Quantization of Non-abelian Gauge Theories: BRST Symmetry

Zhiguang Xiao

May 14, 2020
The gauge fixed Faddeev-Popov Lagrangian is not invariant under a general gauge transformation, though the path integral is invariant.

\[ \mathcal{L}_{FB} = -\frac{1}{4} (F^i_{\mu\nu})^2 + i \bar{\psi} D \psi - m \bar{\psi} \psi - \bar{c}^a \Box c^a - g f^{abc} \bar{c}^a \partial^\mu (A^b_\mu c^c) - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 \]

There is a residue symmetry: To see this, we introduce an auxiliary field \( B \): No kinetic term of \( B \), and can be integrated out to obtain the \( \mathcal{L}_{FB} \).

\[ \mathcal{L}_{BRST} = -\frac{1}{4} (F^i_{\mu\nu})^2 + i \bar{\psi} D \psi - m \bar{\psi} \psi - \bar{c}^a \partial^\mu D^a_\mu c^c + \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu \]

Path integral: \[ \int [dB][d\bar{\psi}][d\psi][dA_\mu][dc][d\bar{c}] \exp \{ i \int d^4x \mathcal{L}_{BRST} \} \]

\[ \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu = \frac{\xi}{2} (B^a + \frac{1}{\xi} \partial^\mu A^a_\mu)^2 - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 \]

EOM of \( B^a \): \( B^a = -\frac{1}{\xi} (\partial^\mu A^{a\mu}) \)
The $L_{BRST}$ has a Global BRST symmetry: infinitesimal parameter $\epsilon$, an anticommuting grassman number, independent of $x$. BRST transformation and classical BRST operator $Q$: ($i\epsilon, c^a, \bar{c}^a$ are real grassman variables, Hermite conjuate for grassman number $(\chi\psi)^\dagger = \psi^\dagger \chi^\dagger$)

$$
\delta A^a_\mu = \epsilon D^{ac}_\mu c^c \equiv \epsilon QA^a_\mu, \\
\delta \psi = igc^a t^a \psi \equiv \epsilon Q\psi, \\
\delta c^a = -\frac{1}{2} g f^{abc} c^b c^c \equiv \epsilon Qc^a, \\
\delta \bar{c}^a = \epsilon B^a \equiv \epsilon Q\bar{c}^a, \\
\delta B^a = 0 \equiv \epsilon QB^a.
$$

$$Q = D^{ac}_\mu c^c(y) \frac{\delta}{\delta A^a_\mu(y)} + igc^a t^a_j \psi^i_\alpha(y) \frac{\delta}{\delta \psi^i_\alpha(y)} - igc^a \bar{\psi}^j_\alpha(y) t^a_{ji} \frac{\delta}{\delta \bar{\psi}^j_\alpha(y)} - \frac{1}{2} g f^{abc} c^b c^c(y) \frac{\delta}{\delta c^a(y)} + B^a(y) \frac{\delta}{\delta \bar{c}^a(y)}$$

(repeated indices are summed, including continuous indices $y$.)

**hw.** (1) $Q^2 = 0$. (2) Check: After integrated out $B$ (substitute the EOM of $B$), the action is still invariant under this transform. Calculate $Q^2\bar{c}$ and $Q^3\bar{c}$. 

$$L_{BRST} \sim L_{YM} - \bar{c}^a \partial^\mu D^{ac}_\mu c^c + \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu$$
\[ \mathcal{L}_{BRST} \sim \mathcal{L}_{YM} - \bar{c}^a \partial^\mu D^{ac}_\mu c^c + \frac{\xi}{2} (B^a)^2 + B^a \partial^\mu A^a_\mu \]

- The transformation of \( A_\mu \) and \( \psi \) is just a gauge transformation with parameter \( \alpha^a(x) = g \epsilon c^a(x) \), \( \mathcal{L}_{YM} \) is invariant.

- \( \delta(B^a \partial^\mu A^a_\mu) = B^a \partial^\mu (\epsilon D^{ac}_\mu c^c) \)

\[
\delta(\bar{c}(-\partial^\mu D^{ac}_\mu c^c)) = -\epsilon B^a \partial^\mu (D^{ac}_\mu c^c) - \bar{c}^a \partial^\mu D^{ac}_\mu \left( -\frac{1}{2} g \epsilon c^{b'} c^b c' \right) \]

\[
\delta A^{b fabc} \]

\[
\delta_{c c} \]

\[
= \ldots + g \epsilon \bar{c}^a \partial^\mu (*) \]

\[
(*) = -\frac{1}{2} \partial^\mu (f^{ab'} c^b' c^c') - \frac{1}{2} g A^d_\mu f^{ad b} f^{c b'} c^c' + (\partial^\mu c^b) f^{abc} c^c + g A^d_\mu f^{b de} c^e f^{abc} c^c
\]

\[
= g A^d_\mu \left( -\frac{1}{2} f^{adc} f^{c b'} c' + f^{c b'} f^{a c c'} \right) c^b' c^c'
\]

\[
= g A^d_\mu \left( -\frac{1}{2} f^{abc} + \frac{1}{2} f^{b' c c a} - \frac{1}{2} f^{c' d c a b} \right) c^b' c^c' = 0
\]
BRST Symmetry

- It can be proved that $Q^2 = 0$, $Q$ carry ghost number 1. Grassman odd operator.

- The $\mathcal{L}_{BRST}$ can be recast into

$$\mathcal{L}_{BRST} = \mathcal{L}_{YM}[A, \psi] + Q\Psi; \quad \Psi = \bar{c}^a \partial^\mu A^a_\mu + \frac{\xi}{2} \bar{c}^a B^a$$

$\mathcal{L}_{YM}[A, \psi]$ is the gauge invariant Yang-Mills Lagrangian. In general, for gauge fixing $G^a[A_\mu] = f^a(A_\mu) - \omega^a(x)$, $\Psi = \bar{c}^a f^a[A_\mu] + \frac{\xi}{2} \bar{c}^a B^a$.

$$Q\Psi = B^a f^a[A_\mu] - \bar{c}^a \frac{\delta f^a[A_\mu]}{\delta A^b_\mu(y)} D^b_\mu \bar{c} \ c^c(y) + \frac{\xi}{2} B^a B^a$$

(repeated indices are summed, including continuous indices $y$.)

- Now the invariance of $\mathcal{L}_{BRST}$ is BRST invariant can be seen easily:

1. Since the BRST trans for $A_\mu$ and $\psi$ is just a special gauge transformation with parameter $:\alpha^a(x) = g\epsilon c^a(x)$, $\mathcal{L}_{YM}$ is invariant.

$$\delta_{BRST} \mathcal{L}_{YM} = \epsilon Q \mathcal{L}_{YM} = 0$$

2. Since $Q^2 = 0$, $\delta_{BRST} Q\Psi = Q^2 \Psi = 0$. 
BRST Symmetry

• In the whole fock state space, including unphysical polarization and ghost excitations, we can define a hermitian operator (quantum operator) $Q$,

$$\delta_{BRST}\phi = \epsilon Q\phi = i[\epsilon Q, \phi]$$

From $Q^2\phi = 0$, we have

$$[Q, [Q, \phi]]_\mp = -Q^2\phi = 0$$

$$\Rightarrow [Q^2, \phi] = Q^2\phi - \phi Q^2 = 0 \text{ for any } \phi.$$ 

Either $Q^2 = 0$ or $Q^2 = I$, $\therefore Q^2$ ghost charge $= 2$, it can not be $I$. So $Q^2 = 0$.

• We want to identify the physical states in the fock state space, in a Lorentz covariant way.

$$[a_\mu(p), a_\mu^\dagger(p')] = -\eta^{\mu\nu} \delta^3(p - p'), \langle 0 | a_0 a_0^\dagger | 0 \rangle = -\langle 0 | 0 \rangle \delta^3(p - p')$$

• The physical amplitude should be invariant under the change of the gauge fixing function, $\sim$ changing of the gauge, or changing of $\xi$ — an arbitrary change of the $\Psi(x)$, $\tilde{\delta}\tilde{\Psi} = \int d^4x \tilde{\delta}\tilde{\Psi}(x)$.

$$0 = \tilde{\delta}\langle \alpha_{ph} | \beta_{ph} \rangle = i\langle \alpha_{ph} | Q\tilde{\delta}\tilde{\Psi} | \beta_{ph} \rangle = -\langle \alpha_{ph} | [Q, \tilde{\delta}\tilde{\Psi}]_+ + | \beta_{ph} \rangle$$

Because $\tilde{\delta}\tilde{\Psi}$ is arbitrary, the only possibility is: $Q|\text{Phy states}\rangle = 0$.

• For any physical state $|\alpha_{ph}\rangle$, we can not physically distinguish it from $|\alpha_{ph}\rangle + Q|\gamma\rangle$, ($\gamma$ any fock state), since

$$\langle \beta_{ph} | \alpha_{ph} + Q|\gamma\rangle = \langle \beta_{ph} | \alpha_{ph} \rangle + \langle \beta_{ph} | Q|\gamma\rangle = \langle \beta_{ph} | \alpha_{ph} \rangle + \langle Q\beta_{ph} | \gamma \rangle = \langle \beta_{ph} | \alpha_{ph} \rangle$$
**BRST Symmetry**

- **Physical state:**
  
  \[
  \begin{align*}
  \text{Physical state condition: } & Q|\text{Phy states}\rangle = 0 \\
  |\alpha_{ph}\rangle & \sim |\alpha_{ph}\rangle + Q|\gamma\rangle
  \end{align*}
  \]

- The whole fock space is divided into 3 parts: \(H_0, H_1, H_2:\)
  
  - \(H_0: |\alpha_0\rangle, Q|\alpha_0\rangle = 0\), but \(|\alpha_0\rangle \neq Q|\gamma\rangle\), \(\mathcal{H}_{ph}\) is a subspace of \(H_0\).
  - \(H_1: |\alpha_1\rangle, Q|\alpha_1\rangle \neq 0\). Unphysical states.
  - \(H_2: |\alpha_2\rangle = Q|\gamma\rangle\).
    
    1. \(Q|\alpha_2\rangle = 0\),
    2. zero norm: \(\langle \alpha_2|\alpha_2\rangle = \langle \gamma|Q^\dagger Q|\gamma\rangle = \langle \gamma|Q^2|\gamma\rangle = 0\),
    3. orthogonal to \(H_0\), \(\langle \alpha_0|\alpha_2\rangle = \langle \alpha_0|Q|\gamma\rangle = \langle Q\alpha_0|\gamma\rangle = 0\).

- **BRST closed states:** \(Q|\alpha\rangle = 0\), Kernel of \(Q\)
  
  **BRST exact states:** \(|\alpha\rangle = Q|\gamma\rangle\), Image of \(Q \subset \text{kernel of } Q\)

  So the physical state space \(\mathcal{H}_{ph}\) is defined as

  \[
  \mathcal{H}_{ph} = \frac{\{|\alpha\rangle: Q|\alpha\rangle = 0\}}{\{|\beta\rangle = Q|\gamma\rangle\}} = \frac{\{\text{Kernel of } Q\}}{\{\text{Image of } Q\}} = \text{Cohomology of } Q
  \]
Consider the asymptotic particle state, in the $t \to \pm \infty$, interactions are adiabatically turned off $g \to 0$. BRST transformation:

$$[Q, A^a_\mu] = -i \partial_\mu c^a, \quad [Q, c^a]_+ = 0, \quad [Q, \bar{c}]_+ = -i B^a \xrightarrow{EOM} -\frac{i}{\xi} \partial_\mu A^{a,\mu}$$

$$A^\mu(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (a^\mu_k e^{-ik \cdot x} + a^{\mu\dagger}_k e^{ik \cdot x})$$

$$c(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (c_k e^{-ik \cdot x} + c^{\dagger}_k e^{ik \cdot x}), \quad \bar{c}(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2E}} (\bar{c}_k e^{-ik \cdot x} + \bar{c}^{\dagger}_k e^{ik \cdot x})$$

$$[Q, a^\mu_k] = k^\mu c^{\dagger}_k, \quad [Q, \bar{c}^{\dagger}_k] = \frac{1}{\xi} k \cdot a^{\dagger}_k, \quad [Q, c^{\dagger}_k] = 0$$
BRST Symmetry

\[ [Q, a_k^\mu \dagger] = k^\mu c_k^\dagger, \quad [Q, \overline{c}_k^\dagger] = \frac{1}{\xi} k \cdot a_k^\dagger, \quad [Q, c_k^\dagger] = 0 \]

The superscript \(a\) and subscript \(k\) are suppressed in the following. A physical state: \(|\psi\rangle\), satisfies \(Q|\psi\rangle = 0\).

- Adding another vector boson, \(|e, \psi\rangle = e_\mu a_\mu^\dagger |\psi\rangle\). Physical condition: 
  \[ 0 = Q|e, \psi\rangle = e_\mu k^\mu c^\dagger |\psi\rangle, \quad \text{so} \quad k^\mu \cdot e = 0. \quad (k \cdot e^+ = 0, \quad k \cdot e^- \neq 0.) \]
  for \(e_\mu \sim e^+\mu\), \(k \cdot e^+ = 0\), \(e^+_\mu a^\mu_\mu^\dagger |\psi\rangle\) satisfies Physical condition.
  for \(e_\mu \sim e^-\mu\), \(k \cdot e^- \neq 0\), \(e^-_\mu a^\mu_\mu^\dagger |\psi\rangle \in \mathcal{H}_1\) is not a physical state.
  for \(e_\mu \sim e^T_\mu\), \(k \cdot e^T = 0\), \(e^T_\mu a^\mu_\mu^\dagger |\psi\rangle\) satisfies physical condition.

- Add a ghost: \(c_\dagger^\mu(k) |\psi\rangle = Q|e, \psi\rangle / (e \cdot k)\), \((\forall \ e \cdot k \neq 0,)\) is BRST exact \(\sim 0\), \(\in \mathcal{H}_2\).

- \(\therefore Q\overline{c}^\dagger |\psi\rangle = \frac{1}{\xi} k^\mu a^\mu_\mu^\dagger |\psi\rangle\), BRST exact, \(e^+\mu a^\mu_\mu^\dagger |\psi\rangle \in \mathcal{H}_2\).

\(\therefore,|e_\mu + \alpha k^\mu, \psi\rangle = |e_\mu, \psi\rangle + \xi Q\alpha \overline{c}^\dagger |\psi\rangle \sim |e_\mu, \psi\rangle\), this is the gauge equivalent condition for the vector boson.

- \(Q\overline{c}^\dagger |\psi\rangle = \frac{1}{\xi} k^\mu a^\mu_\mu^\dagger |\psi\rangle \neq 0 \Rightarrow \overline{c}^\dagger |\psi\rangle\) is not physical \(\in \mathcal{H}_1\).

\(e^T \cdot a^\dagger |\psi\rangle \in \mathcal{H}_0, \quad \overline{c}^\dagger |\psi\rangle \in \mathcal{H}_1, \quad e^- \cdot a^\dagger |\psi\rangle \in \mathcal{H}_1,
\(e^+ \cdot a^\dagger |\psi\rangle \in \mathcal{H}_2, \quad c^\dagger |\psi\rangle \in \mathcal{H}_2\).

For each equivalent class of states, we choose \(e^T \cdot a^\dagger |\psi\rangle\) to represent the class, \(\mathcal{H}_{ph}\) can be restricted to states with only transverse polarizations \(|A; tr\rangle \sim e^T \cdot a^\dagger |\psi\rangle\). So the physical \(\mathcal{H}_{ph} \subset H_0\) is ghost free.
We can choose the external polarizations containing only the transverse polarizations.

Because the hamiltonian $H$ commutes with $Q$, then $QS = SQ$. $QS|A, tr\rangle = SQ|A, tr\rangle = 0 \in \mathcal{H}_{ph}$, $S|A, tr\rangle = |C; tr\rangle + Q\text{exact}$.

The $S$ matrix in the full fock space is (pseudo-)unitary: $S \cdot S^\dagger = S^\dagger \cdot S = 1$. (intermediate states include a set of orthogonal independent basis $\{C\}$ of the whole fock space $H_0 + H_1 + H_2$).

The restricted $S$-matrix on the transversely polarized states is also Unitary: $S|A; tr\rangle = |C; tr\rangle + Q|\chi\rangle$, $S|B; tr\rangle = |D; tr\rangle + Q|\psi\rangle$

$|C; tr\rangle = \sum_{X; tr} |X; tr\rangle \langle X; tr|S|A; tr\rangle$, $|D; tr\rangle = \sum_{X; tr} |X; tr\rangle \langle X; tr|S|B; tr\rangle$

$$\delta(A, B) = \langle A; tr|B; tr\rangle = \langle A; tr|S^\dagger \cdot S|B; tr\rangle = \langle C; tr + Q\chi|D; tr + Q\psi\rangle$$

$$= \langle C; tr|D; tr\rangle = \sum_{X; tr} \langle A; tr|S|X; tr\rangle \langle X; tr|S|B; tr\rangle$$

(See Kugo and Ojima, Prog. Theo. Phys., V60, NO.6 1869 for details.)