bar{\psi} \phi \psi - e \bar{\psi} A \psi - m \bar{\psi} \psi \]

Local gauge symmetry: \( \psi \rightarrow e^{i\alpha(x)} \psi(x), A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha \)

- We know Lagrangian first, and then find symmetry.
- On the other hand, if at first we have a theory:

\[ \mathcal{L} = i \bar{\psi} \phi \psi - m \bar{\psi} \psi \]

invariant under \( \psi \rightarrow e^{i\alpha \psi(x)} \) global symmetry, \( \alpha \) independent of \( x \). Then change the symmetry to a local one \( \alpha(x) \), and want the Lagrangian invariant under this local \( U(1) \).

- \( m \bar{\psi} \psi \) is invariant. However \( i \bar{\psi} \phi \psi \) is not invariant.

\[ \partial_\mu \psi \rightarrow \partial_\mu (e^{i\alpha(x)} \psi(x)) = e^{i\alpha(x)} (\partial_\mu + i \partial_\mu \alpha(x)) \psi(x) \]

We want a covariant derivative, and require it transform as:

\[ D_\mu \psi \rightarrow D'_\mu e^{i\alpha(x)} \psi = e^{i\alpha(x)} D_\mu \psi \]

Introduce another field \( A_\mu \), covariant derivative \( D_\mu = \partial_\mu + i e A_\mu \), and \( A_\mu \) change under the gauge transformation \( A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha \).

- Kinetic term of \( A_\mu \):

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad \mathcal{L} \sim -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]

Local gauge symmetry determine the interaction between \( \psi \) and \( A_\mu \), and also the kinetic term of \( A_\mu \).
The geometry of Gauge Invariance

Another way, more geometric view of $U(1)$ gauge invariance.

- Derivative involves subtraction of fields at different points, which transform differently: $(n^\mu$ a direction) $n^\mu \partial_\mu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\psi(x + \epsilon n) - \psi(x)]$

- Comparator: We introduce a $U(y, x)$, group element of $U(1)$, $U(y, x) = e^{i\phi(y, x)}$, satisfies $U(x, x) = 1, \phi(x, x) = 0$.

  Transformation: $U(y, x) \to e^{i\alpha(y)} U(y, x) e^{-i\alpha(x)}$.

- Then $U(y, x)\psi(x)$ transforms the same as $\psi(y)$:

  $U(y, x)\psi(x) \to e^{i\alpha(y)} U(y, x)\psi(x)$

- $\psi(y) - U(y, x)\psi(x)$ has definite transformation law.

  Define covariant derivative on $n^\nu$ direction:

  $n^\nu D_\nu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x)]$

- Expand $U(x + \epsilon n, x)$ w.r.t $\epsilon$: ( $U(y, x)$ continuous w.r.t. $x$ and $y$)

  $U(x + \epsilon n, x) = \exp\{-i\epsilon n^\mu eA_\mu(x) + O(\epsilon^2)\} = 1 - i\epsilon n^\mu eA_\mu(x) + O(\epsilon^2)$

  $A_\mu$ a new vector field — connection.

- Covariant derivative: $D_\mu \psi(x) = (\partial_\mu + ieA_\mu)\psi(x)$. 
The geometry of Gauge Invariance

Transformation of $A_\mu$:

$$U(x + \epsilon n, x) = \exp\{-i\epsilon n^\mu eA_\mu(x) + O(\epsilon^2)\} = 1 - i\epsilon n^\mu eA_\mu(x) + O(\epsilon^2)$$

$$U(y, x) \rightarrow e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)} = \exp\{-i\epsilon n^\mu eA_\mu(x) + i\alpha(x + \epsilon n) - i\alpha(x)\}$$

$$\Rightarrow -i\epsilon n^\mu eA_\mu(x) \rightarrow -i\epsilon n^\mu eA_\mu(x) + i\alpha(x + \epsilon n) - i\alpha(x)$$

$$\Rightarrow A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

Then $D_\mu \psi(x) = (\partial_\mu + ieA_\mu)\psi(x)$ transforms as :

$$D_\mu \psi(x) \rightarrow [\partial_\mu + ie(A_\mu - \frac{1}{e}\partial_\mu \alpha)]e^{i\alpha(x)}\psi(x) = e^{i\alpha(x)}D_\mu \psi(x).$$

$L \sim \bar{\psi}D\psi(x)$ is invariant under the local symmetry.

- Global symmetry → Local gauge symmetry: requires a new vector field $A_\mu$
  and determines its coupling form with $\psi$.

- A general way to construct a local gauge invariant Lagrangian from a
  globally invariant theory $\partial_\mu \rightarrow D_\mu$. 
The geometry of Gauge Invariance

Kinetic term for $A_\mu$, requirements:

1. local gauge invariant,  
2. contain derivatives of $A_\mu$, no $\psi$.

Two ways:

(1) integrally from $U(y, x)$, or (2) infinitesimally from covariant derivative.

- (1) From $U(y, x)$:
  Requirement: $U(y, x) \in U(1)$ pure phase & $U(x, y)\dagger = U(y, x)$. Going away and then returning along the same path gives phase 1. Infinitesimally

  \[ U(x + \epsilon n, x) = \exp\{-i\epsilon n^\mu eA_\mu(x + \frac{\epsilon}{2} n) + O(\epsilon^3)\} \]

Consider $U$ goes around a small square, in $\hat{1}$ and $\hat{2}$ direction:

\[ U(x) \equiv U(x, x + \epsilon \hat{2})U(x + \epsilon \hat{2}, x + \epsilon \hat{1} + \epsilon \hat{2}) \times U(x + \epsilon \hat{1} + \epsilon \hat{2}, x + \epsilon \hat{1})U(x + \epsilon \hat{1}, x). \]

$U(x)$ is invariant under local gauge symmetry. Taking $\epsilon \to 0$ we will have an invariant constructed from $A_\mu$. 

The geometry of Gauge Invariance

\[^1\]

A \[1^\, A \]
\[^2\]
A \[1^\, A \]
\[^2\]

\[x + \epsilon \hat{2}\]

\[x \]

\[x + \epsilon \hat{1}\]

\[\hat{1}^\mu A_\mu \equiv A_1, \hat{2}^\mu A_\mu \equiv A_2\]

\[U(x) = \exp \left\{ -i \epsilon \left( -A_2 \left( x + \frac{\epsilon}{2} \hat{2} \right) - A_1 \left( x + \frac{\epsilon}{2} \hat{1} + \epsilon \hat{2} \right) \right. \right. \]
\[+ \left. \left. A_2 \left( x + \epsilon \hat{1} + \frac{\epsilon}{2} \hat{2} \right) + A_1 \left( x + \frac{\epsilon}{2} \hat{1} \right) \right) + O(\epsilon^3) \right\} \]
\[= \exp \left\{ -i \epsilon \left( \epsilon \partial_1 A_2 \left( x + \frac{\epsilon}{2} \hat{2} \right) - \epsilon \partial_2 A_1 \left( x + \frac{\epsilon}{2} \hat{1} \right) \right) + O(\epsilon^3) \right\} \]
\[= \exp \left\{ -i \epsilon^2 \left( \partial_1 A_2(x) - \partial_2 A_1(x) \right) + O(\epsilon^3) \right\} \]
\[= 1 - i \epsilon^2 \left( \partial_1 A_2(x) - \partial_2 A_1(x) \right) + O(\epsilon^3) \]

\[F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\] is locally invariant — Geometric origin of the field strength.

Any function of \[F_{\mu\nu}\] is locally invariant.
The geometry of Gauge Invariance

The second way : Using covariant derivative

- \( D_\mu = \partial_\mu + ieA_\mu(x) \).

\[
[D_\mu, D_\nu]\psi \rightarrow e^{i\alpha(x)}[D_\mu, D_\nu]\psi = e^{i\alpha(x)}[D_\mu, D_\nu]e^{-i\alpha}e^{i\alpha}\psi
\]

\[
\]

\[
= ie(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) = ieF_{\mu\nu}
\]

- \( F_{\mu\nu} \) is not a derivative operator.

- Transform as

\[
[D_\mu, D_\nu] \rightarrow e^{i\alpha(x)}[D_\mu, D_\nu]e^{-i\alpha}
\]

\[
F_{\mu\nu} \rightarrow e^{i\alpha(x)}F_{\mu\nu}e^{-i\alpha} = F_{\mu\nu}
\]

Any function of \( F_{\mu\nu} \) is invariant under the gauge trans.
**The geometry of Gauge Invariance**

The most general *renormalizable* Lagrangian invariant under local gauge transformation:

\[ \mathcal{L}_4 = -\frac{1}{4}(F^{\mu\nu})^2 + i\bar{\psi}D\psi - m\bar{\psi}\psi - c\epsilon^{\mu\nu\rho\delta}F_{\mu\nu}F_{\rho\delta} \]

- \( c\epsilon^{\mu\nu\rho\delta}F_{\mu\nu}F_{\rho\delta} \) violate \( P \) and \( T \) symmetry. Excluded by \( P \) and \( T \) requirement.

**Parity:** \( \bar{\psi}\gamma^\mu\psi, \ (-1)^\mu = \begin{cases} 1, & \mu = 0 \\ -1, & \mu = 1, 2, 3 \end{cases} ; A_\mu, (-1)^\mu; E, -1; B, +1. \)

**Time reversal:** \( \bar{\psi}\gamma^\mu\psi, \ (-1)^\mu ; A_\mu, (-1)^\mu; E, +1; B, -1. \)

- Two free parameters \( e \) and \( m \).

- Higher dimensional nonrenormalizable interaction terms, irrelevant operators, irrelevant for low energy physics.

- local gauge invariant + renormalizable+ \( P \) and \( T \) invariant \( \Rightarrow \) QED. 
  *Symmetry determines the dynamics!*
**Yang-Mills Lagrangian**

Consider a doublet of Dirac spinor fields: \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \)

- At first we have a Lagrangian:
  \[
  \mathcal{L} = i \bar{\psi} \slashed{\partial} \psi - m \bar{\psi} \psi = i(\bar{\psi}_1, \bar{\psi}_2) \slashed{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - m(\bar{\psi}_1, \bar{\psi}_2) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
  \]

  invariant under global trans.: \( \psi \to V \psi; \ V: 2 \times 2 \text{ matrix}, \) satisfies \( V^\dagger V = 1. \ V \in U(2) = U(1) \times SU(2), \) independent of \( x. \)

- Promote the global SU(2) symmetry \( \to \) local symmetry .

  Requirement: \( \det V(x) = 1; \) local trans: \( \psi \to V(x) \psi(x). \)

  A general \( SU(2) \) element \( V(x) = \exp \{ i \frac{\sigma^i}{2} \alpha^i(x) \}, \ \sigma^i \) pauli matrix.

- \( \bar{\psi}(x) \psi(x) \) term is invariant. \( \bar{\psi} \slashed{\partial} \psi \) is not inv.

  \[
  \bar{\psi} \slashed{\partial} \psi \to \bar{\psi}(V^\dagger \slashed{\partial} V) \psi + \bar{\psi} \slashed{\partial} \psi
  \]

\[
tr(V^\dagger \partial_\mu V) = 0 \text{ and } (V^\dagger \partial_\mu V)^\dagger = -V^\dagger \partial_\mu V \text{ can be expanded using } i \sigma^i.
\]

- Introduce covariant derivative \( D_\mu = \partial_\mu - igA_\mu, \ A_\mu = A^i_\mu \frac{\sigma^i}{2} \) and require its transformation: \( D_\mu \psi \to D'_\mu (V \psi) = VD_\mu \psi \Rightarrow A_\mu \) transformation:

\[
D'_\mu (V \psi) = ((\partial_\mu V) + V \partial_\mu - igA'_\mu V) \psi = VD_\mu \psi = (V \partial_\mu - igVA_\mu) \psi
\]

\[
\Rightarrow -igA'_\mu = -igVA_\mu V^\dagger - (\partial_\mu V)V^\dagger
\]

\[
A'_\mu = VA_\mu V^\dagger + \frac{i}{g} V(\partial_\mu V^\dagger)
\]
**Yang-Mills Lagrangian**

Geometric construction of the covariant derivative:
\[ \psi \rightarrow V(x)\psi(x), \quad \psi \rightarrow V(y)\psi(y), \] transform differently.

**Derivative: No good transformation property.**

- Introduce Comparator \( U(y, x) \in SU(2), \) \( U(x, x) = 1 \), which transforms as \( U(y, x) \rightarrow V(y)U(y, x)V^\dagger(x) \).
- \( U(y, x)\psi(x) \rightarrow V(y)U(y, x)V^\dagger(x)V(x)\psi = V(y)U(y, x)\psi(x) \). Transforms the same as \( \psi(y) \).
- Define covariant derivative in direction \( n^\mu \):
\[
n^\nu D_\nu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \psi(x + \epsilon n) - U(x + \epsilon n, x)\psi(x) \right]
\]
- Expand \( U(x + \epsilon n, x) \) w.r.t \( \epsilon \):
\[
U(x + \epsilon n, x) = 1 + i\epsilon n^\mu gA^i_\mu(x)\frac{\sigma^i}{2} + O(\epsilon^2)
\]
- Covariant derivative:
\[
n^\nu D_\nu \psi = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \psi(x + \epsilon n) - \psi(x) - i\epsilon n^\mu gA^i_\mu(x)\frac{\sigma^i}{2}\psi(x) \right]
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \epsilon n^\mu \partial_\mu \psi(x) - i\epsilon n^\mu gA^i_\mu(x)\frac{\sigma^i}{2}\psi(x) \right]
\]
\[
D_\mu \psi = (\partial_\mu - igA_\mu)\psi(x), \quad A_\mu = A^i_\mu \frac{\sigma^i}{2}
\]
Yang-Mills Lagrangian

- Covariant derivative: \( D_\mu \psi = (\partial_\mu - igA_\mu)\psi(x), \ A_\mu = A_\mu^i \frac{\sigma^i}{2} \)
- Transformation of \( A_\mu \): \( U(x + \epsilon n, x) \rightarrow V(x + \epsilon n)U(x + \epsilon n, x)V(x)^\dagger \)

\[
1 + i\epsilon n^\mu gA_\mu(x) \rightarrow V(x + \epsilon n)(1 + i\epsilon n^\mu gA_\mu(x))V(x)^\dagger
\]

\[
= V(x + \epsilon n)V(x)^\dagger + i\epsilon n^\mu gV(x + \epsilon n)A_\mu(x)V(x)^\dagger
\]

\[
= 1 + i\epsilon n^\mu g \left( -\frac{i}{g} \partial_\mu V(x)V(x)^\dagger + V(x)A_\mu(x)V(x)^\dagger \right) + O(\epsilon^2)
\]

\[
= 1 + i\epsilon n^\mu g \left( \frac{i}{g} V(x)\partial_\mu V(x)^\dagger + V(x)A_\mu(x)V(x)^\dagger \right) + O(\epsilon^2)
\]

\[
A_\mu \rightarrow V(x) \left( \frac{i}{g} \partial_\mu + A_\mu(x) \right) V(x)^\dagger
\]

- \( V(x), \ A_\mu(x) \) matrix, not commute.
**Yang-Mills Lagrangian**

\[ A_\mu \rightarrow V(x) \left( \frac{i}{g} \partial_\mu + A_\mu (x) \right) V(x)^\dagger \]

- **Infinitesimal trans.:** \( V = \exp \left\{ i \frac{\sigma^i}{2} \alpha^i (x) \right\} \sim 1 + i \frac{\sigma^i}{2} \alpha^i (x) + O(\alpha^2) \)

\[
A_\mu = A_\mu \frac{\sigma^i}{2} \rightarrow (1 + i \frac{\sigma^i}{2} \alpha^i (x)) \left( \frac{i}{g} \partial_\mu + A_\mu (x) \right) (1 - i \frac{\sigma^i}{2} \alpha^i (x)) + \ldots
\]

\[
= A_\mu + \frac{1}{g} \partial_\mu \alpha^i (x) \frac{\sigma^i}{2} + i \alpha^i \left[ \frac{\sigma^i}{2}, A_\mu \right] + \ldots
\]

\[
= A_\mu \frac{\sigma^i}{2} + \frac{1}{g} \partial_\mu \alpha^i (x) \frac{\sigma^i}{2} + i \alpha^i \left[ \frac{\sigma^i}{2}, A_\mu \frac{\sigma^j}{2} \right] + \ldots
\]

\[
= \frac{\sigma^i}{2} \left( A_\mu \frac{1}{g} \partial_\mu \alpha^i (x) + \epsilon^{ijk} A_\mu^j \alpha^k + \ldots \right)
\]
Yang-Mills Lagrangian

Kinetic part of $A_\mu$:

- Using covariant derivative, look at the commutator:

\[
[D_\mu, D_\nu] \psi \rightarrow [D'_\mu, D'_\nu] V \psi = V [D_\mu, D_\nu] \psi
\]

\[
[D_\mu, D_\nu] \psi = [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \psi
\]

\[
= ([\partial_\mu, \partial_\nu] - ig[\partial_\mu, A_\nu] + ig[\partial_\nu, A_\mu] - g^2 [A_\mu, A_\nu]) \psi
\]

\[
= -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) \psi
\]

\[
\equiv -igF_{\mu\nu} \psi
\]

$F_{\mu\nu}$ matrix, $F_{\mu\nu} = F^i_{\mu\nu} \frac{\sigma^i}{2}$;

components: $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu$.

- Transformation of $F_{\mu\nu}$:

\[
[D_\mu, D_\nu] \psi \rightarrow [D'_\mu, D'_\nu] \psi' = [VD_\mu V^\dagger, VD_\nu V^\dagger] \psi' = V[D_\mu, D_\nu] V^\dagger \psi'
\]

\[
F_{\mu\nu}(x) \rightarrow V(x) F_{\mu\nu}(x) V^\dagger(x).
\]
Yang-Mills Lagrangian

\[ F_{\mu\nu}(x) \rightarrow V(x)F_{\mu\nu}(x)V^\dagger(x). \]

Infinitesimal: \( V = \exp\{i \frac{\sigma^i}{2} \alpha^i(x)\} \sim 1 + i \frac{\sigma^i}{2} \alpha^i(x) + O(\alpha^2) \)

\[
F_{\mu\nu}^i(x) \frac{\sigma^i}{2} \rightarrow \left(1 + i \frac{\sigma^i}{2} \alpha^i(x)\right) F_{\mu\nu}(x) \left(1 - i \frac{\sigma^i}{2} \alpha^i(x)\right) + \ldots \\
= F_{\mu\nu}^i(x) \frac{\sigma^i}{2} + [i\alpha^i \frac{\sigma^i}{2}, F_{\mu\nu}^j \frac{\sigma^j}{2}] + \ldots \\
= \frac{\sigma^i}{2} \left( F_{\mu\nu}^i(x) - \epsilon^{ijk} \alpha^j F_{\mu\nu}^k + \ldots \right)
\]

Unlike in \( U(1) \) case, \( F_{\mu\nu}^i(x) \) is not invariant, but transform in the adjoint rep. of SU(2).
Yang-Mills Lagrangian

- Construct invariant kinetic part of $A_\mu$: $F_{\mu\nu}(x) \rightarrow V(x)F_{\mu\nu}(x)V^\dagger(x)$;

$$\mathcal{L} = -\frac{1}{2} tr[(F_{\mu\nu})^2] = -\frac{1}{4} F_{\mu\nu}^i F^{i,\mu\nu}$$

— Yang-Mills theory. Because $F_{\mu\nu}^i = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^j_\mu A^k_\nu$, there are cubic and quartic self-interaction terms in $A_\mu$.

- Put all together, a theory of Yang-Mills interacting with fermions ($P$ and $T$ symmetric and renormalizable): $\partial_\mu \rightarrow D_\mu$.

$$\mathcal{L}_4 = -\frac{1}{4} (F_{\mu\nu}^i)^2 + i\bar{\psi} D\psi - m\bar{\psi}\psi$$

Like QED, only depends on 2 free parameters, $g$, $m$.

- EOM:
  $\delta\psi$, covariant Dirac eq.: $iD\psi = m\psi$
  $\delta A^i_\mu$: $\partial^\mu F_{\mu\nu}^i + g\epsilon^{ijk} A^j_\mu F_{\mu\nu}^k = -g\bar{\psi}\gamma_\nu \frac{\sigma^i}{2} \psi$
  $\bar{\psi}\gamma_\nu \frac{\sigma^i}{2} \psi$ is the original current for global symmetry. No longer conserved.
  
  We can still define a conserved current: $J^\mu = \bar{\psi} \gamma^\mu \frac{\sigma^i}{2} \psi + \epsilon^{ijk} A^j_\nu F_{k,\nu\mu}$

- Bianchi Id.: $[D_\mu, [D_\nu, D_\rho]] = \epsilon^{\mu\nu\rho\delta} [D_\mu, [D_\nu, D_\rho]] = 0$,

$$\epsilon^{\mu\nu\rho\delta} [D_\mu, F_{\nu\rho}] = \epsilon^{\mu\nu\rho\delta} \frac{\sigma^i}{2} (\partial_\mu F_{\nu\rho}^i + g\epsilon^{ijk} A^j_\mu F_{\nu\rho}^k) = 0$$
Previously, we discussed $SU(2)$ group as the gauge symmetry. In general, we can change $SU(2)$ to any compact Lie group of transformation. Any group element can be represented in a $n \times n$ unitary matrix. $\psi$ a $n$-plet, and transforms according to

$$\psi(x) \rightarrow V(x)\psi(x),$$

and infinitesimally

$$V(x) = \exp\{i\alpha^a(x)t^a\} = 1 + i\alpha^a(x)t^a + O(\alpha^2),$$

- $t^a: n \times n$ Hermitian matrices, generators of the Lie group in the corresponding representation.
- Generators satisfy: $[t^a, t^b] = i f^{abc}t^c$, structure constant $f^{abc}$, can be chosen to be completely antisymmetric.
- All previous discussion can be used. $\frac{\sigma^i}{2} \rightarrow t^a, \epsilon^{ijk} \rightarrow f^{abc}$
**Yang-Mills Lagrangian**

- **Covariant derivative:** $D_\mu \psi = (\partial_\mu - igA_\mu)\psi(x)$, $A_\mu = A_\mu^a t^a$

- **Local gauge transformations for $\psi$ and $A_\mu$:**
  
  $V(x) = e^{i\alpha^a(x)t^a} = 1 + i\alpha^a(x)t^a + O(\alpha^2),$

  $\psi \rightarrow V(x)\psi \xrightarrow{\text{infinitesimal}} (1 + i\alpha^a t^a)\psi$

  $A_\mu^a \rightarrow V(x)(A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu)\psi \xrightarrow{\text{infinitesimal}} A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c$

- **Field strength:** $[D_\mu, D_\nu] = -igF_{\mu\nu} = -igF_{\mu\nu}^a t^a$

  **Transformation:** $F_{\mu\nu} \rightarrow VF_{\mu\nu}V^\dagger$

  $$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c.$$

  **Infinitesimal:** $F_{\mu\nu}^a(x) \rightarrow F_{\mu\nu}^a(x) - f^{abc}\alpha^b F_{\mu\nu}^c$

- **Lagrangian of Yang-Mills interacting with fermions ($P$ and $T$ symmetric and renormalizable):**

  $$\mathcal{L}_4 = -\frac{1}{4}(F_{\mu\nu}^a)^2 + i\bar{\psi}D\psi - m\bar{\psi}\psi$$

- **EOM:** $\partial^\mu F_{\mu\nu}^a + gf^{abc}A_{\mu}^b F_{\mu\nu}^c = -g\bar{\psi}\gamma_\nu t^a \psi$

- **Bianchi Id.:** $\epsilon^{\mu\nu\rho\delta}[D_\mu, F_{\nu\rho}] = \epsilon^{\mu\nu\rho\delta} t^a (\partial_\mu F_{\nu\rho}^a + gf^{abc}A_\mu^b F_{\nu\rho}^c) = 0$

- **Local gauge symmetry determines the interaction between fermions and the gauge field and also the self-interaction of Yang-Mills.**
Wilson line and Wilson loop in $U(1)$

We considered the comparator $U(y, x)$: $x$ and $y$ are infinitesimally separated. Now we consider finitely separated $x$ and $y$. First in $U(1)$ gauge theory.

- Previously, $U(1)$ case we have $U(x) = \exp\{-ie\epsilon^2 F_{12}\} \neq 1$ which means $U(y, x)$ depends on path.
- Given a path $P$, $x \rightarrow y$, evenly separated into $N$ segments (length $\epsilon = l/N$, point $y_i$, $i = 1, \ldots, N + 1$, $y_1 = x$, $y_{N+1} = y$). Normalized tangent vector at each point $n_i^\mu$, ($i = 1, \ldots, N$), Introduce $U_P(y, x)$,

$$U_P(y, x) = \lim_{N \rightarrow \infty, \epsilon \rightarrow 0} U(y, y_N)U(y_N, y_{N-1}) \ldots U(y_2, y_1)$$

$$= \lim_{N \rightarrow \infty} \exp\{-ie\epsilon\left(n_N \cdot A_\mu \left(\frac{1}{2}(y + y_N)\right) + \cdots + n_1 \cdot A_\mu \left(\frac{1}{2}(y_2 + x)\right)\right)\}$$

$$= \exp\{-ie \int_P dx^\mu A_\mu\}$$

The integral on the exponential is path dependent.
- $U_P(y, x)$ is a explicit expression for $U(y, x)$. Transformation:

$$U(y, x) \rightarrow \exp\{-ie \int_P dx^\mu \left(A_\mu - \frac{1}{e} \partial_\mu \alpha(x)\right)\}$$

$$\rightarrow \exp\{-ie \int_P dx^\mu A_\mu + i\alpha(y) - i\alpha(x)\} \rightarrow e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)}$$

$U_P(y, x)U_P(x, y) = U_P(y, x)U_P^\dagger(y, x) = 1$
Wilson line and Wilson loop in $U(1)$

Wilson Loop:

- If the path $P$ is closed, form a loop — Wilson loop. $U_P(x, x)$
- $U_P(x, x)$ is locally gauge invariant. $U_P(x, x) \rightarrow e^{+i\alpha(x)}U_P(x, x)e^{-i\alpha(x)} = U_P(x, x)$
- From Stokes theorem: $\frac{1}{2} \int_{\Sigma} dx^\mu \wedge dx^\nu F_{\mu\nu} = \oint dx^\mu A_\mu(x)$, $\Sigma$ a surface enclosed by the loop.

$$U_P(y, y) = \exp \left[ -i \frac{e}{2} \int_{\Sigma} d\sigma^{\mu\nu} F_{\mu\nu} \right]$$

- Loop $P = P_2 \cdot P_1$, $U_P = U_{P_2}(y, x)U_{P_1}(x, y)$.
  If $\frac{1}{2} \int_{\Sigma} dx^\mu \wedge dx^\nu F_{\mu\nu} \neq 0$, $U_P \neq 1 \Rightarrow U_{P_2}(y, x) \neq U_{P_1}^\dagger(x, y) = U_{P_1}(y, x)$
  $U_P$ depends on path.
Wilson line and Wilson loop: Non-abelian

The $U(1)$ Wilson line can be generalized to Non-abelian case. Now construct $U(y, x)$, given gauge field $A_\mu$, matrices.

$$U(x + \epsilon n, x) = I + i\epsilon n^\mu g A_\mu(x) + O(\epsilon^2)$$

Given a path $P$, $x^\mu(s)$, $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$.

The same as in Abelian case, separated into $N$ segments, points $y_{N+1} = y, \ldots, y_1 = x$.

And define Wilson line:

$$U_P(y, x) = \lim_{N \to \infty, \epsilon \to 0} U(y, y_N)U(y_N, y_{N-1}) \ldots U(y_2, y_1)$$

$$= \lim_{N \to \infty} \prod_{i=N}^{1} \left( I + i\epsilon n^\mu_i g A_\mu \left( \frac{1}{2} (y_i + y_{i+1}) \right) + \cdots \right)$$

$$\equiv P \exp \{ ig \int_x^y dx^\mu A_\mu \} = P \exp \{ ig \int_0^1 ds \frac{dx^\mu}{ds} A_\mu \}$$

$P$: Path ordered.
Wilson line and Wilson loop: Non-abelian

Path ordered integral, (like $U(t, t_0) = T \exp\{-i \int_{t_0}^{t} dt' H_I(t')\}$, time ordered product):

$$P \exp\{ig \int_{0}^{1} \frac{dx^\mu}{ds} A_\mu\}$$

$$= \sum_{n=0}^{\infty} (ig)^n \int_{s_1 \geq s_2 \geq \cdots \geq s_n} ds_1 \cdots ds_n A_{\mu_1}(x(s_1)) \frac{dx^{\mu_1}}{ds_1} \cdots A_{\mu_n}(x(s_n)) \frac{dx^{\mu_n}}{ds_n}$$

$$= 1 + ig \int_{0}^{1} ds_1 A_{\mu_1}(x(s_1)) \frac{dx^{\mu_1}}{ds_1} + (ig)^2 \int_{0}^{1} ds_1 \int_{0}^{s_1} ds_2 A_{\mu_1}(x(s_1)) \frac{dx^{\mu_1}}{ds_1} A_{\mu_2}(x(s_2)) \frac{dx^{\mu_2}}{ds_2} + \cdots$$

$$= 1 + ig \int_{0}^{1} ds_1 A_{\mu_1}(x(s_1)) \frac{dx^{\mu_1}}{ds_1} + \frac{(ig)^2}{2} P \int_{0}^{1} ds_1 \int_{0}^{1} ds_2 A_{\mu_1}(x(s_1)) \frac{dx^{\mu_1}}{ds_1} A_{\mu_2}(x(s_2)) \frac{dx^{\mu_2}}{ds_2} + \cdots$$

where $\int_{0}^{1} ds_1 \int_{0}^{s_1} ds_2 = \int_{0}^{1} ds_2 \int_{s_2}^{1} ds_1$ is used and then rename $s_1 \leftrightarrow s_2$.

Like $U(t, t_0)$ which satisfies $\frac{d}{dt} U(t, t_0) = -i H_I(t) U(t, t_0)$, $U_P(x(s), x(0))$ satisfies:

$$\frac{d}{ds} U_P(x(s), x(0)) = ig \frac{dx^\mu}{ds} A_\mu(x(s)) U_P(x(s), x(0))$$

It is easy to see this using previous equation.
**Wilson line and Wilson loop: Non-abelian**

\( U_P(x(s), x(0)) \) satisfies the transformation law under gauge transformation:

\[ U_P(x(s), x(0), A') = V(x(s))U_P(x(s), x(0), A)V^\dagger(0) \]

Proof: They satisfy the same first order equation and the same initial condition.

(1) equation of \( U_P(x(s), x(0), A) \)

\[
\frac{d}{ds} U_P(x(s), x(0), A) = \frac{dx^\mu}{ds} \frac{\partial}{\partial x^\mu} U_P(x(s), x(0), A) \\
= ig \frac{dx^\mu}{ds} A_\mu(x(s))U_P(x(s), x(0), A)
\]

the second eq. used the result of previous slide. i.e.

\[
\frac{dx^\mu}{ds} \left( \frac{\partial}{\partial x^\mu} - igA_\mu(x(s)) \right) U_P(x(s), x(0), A) = \frac{dx^\mu}{ds} (D_\mu U_P(x, x(0), A)) = 0
\]

Equation for \( V(x(s))U_P(x(s), x(0), A)V^\dagger(x(0)) \)

\[
\frac{dx^\mu}{ds} D'_\mu \left( V(x(s))U_P(x(s), x(0), A) \right)V^\dagger(x(0)) = \frac{dx^\mu}{ds} V(x(s))(D_\mu U_P(x(s), x(0), A))V^\dagger(x(0)) \\
= 0
\]

Equation for \( U_P(x(s), x(0), A') \)

\[
\frac{dx^\mu}{ds} D'_\mu (U_P(x(s), x(0), A')) = 0
\]

(2) Initial condition :

\[ U_P(x(0), x(0), A') = I \ , \ V(x(0))U_P(x(0), x(0), A)V^\dagger(x(0)) = I. \]
Wilson line and Wilson loop: Non-abelian

Non-abelian Wilson loop:

- If the path is a loop,

\[ U_P(y, y) = P \exp\{ig \int_P dx^\mu A_\mu(x)\} \]

- Now \( U_P(y, y) \) is a matrix, and the integral is path ordered. We can not use stokes's theorem.

- Transformation: \( U_P(y, y) \rightarrow V(y)U_P(y, y)V^\dagger(y) \), not invariant because the matrices in general do not commute.

- Construct gauge inv.:

\[ tr(U_P(y, y)) \rightarrow tr(V(y)U_P(y, y)V^\dagger(y)) = tr(U_P(y, y)), \text{ gauge invariant.} \]
Relation of Wilson loop and Field strength:
As in $U(1)$, take an infinitesimal square:
Infinitesimally, the same as before

$$U(x + \epsilon n, x) = \exp \{ i \epsilon n^\mu gA_\mu (x + \frac{\epsilon}{2} n) + O(\epsilon^3) \}$$

$$= I + i \epsilon n^\mu gA_\mu (x + \frac{\epsilon}{2} n) - \frac{\epsilon^2}{2}(n^\mu gA_\mu (x + \frac{\epsilon}{2} n))^2 + O(\epsilon^3).$$

$$U(x) = (I - i \epsilon gA_2 (x + \frac{\epsilon}{2} \hat{2}) - \frac{\epsilon^2}{2} (gA_2 (x + \frac{\epsilon}{2} \hat{2}))^2)$$

$$\times (I - i \epsilon gA_1 (x + \frac{\epsilon}{2} \hat{1} + \epsilon \hat{2}) - \frac{\epsilon^2}{2} (gA_1 (x + \frac{\epsilon}{2} \hat{1} + \epsilon \hat{2}))^2)$$

$$\times (I + i \epsilon gA_2 (x + \frac{\epsilon}{2} \hat{2} + \epsilon \hat{1}) - \frac{\epsilon^2}{2} (gA_2 (x + \frac{\epsilon}{2} \hat{2} + \epsilon \hat{1}))^2)$$

$$\times (I + i \epsilon gA_1 (x + \frac{\epsilon}{2} \hat{1}) - \frac{\epsilon^2}{2} (gA_1 (x + \frac{\epsilon}{2} \hat{1}))^2) + O(\epsilon^3)$$

$$= 1 + ig\epsilon^2 \left( \partial_1 A_2(x) - \partial_2 A_1(x) - igA_1(x)A_2(x) + igA_2(x)A_1(x) \right) + O(\epsilon^3)$$

$$= 1 + ig\epsilon^2 F_{12} + O(\epsilon^3)$$

$\epsilon^2 A^2$ cancelled. Thus, from the transformation law of

$$U(x, x) \rightarrow V(x)U(x, x)V^\dagger(x),$$

the transformation of $F_{\mu\nu}$ is

$$F_{\mu\nu} \rightarrow V(x)F_{\mu\nu}V^\dagger(x).$$
Basic facts about Lie algebras

- Abstract group: A set of elements $G = \{a, b, c \ldots\}$, with a product operation $\cdot : G \times G \mapsto G$
  - closure: $a \cdot b \in G$
  - Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
  - Identity element $e$: $e \cdot a = a \cdot e = a$
  - Inverse $a^{-1} \cdot a = a \cdot a^{-1} = e$ for $\forall a \in G$.

- Abelian group: $a \cdot b = b \cdot a$, $\forall a, b \in G$, e.g. $U(1)$, $e^{i\alpha}$.

- Non-Abelian group: $\exists a, b$, $a \cdot b \neq b \cdot a$. 

Basic facts about Lie algebras

- Lie group: $G$ is an abstract group.
  - $G$ is an analytic manifold.
  - map $(x, y) \mapsto x \cdot y^{-1}$ is analytic. $(x \mapsto x^{-1}, (x, y) \mapsto z = x \cdot y$ are analytic.)

  e.g.
  (1) $U(1)$, $e^{i\alpha}, \alpha \in [0, 2\pi)$, a circle — a manifold. $e^{i\alpha_1}e^{i\alpha_2} = e^{i\alpha_3}$, $\alpha_3 = (\alpha_1 + \alpha_2) \mod 2\pi$
  (2) $SU(2)$, any element can be written as $M(a) = a^0 I + ia^i \sigma^i$, det $M = 1$ \Rightarrow $\sum(a^\mu)^2 = 1$, $SU(2) = S^3$. $M(a)M(b) = M(c)$.

- Composite function: $g_1(\alpha_1) \cdot g_2(\alpha_2) \mapsto g_3(\alpha_3)$, $\alpha_i \in R^n$ coordinate of the elements. $\alpha_3 = f(\alpha_1, \alpha_2)$ composite function.

  e.g. In $U(1)$ example: $\alpha_3 = (\alpha_1 + \alpha_2) \mod 2\pi$ is the composite function. In $SU(2)$, $c = f(a, b)$ is the composite function.

  If we choose the coordinate of $e$ to be $\alpha^i(e) = 0$:
  $f^i(\alpha, 0) = f^i(0, \alpha) = \alpha^i$, $f(\alpha, f(\beta, \gamma)) = f(f(\alpha, \beta), \gamma)$.
  $f(\alpha, \beta)$ can be expanded around $(0, 0)$,
  $f^i(\alpha, \beta) = \alpha^i + \beta^i + h^i_{jk} \alpha^j \beta^k + \ldots$
Basic facts about Lie algebras

- A matrix representation of a group: for $\forall g$ there is a matrix $U_g$, the product of group is mapped to the matrix product, and is a homomorphism, i.e. $U_{g_1} U_{g_2} = U_{g_1 \cdot g_2}$.
- Using coordinates $\alpha_i$ for $g_i$: $U(\alpha_1) U(\alpha_2) = U(f(\alpha_1, \alpha_2))$.
- Matrices $U(\alpha)$ can be expanded around the identity element, $\alpha(e) = 0$, $U(0) = I$:

$$U(x) = I + ix^a t^a + \frac{1}{2} x^a x^b t_{ab} + \ldots$$

It can be proved that the commutator $[t_a, t_b] = iC^c_{ab} t_c$, $C^c_{ab} = -h^a_{bc} + h^a_{cb}$ is independent of the coordinates and is called structure constants.
All $t^a$ are called generators of the Lie group and constitute a Lie algebra.
Basic facts about Lie algebras

Lie Algebra $\mathfrak{g}$: elements $t_a$

- all $t_a$ generate a linear vector space. (field $K$) A set of linear independent $t_a$.
- Lie product: $\forall X, Y \in \mathfrak{g}$, $(X, Y) \mapsto [X, Y] \in \mathfrak{g}$. satisfies:
  - (a) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$;
  - (b) $[X, Y] = -[Y, X]$.
  - (c) Jacobi id.: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Properties:

- The dimension of the Lie algebra $\equiv$ the number of linearly independent elements (Lie algebra as a linear vector space).
- Abelian Lie algebra $[X, Y] = 0$, for $\forall X, Y \in \mathfrak{g}$.
- Real Lie algebra: $K = \mathbb{R}$; Complex Lie algebra: $K = \mathbb{C}$.
- Matrix representation: $\forall X \in \mathfrak{g}$, there is a matrix $X$, $X \mapsto X$, which is a vector space isomorphism. And Lie product, $[X, Y] \mapsto XY - YX \equiv [X, Y]$, which is a homomorphism.
- In matrix representation, near identity of the Lie group, every element of the corresponding Lie algebra is mapped to an element of the Lie group through the Exponential map of matrix: $a^i X^i \mapsto \exp\{a^i X^i\}$
Basic facts about Lie algebras

- Adjoint representation: Choose a set of basis of the Lie algebra \( \{X_a\} \), the Lie product \([X_a, X_b] = iC^c_{ab} X_c\) can be viewed as a linear transformation of \(X_a\) on \(X_b\):
  \[\hat{X}_a (X_b) = iC^c_{ab} X_c,\]
  in matrix rep: \((X_a)^c_b = iC^c_{ab}\)

- From Jacobi id:
  \[ [X_a, [X_b, X_c]] - [X_b, [X_a, X_c]] = [[X_a, X_b], X_c] \]
  \( (X_b)^d_c (X_a)^e_d - (X_a)^d_c (X_b)^e_d = iC^d_{ab} (X_d)^e_c \Rightarrow [X_a, X_b] = iC^c_{ab} X_c \)
  \(X_a\) is a matrix representation of the Lie algebra — adjoint representation.

- The adjoint representation of the Lie algebra generates the adjoint representation of the corresponding Lie group.

- the dimension of the adjoint rep. = the dimension of the Lie algebra.
Basic facts about Lie algebras

- Cartan metric of the adjoint rep.: $X_a$ a set of basis, $g_{ab} = \text{tr}(X_a X_b)$. Killing form: For Real Lie algebra: $\forall X = x^i X_i, Y = y^i X_i$, we can define inner product

\[
(X, Y) = \text{tr}(XY) = g_{ij} x^i y^j,
\]

this is called killing form.

- it can be proved that $\tilde{C}_{ijk} \equiv g_{il} C^l_{jk}$ is antisymmetric in $i, j, k$.

- Semisimple Lie group: simply speaking, can not be decomposed to $G \times U(1)$.

- Semisimple Lie algebra: $\exists X, s.t. [X, Y] = 0, \forall Y \in \mathfrak{g}$

- Simple Lie group: can not be decomposed to $G_1 \times G_2$.

- Simple Lie algebra: can not be decomposed to $X \oplus Y$. $X, Y$ two sub-Liealgebras, generators of the two sets $[X_i, Y_j] = 0$. e.g.: $SU(N)$ simple algebra. $U(2)$ not semisimple, $U(2) = SU(2) \times U(1)$, $so(4) = su(2) \oplus su(2)$ semisimple but not simple. The standard model group $SU(2) \times SU(3) \times U(1)$ not semisimple.
Basic facts about Lie algebras

Finite dimensional semisimple Compact Lie group: Compact, the manifold of the parameters of the group is compact.

- The killing form is positive definite.
- We can choose the basis, $g_{ij} = C(G)\delta_{ij}$. So define $C_{ijk} = \delta_{il}C^l_{jk} = \frac{1}{C(G)}g_{il}C^l_{jk} = C^i_{jk}$ is antisymmetric in $i, j, k$. We do not distinguish upper indices and lower indices now.
- ∃ finite dimensional unitary representation.

From now on, we only consider finite dimensional semisimple Compact Lie group in our discussion of Nonabelian gauge field theory.
Basic facts about Lie algebras

Matrix rep of Lie groups and Lie algebra (finite dim Semisimple compact Lie group):

\( \exists \) unitary finite dim representation for Lie group.

Unitary matrices \( U_g : g \in G \mapsto U_g \) is a homomorphism:

- Irreducible representation \( U \): There does not exist a set of basis such that all the representation matrices are of the same block diagonal form at the same time.

- In any unitary rep \( r \), the generator \( t_r^a \) for the semisimple group is hermitian and traceless. \( \text{tr}(t_r^a t_r^b) = D^{ab} \) is positive definite. We can choose a set of basis \( t_r^a \) to diagonalize \( D^{ab} \). If we choose these basis, in other representations \( D^{ab} \) is also diagonalized. We define the normalization for each representation

\[
\text{tr}(t_r^a t_r^b) = C(r) \delta^{ab}.
\]

In rep. \( r \), the structure constants can be represented as

\[
f^{abc} = -\frac{i}{C(r)} \text{tr}\{[t_r^a, t_r^b] t_r^c\}.
\]
For rep \( r \), there is a associated **Conjugate representation** \( \bar{r} \). Complex Conjugate \( M^*_g \):

\[
M_{g_1}M_{g_2} = M_{g_1 \cdot g_2} \Rightarrow M^*_g M^*_g = M^*_{g_1 \cdot g_2}
\]

is also a representation.

Infinitesimal trans for \( r \): \( \phi_r \rightarrow (1 + i\alpha^a t^a_r)\phi_r \)

Generator for Complex conjugate rep: from \( \phi^*_r \rightarrow (1 - i\alpha^a t^{a*}_r)\phi^*_r \),

\[
t^a_{\bar{r}} = -(t^a_r)^* = -(t^a_r)^T.
\]

\( \phi^*_r, \phi_r \) is invariant.

Real irrep: if rep \( r \) and \( \bar{r} \) are equivalent.

\( \exists U \) unitary, s.t. \( t^a_{\bar{r}} = U t^a_r U^\dagger \), for \( \forall t^a_r \). A real rep. \( \exists G_{ab} \), s.t. \( \psi^a_r \phi^b_r G_{ab} \)

invariant.

If \( G_{ab} \) is symmetric — strictly real. We can find a set of basis s.t. \( t^a_{\bar{r}} = t^a_r \).

If \( G_{ab} \) is antisymmetric — peudo-real. e.g. SU(2) fundamental rep is peudoreal. ( see Srednicki for details).

Adjoint representation is always a real representation (strictly real):

\[
(t^b_G)_{ac} = (X^b)_{ac} = i f_{abc}, \quad ((t^*_G)^b)_{ac} = -(t^b_G)_{ac} = i f_{abc} = (t^b_G)_{ac}
\]
**Basic facts about Lie algebras**

**Adjoint rep.:**

- For fields $\phi_a$ in the representation space of adjoint representation. It transforms infinitesimally: $\phi_a \rightarrow (1 + i\alpha^b t^b_G)_{ac} \phi_c = (\delta_{ac} - \alpha^b f^{abc}) \phi_c$.

  In matrix form: $\phi = \phi_a t^a_r$, transformation $V = \exp\{i\alpha^a t^a_r\}$

  $\phi \rightarrow V \phi V^\dagger = (1 + i\alpha^b t^b_r) \phi (1 - i\alpha^b t^b_r) + O(\alpha^2) = \phi + i[\alpha, \phi] + O(\alpha^2)$

- Covariant derivative for adjoint rep.:

  \[
  D_\mu \phi_a = \partial_\mu \phi_a - igA^b_\mu (t^b_G)_{ac} \phi_c = \partial_\mu \phi_a + g f^{abc} A^b_\mu \phi_c.
  \]

  In matrix form: $D_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi] = [D_\mu, \phi]$.

- The form of infinitesimal gauge transformation for $A^a_\mu : \alpha^a(x)$ in adjoint representation:

  $A^a_\mu \rightarrow A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = A^a_\mu + \frac{1}{g} (D_\mu \alpha)^a$

- The EOM of $A^a_\mu$: $\partial^\mu F^a_{\mu \nu} + g f^{abc} A^b_\mu F^c_{\mu \nu} = -g j^a_\nu \Rightarrow (D^\mu F^a_{\mu \nu})^a = -g j^a_\nu$

  \[
  D^\nu j^a_\nu = -\frac{1}{g} (D^\nu D^\mu F^a_{\mu \nu})^a = -\frac{1}{2g} ([D^\nu, D^\mu] F^a_{\mu \nu})^a
  \]

  \[
  = -\frac{i}{2} F^{b, \mu \nu} (t^b_G)_{ac} F^c_{\mu \nu} = \frac{1}{2} f^{abc} F^{b, \mu \nu} F^c_{\mu \nu} = 0
  \]

- Bianchi Id.: From Jacobi id

  \[
  [D_\mu, [D_\nu, D_\rho]] \epsilon^{\mu \nu \rho \delta} = 0 \Rightarrow (D_\mu F_{\nu \rho})^a \epsilon^{\mu \nu \rho \delta} = 0
  \]
Basic facts about Lie algebras

Classical group: SU(N), SO(N), Sp(N) groups

- U(N), Unitary transformation of N-dimensional vectors: $\xi, \eta$ two complex N-vectors.
  Unitary transformations $\xi_a \rightarrow U_{ab}\xi_b$, $\eta_a \rightarrow U_{ab}\eta_b$: preserve the inner product: $\xi_a^*\eta_a$.
  $U(1)$ subgroup of $U(N)$: $\xi_a \rightarrow e^{i\alpha}$
  $SU(N)$: $\det U = 1$, simple group.
  Generators of $SU(N)$: $t^a$, traceless hermitian matrices, dimension $N^2 - 1$.
  The $N$-vector space ($N$): fundamental representation (space), dimension $N$.
  Conjugate rep: anti-fundamental representation ($\bar{N}$).

- O(N), Orthogonal transformations of N-dim real vectors: real $N$-vectors.
  Subgroup of $U(N)$, preserves $\xi_a E_{ab}\eta_b$, $E_{ab} = \delta_{ab}$.
  $SO(N)$: Rotations $\det O = 1$. Different from $O(N)$ by some reflections.
  Generators: $t^a$, antisymmetric hermitian matrices, dimension $N(N - 1)/2$.
  Fundamental rep: $N$-dim real space, Real representation.

- Sp(N): Symplectic transformation.

Exceptional group: $G_2, F_4, E_6, E_7, E_8$. 
Basic facts about Lie algebras

Casmir operator: $T^a$ generator in any representation. Define operator $T^2 = T^a T^a$. Commutes with all generators:

$$ [T^b, T^a T^a] = (i f^{bac} T^c) T^a + T^a (i f^{bac} T^c) = i f^{bac} \{T^c, T^a\} = 0 $$

In any irrep $r$, \( t^a_r t^a_r = C_2(r) \cdot 1_r \), $C_2(r)$ constant. $T^2$: quadratic Casimir operator.

- **Adjoint rep:** $i f^{aecd} i f^{decb} = f^{aecd} f^{bcde} = C_2(G) \delta^{ab}$.
- We have defined the normalization: $tr(t^a_r t^b_r) = C(r) \delta^{ab}$. From $t^a_r t^a_r = C_2(r) \cdot 1_r$

  $$ \sum_{a,b} \delta_{ab} (\cdots): \Rightarrow d(r) C_2(r) = d(G') C(r). $$

- SU(N): fundamental rep. N: $d(N) = N$, $d(G) = N^2 - 1$,
  Normalization $tr(t^a_N t^b_N) = \frac{1}{2} \delta^{ab}$, $\Rightarrow C_2(N) = \frac{N^2 - 1}{2N}$.
- Adjoint rep: SU(N): $C_2(G') = C(G') = N$
Basic facts about Lie algebras

Adjoint rep: SU(N): $C_2(G) = C(G) = N$

* Need to look at the product rep of $N \otimes \bar{N} = 1 \oplus \text{Adj}$,

In general: $r_1 \otimes r_2 = \sum \oplus r_i'$

Generators for product rep: $t^a_{r_1 \otimes r_2} = t^a_{r_1} \otimes 1_{r_2} + 1_{r_1} \otimes t^a_{r_2}$.

$$
(t^a_{r_1 \otimes r_2})^2 = (t^a_{r_1})^2 \otimes 1_{r_2} + 2 t^a_{r_1} \otimes t^a_{r_2} + 1_{r_1} \otimes (t^a_{r_2})^2
= C_r(r_1) 1_{r_1} \otimes 1_{r_2} + 2 t^a_{r_1} \otimes t^a_{r_2} + 1_{r_1} \otimes 1_{r_2} C_2(r_2)
= \sum_i \oplus (t^a_{r_i})^2 = \sum_i \oplus (C_2(r_i)) 1_{r_i}
$$

For Casimir for product rep:

$$
tr((t^a_{r_1 \otimes r_2})^2) = (C_2(r_1) + C_2(r_2)) d(r_1) \cdot d(r_2) = \sum C_2(r_i') d(r_i').
$$

Used on $N \otimes \bar{N} = 1 \oplus \text{Adj}$: ($C_2(1) = 0$):

$$
(2 \cdot \frac{N^2 - 1}{2N}) N^2 = 0 + C_2(G)(N^2 - 1) \Rightarrow C_2(G) = C(G') = N.
$$
V = \exp \{ i\alpha^a(x)\sigma^a \}, prove \ tr(V^\dagger \partial_\mu V) = 0. Expand \ tr(V^\dagger \partial_\mu V)

\[ tr(V^\dagger \partial_\mu V) = tr \left( \sum_{n=0}^{\infty} \frac{1}{n!} (-i\alpha^a(x)\sigma^a)^n \partial_\mu \sum_{m=0}^{\infty} \frac{1}{m!} (i\alpha^a(x)\sigma^a)^m \right) \]

\[ = tr \left( \sum_{n=0}^{\infty} \frac{1}{n!} (-i\alpha^a(x)\sigma^a)^n \sum_{m=1}^{\infty} \sum_{l=0}^{m-1} \frac{1}{m!} (i\alpha^a(x)\sigma^a)^l (i\partial_\mu \alpha^a(x)\sigma^a) (i\alpha^a(x)\sigma^a)^{m-l-1} \right) \]

\[ = tr \left( \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{\infty} \sum_{l=0}^{m-1} (-1)^n \frac{1}{(m-1)!} (i\partial_\mu \alpha^a(x)\sigma^a) (i\alpha^a(x)\sigma^a)^{m+n-1} \right) \]

\[ = tr \left( (i\partial_\mu \alpha^a(x)\sigma^a) \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{\infty} \frac{(-1)^n}{m!} (i\alpha^a(x)\sigma^a)^{m+n} \right) \]

\[ = tr \left( (i\partial_\mu \alpha^a(x)\sigma^a) VV^\dagger \right) \]

\[ = tr \left( (i\partial_\mu \alpha^a(x)\sigma^a) \right) \]

\[ = 0 \]