Quantization of Non-abelian Gauge Theories: BRST Symmetry

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May 14, 2019
The gauge fixed Faddeev-Popov Lagrangian is not invariant under a general gauge transformation, though the path integral is invariant.

\[
L_{FB} = -\frac{1}{4} (F_{\mu\nu}^i)^2 + i \bar{\psi} \slash{D} \psi - m \bar{\psi} \psi - \bar{c}^a \Box c^a - gf^{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2
\]

There is a residue symmetry: To see this, we introduce an auxiliary field \( B \): No kinetic term of \( B \), and can be integrated out to obtain the \( L_{FB} \).

\[
L_{BRST} = -\frac{1}{4} (F_{\mu\nu}^i)^2 + i \bar{\psi} \slash{D} \psi - m \bar{\psi} \psi - \bar{c}^a \partial^\mu D^{ac}_\mu c^c + \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu
\]

Path integral: \( \int [dB][d\bar{\psi}][d\psi][dA_\mu][dc][d\bar{c}] \exp \{ i \int d^4x \mathcal{L}_{BRST} \} \)

\[
\frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu = \frac{\xi}{2} (B^a + \frac{1}{\xi} \partial^\mu A^a_\mu)^2 - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2
\]

EOM of \( B^a \): \( B^a = -\frac{1}{\xi} (\partial^\mu A^{a\mu}) \)
**BRST Symmetry**

\[
\mathcal{L}_{BRST} \sim \mathcal{L}_{YM} - \bar{c}^a \partial^\mu D^{ac}_\mu c^c + \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu
\]

- The \( \mathcal{L}_{BRST} \) has a Global BRST symmetry: infinitesimal parameter \( \epsilon \), an anticommuting grassman number, independent of \( x \). BRST transformation and classical BRST operator \( Q \): (\( i\epsilon, c^a, \bar{c}^a \) are real grassman variables, Hermite conjugate for grassman number \((\chi_\psi)^\dagger = \psi^\dagger \chi^\dagger\))

\[
\delta A^a_\mu = \epsilon D^{ac}_\mu c^c = \epsilon QA^a_\mu, \quad \delta \psi = i\epsilon c^a \bar{t}^a \psi = \epsilon Q\bar{\psi}, \quad \delta \bar{\psi} = - i\epsilon c^a \bar{t}^a \psi = \epsilon Q\bar{\psi},
\]

\[
\delta c^a = - \frac{1}{2} g f^{abc} c^b c^c = \epsilon Qc^a, \quad \delta \bar{c}^a = \epsilon B^a = \epsilon Q\bar{c}^a,
\]

\[
\delta B^a = 0 \equiv \epsilon QB^a.
\]

\[
Q = D^{ac}_\mu c^c(y) \frac{\delta}{\delta A^a_\mu(y)} + igc^a t^a_{ij} \psi^i_\alpha(y) \frac{\delta}{\delta \psi^i_\alpha(y)} + igc^a \bar{\psi}^i_\alpha(y) t^a_{ji} \frac{\delta}{\delta \bar{\psi}^i_\alpha(y)}
\]

\[- \frac{1}{2} g f^{abc} c^b c^c(y) \frac{\delta}{\delta c^a(y)} + B^a(y) \frac{\delta}{\delta \bar{c}^a(y)}
\]

(repeated indices are summed, including continuous indices \( y \).)

**hw.** (1) \( Q^2 = 0 \). (2) Check: After integrated out \( B \) (substitute the EOM of \( B \)), the action is still invariant under this transform. Calculate \( Q^2 \bar{c} \) and \( Q^3 \bar{c} \).
The transformation of $A_\mu$ and $\psi$ is just a gauge transformation with parameter $\alpha^a(x) = g \epsilon c^a(x)$, $\mathcal{L}_{YM}$ is invariant.

$$
\delta(B^a \partial^\mu A^a_\mu) = B^a \partial^\mu (\epsilon D^a_{\mu c} c^c)
$$

$$
\delta(\bar{c}(-\partial^\mu D^a_{\mu c} c^c)) = -\epsilon B^a \partial^\mu (D^a_{\mu c} c^c) - \bar{c}^a \partial^\mu D^a_{\mu c} \left( -\frac{1}{2} g \epsilon f^{c b' c' b' c'} \right)
$$

$$
= \ldots + g \epsilon \bar{c}^a \partial^\mu (\star)
$$

$$
\star = -\frac{1}{2} \partial_{\mu} (f^{a b' c'} c^b c'^c) - \frac{1}{2} g A^d_\mu f^{a d c} f^{c b' c'} c^b c'^c
$$

$$
+ (\partial_{\mu} c^b) f^{a b c} c^c + g A^d_\mu f^{b d e} c^e f^{a b c} c^c
$$

$$
= g A^d_\mu ( -\frac{1}{2} f^{a d c} f^{c b' c'} + f^{c d b} f^{a c' c'} ) c^b c'^c
$$

$$
= g A^d_\mu ( -\frac{1}{2} f^{a d c} f^{c b' c'} - \frac{1}{2} f^{b' d c} f^{c c' a} - \frac{1}{2} f^{c' d c} f^{c a b'} ) c^b c'^c = 0
$$
BRST Symmetry

- It can be proved that $Q^2 = 0$, $Q$ carry ghost number 1. Grassman odd operator.
- The $\mathcal{L}_{BRST}$ can be recast into

$$
\mathcal{L}_{BRST} = \mathcal{L}_{YM}[A, \psi] + Q\Psi; \quad \Psi = \bar{c}^a \partial^\mu A^a_\mu + \frac{\xi}{2} \bar{c}^a B^a 
$$

$\mathcal{L}_{YM}[A, \psi]$ is the gauge invariant Yang-Mills Lagrangian. In general, for
gauge fixing $G^a[ A_\mu ] = f^a( A_\mu ) - \omega^a(x)$, $\Psi = \bar{c}^a f^a[ A_\mu ] + \frac{\xi}{2} \bar{c}^a B^a$.

$$
Q\Psi = B^a f^a[ A_\mu ] - \bar{c}^a \frac{\delta f^a[ A_\mu ]}{\delta A^b_\mu(y)} D^b_{\mu} c^c(y) + \frac{\xi}{2} B^a B^a
$$

(repeated indices are summed, including continuous indices $y$.)
- Now the invariance of $\mathcal{L}_{BRST}$ is BRST invariant can be seen easily:
  (1) Since the BRST trans for $A_\mu$ and $\psi$ is just a special gauge transformation with parameter $:\alpha^a(x) = g{c}^a(x)$, $\mathcal{L}_{YM}$ is invariant.

$$
\delta_{BRST} \mathcal{L}_{YM} = \epsilon Q \mathcal{L}_{YM} = 0
$$

(2) Since $Q^2 = 0$, $\delta_{BRST} Q\Psi = Q^2 \Psi = 0$. 

BRST Symmetry

- In the whole fock state space, including unphysical polarization and ghost excitations, we can define a hermitian operator (quantum operator) $Q$,

$$\delta_{BRST}\phi = \epsilon Q\phi = i[\epsilon Q, \phi]$$

From $Q^2\phi = 0$, we have

$$[Q, [Q, \phi]_{\pm}]_\mp = -Q^2\phi = 0$$

$\Rightarrow [Q^2, \phi] = Q^2\phi - \phi Q^2 = 0$ for any $\phi$.

Either $Q^2 = 0$ or $Q^2 = I$, $Q^2$ ghost charge $= 2$, it can not be $I$. So $Q^2 = 0$.

- We want to identify the physical states in the fock state space, in a Lorentz covariant way.

- The physical amplitude should be invariant under the change of the gauge fixing function, changing of the gauge, or changing of $\xi$ — an arbitrary change of the $\Psi$, $\tilde{\delta}\Psi$.

$$0 = \tilde{\delta}\langle \alpha_{\text{ph}} | \beta_{\text{ph}} \rangle = i\langle \alpha_{\text{ph}} | Q\tilde{\delta}\Psi | \beta_{\text{ph}} \rangle = -\langle \alpha_{\text{ph}} | [Q, \tilde{\delta}\Psi]_+ | \beta_{\text{ph}} \rangle$$

Because $\tilde{\delta}\Psi$ is arbitrary, the only possibility is: $Q|\text{Phy states}\rangle = 0$.

- For any physical state $|\alpha_{\text{ph}}\rangle$, we can not physically distinguish it from $|\alpha_{\text{ph}}\rangle + Q|\gamma\rangle$, ($\gamma$ any fock state), since

$$\langle \beta_{\text{ph}} | \alpha_{\text{ph}} + Q|\gamma\rangle = \langle \beta_{\text{ph}} | \alpha_{\text{ph}} \rangle + \langle \beta_{\text{ph}} | Q|\gamma\rangle = \langle \beta_{\text{ph}} | \alpha_{\text{ph}} \rangle + \langle Q\beta_{\text{ph}} | \gamma \rangle = \langle \beta_{\text{ph}} | \alpha_{\text{ph}} \rangle$$
**BRST Symmetry**

- **Physical state:**
  
  \[
  \begin{aligned}
  \text{Physical state condition: } & Q|\text{Phy states}\rangle = 0 \\
  |\alpha_{ph}\rangle & \sim |\alpha_{ph}\rangle + Q|\gamma\rangle
  \end{aligned}
  \]

- The whole fock space is divided into 3 parts: $H_0$, $H_1$, $H_2$:
  
  $H_0$: $|\alpha_0\rangle$, $Q|\alpha_0\rangle = 0$, but $|\alpha_0\rangle \neq Q|\gamma\rangle$, $\mathcal{H}_{ph}$ is a subspace of $H_0$.
  
  $H_1$: $|\alpha_1\rangle$, $Q|\alpha_1\rangle \neq 0$. Unphysical states.
  
  $H_2$: $|\alpha_2\rangle = Q|\gamma\rangle$.
  
  (1) $Q|\alpha_2\rangle = 0$, (2) zero norm: $\langle \alpha_2|\alpha_2\rangle = \langle \gamma|Q^\dagger Q|\gamma\rangle = \langle \gamma|Q^2|\gamma\rangle = 0$,
  
  (3) orthogonal to $H_0$, $\langle \alpha_0|\alpha_2\rangle = \langle \alpha_0|Q|\gamma\rangle = \langle Q\alpha_0|\gamma\rangle = 0$.

- **BRST closed states:** $Q|\alpha\rangle = 0$, Kernel of $Q$
  
  BRST exact states: $|\alpha\rangle = Q|\gamma\rangle$, Image of $Q \subset$ kernel of $Q$

  So the physical state space $\mathcal{H}_{ph}$ is defined as

  \[
  \mathcal{H}_{ph} = \frac{\{|\alpha\rangle : Q|\alpha\rangle = 0\}}{\{|\beta\rangle = Q|\gamma\rangle\}} = \frac{\text{Kernel of } Q}{\text{Image of } Q} = \text{Cohomology of } Q
  \]
BRST Symmetry

(reference Weinberg, The Quantum theory of fields, II)
Consider the asymptotic particle state, in the $t \to \pm \infty$, interactions are adiabatically turned off $g \to 0$. BRST transformation:

$$[Q, A^a_{\mu}] = -i \partial_\mu c^a, \quad [Q, c^a]_+ = 0, \quad [Q, \bar{c}]_+ = -i B^a \xrightarrow{EOM} -\frac{i}{\xi} \partial_\mu A^{a,\mu}$$

$$A^\mu(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (a^\mu_k e^{-ik \cdot x} + a^\mu_k^\dagger e^{ik \cdot x})$$

$$c(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (c_k e^{-ik \cdot x} + c_k^\dagger e^{ik \cdot x}), \quad \bar{c}(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (\bar{c}_k e^{-ik \cdot x} + c_k^\dagger e^{ik \cdot x})$$

$$[Q, a^\mu_k] = k^\mu c_k^\dagger, \quad [Q, c_k^\dagger] = \frac{1}{\xi} k \cdot a_k^\dagger, \quad [Q, c_k] = 0$$
\[ [Q, a_{k}^{\mu}] = k^{\mu} c_{k}^{\dagger}, \quad [Q, c_{k}^{\dagger}] = \frac{1}{\xi} k \cdot a_{k}^{\dagger}, \quad [Q, c_{k}^{\dagger}] = 0 \]

The superscript \( a \) and subscript \( k \) are suppressed in the following. A physical state: \( |\psi\rangle \), satisfies \( Q|\psi\rangle = 0 \).

- Adding another vector boson, \( |e, \psi\rangle = e_{\mu} a_{\mu}^{\dagger} |\psi\rangle \). Physical condition: \( 0 = Q|e, \psi\rangle = e_{\mu} k^{\mu} c_{k}^{\dagger} |\psi\rangle \), so \( k^{\mu} \cdot e = 0 \). ( \( k^{\mu} \cdot e^{+} = 0, k^{\mu} \cdot e^{-} \neq 0 \).)
  for \( e_{\mu} \sim e^{+\mu} \), \( k^{\mu} \cdot e^{+} = 0 \), \( e^{+\mu} a_{\mu}^{\dagger} |\psi\rangle \) satisfies physical condition.
  for \( e_{\mu} \sim e^{-\mu} \), \( k^{\mu} \cdot e^{-} \neq 0 \), \( e^{-\mu} a_{\mu}^{\dagger} |\psi\rangle \notin \mathcal{H}_{1} \) is not a physical state.
  for \( e_{\mu} \sim e_{T\mu} \), \( k^{\mu} \cdot e_{T} = 0 \), \( e_{T\mu} a_{\mu}^{\dagger} |\psi\rangle \) satisfies physical condition.
- Add a ghost: \( c_{k}^{\dagger} |\psi\rangle = Q|e, \psi\rangle / (e \cdot k) \), (\( e \cdot k \neq 0, \)) is BRST exact \( \sim 0 \), \( \in \mathcal{H}_{2} \).
- \( \therefore Qc_{k}^{\dagger} |\psi\rangle = \frac{1}{\xi} k^{\mu} a_{\mu}^{\dagger} |\psi\rangle \), BRST exact, \( e^{+\mu} a_{\mu}^{\dagger} |\psi\rangle \in \mathcal{H}_{2} \).
  \( \therefore |e_{\mu} + \alpha k^{\mu}, \psi\rangle = |e_{\mu}, \psi\rangle + \xi Qa_{\mu} c_{k}^{\dagger} |\psi\rangle \sim |e_{\mu}, \psi\rangle \), this is the gauge equivalent condition for the vector boson.
- \( Qc_{k}^{\dagger} |\psi\rangle = \frac{1}{\xi} k^{\mu} a_{\mu}^{\dagger} |\psi\rangle \neq 0 \Rightarrow c_{k}^{\dagger} |\psi\rangle \) is not physical \( \in \mathcal{H}_{1} \).
  \( e_{T} \cdot a_{k}^{\dagger} |\psi\rangle \in \mathcal{H}_{0}, \quad c_{k}^{\dagger} |\psi\rangle \in \mathcal{H}_{1}, \quad e^{-} \cdot a_{k}^{\dagger} |\psi\rangle \in \mathcal{H}_{1}, \)
  \( e^{+} \cdot a_{k}^{\dagger} |\psi\rangle \in \mathcal{H}_{2}, \quad c_{k}^{\dagger} |\psi\rangle \in \mathcal{H}_{2} \).

For each equivalent class of states, we choose \( e_{T} \cdot a_{k}^{\dagger} |\psi\rangle \) to represent the class, \( \mathcal{H}_{ph} \) can be restricted to states with only transverse polarizations \( |A; tr\rangle \sim e_{T} \cdot a_{k}^{\dagger} |\psi\rangle \). So the physical \( \mathcal{H}_{ph} \subset H_{0} \) is ghost free.
We can choose the external polarizations contain only the transverse polarizations.

Because the hamiltonian $H$ commutes with $Q$, then $QS = SQ$. $QS|A, tr\rangle = SQ|A, tr\rangle = 0 \in \mathcal{H}_{ph}$, $S|A, tr\rangle = |C; tr\rangle + Q_{\text{exact}}$.

The $S$ matrix in the full fock space is (pseudo-)unitary: $S \cdot S^\dagger = S^\dagger \cdot S = 1$. (intermediate states include a set of orthogonal independent basis $\{C\}$ of the whole fock space $H_0 + H_1 + H_2$).

The restricted $S$-matrix on the transversely polarized states is also Unitary: $S|A; tr\rangle = |C; tr\rangle + Q|x\rangle$, $S|B; tr\rangle = |D; tr\rangle + Q|\psi\rangle$

$|C; tr\rangle = \sum |X; tr\rangle |X; tr\rangle \langle X; tr|S|A; tr\rangle$, $|D; tr\rangle = \sum |X; tr\rangle |X; tr\rangle \langle X; tr|S|B; tr\rangle$

$$\delta(A, B) = \langle A; tr|B; tr\rangle = \langle A; tr|S^\dagger \cdot S|B; tr\rangle = \langle C; tr + Q|x|D; tr + Q|\psi\rangle$$

$$= \langle C; tr|D; tr\rangle = \sum |X; tr\rangle \langle A; tr|S|X; tr\rangle \langle X; tr|S|B; tr\rangle$$

(See Kugo and Ojima, Prog. Theo. Phys., V60, NO.6 1869 for details.)