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SHARP SIZE ESTIMATES FOR CAPILLARY FREE SURFACES WITHOUT GRAVITY

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For the equation of constant mean curvature with prescribed constant contact angle boundary condition, using the unique continuation of analytic function, we get a minimum principle for a combination of the solution and its gradient. Thus we get the endpoint case for *P*-function (Sperb, 1981) and in fact answer an open question which appeared twenty years ago in Payne & Philippin, 1977, 1979 and Sperb, 1981. As an application, sharp size and shape estimetes for capillary free surface without gravity are obtained.

1. Introduction and Results.

The capillary surface of a liquid contained in a vertical tube with arbitrary cross section Ω in the outer space has the shape of surface of constant mean curvature with constant contact angle θ_o against the wall of the tube. Let the capillary surface be expressed non-parametrically as the graph of a function u defined over the cross section Ω. How does the boundary geometry of Ω and the contact angle θ_o influence the size and shape of the capillary free surface?

For the convexity of the capillary free surface, in [2], Chen and Huang have shown if Ω is a bounded convex domain in the plane and $\theta_o = 0$, then the corresponding capillary surface is also convex. Finn [3] provided an example to show if $\theta_o \neq 0$ the result is in general false.

In [1, 10], Chen and Sakaguchi showed if Ω be a bounded smooth convex domain in R^2, $0 < \theta_o < \frac{\pi}{2}$, the capillary free surface over Ω has only one minimal point. From the convexity of the surface as $\theta_o = 0$, we know for any θ_o $(0 \leq \theta_o < \frac{\pi}{2}$), the minimal point is unique.

In this paper we consider the influence of boundary geometry and the contact angle θ_o $(0 \leq \theta_o < \frac{\pi}{2})$ on the size and shape for the capillary free surface without gravity. Precisely, let Ω be a bounded convex domain in R^2 with smooth boundary $\partial\Omega$. Give a positive constant H, consider the following equations:

$$\sum_{i=1}^{2} D_i\left(\frac{u_i}{\sqrt{1+|Du|^2}}\right) = 2H \qquad \text{in} \qquad \Omega \tag{1.1}$$

$$\frac{u_n}{\sqrt{1+|Du|^2}} = \cos\theta_o \qquad \text{on} \qquad \partial\Omega. \tag{1.2}$$

Where $u_i, i = 1, 2$ are partial derivatives of u, n denotes the unit outer normal to $\partial\Omega$, u_n denotes the direction derivative of u along n, and θ_o ($0 \le \theta_o < \frac{\pi}{2}$) is the constant with $2H|\Omega| = \cos\theta_o|\partial\Omega|$ ($|\Omega|$ is the area of Ω and $|\partial\Omega|$ is the length of $\partial\Omega$). The graph of solution u to (1.1)-(1.2) described a capillary free surface without gravity over the cross section Ω.

Let $A \in \partial\Omega$ be a point corresponding to a minimum boundary value of u, $B \in \partial\Omega$ be a point corresponding to a maximum boundary value of u, $C \in \Omega$ be the unique minimal (critical) point of u and $k(x)$ be the curvature of $\partial\Omega$ at $x \in \partial\Omega$. Now we state our theorems:

Theorem 1. *Let $u \in C^3(\bar\Omega)$ be a solution to (1.1)-(1.2), then the following inequalities hold*

1):

$$u(A) - u(C) \le \frac{1-\sin\theta_o}{H} \tag{1.3}$$

$$k(A) \le \frac{H}{\cos\theta_o}, \tag{1.4}$$

2):

$$u(B) - u(C) \ge \frac{1-\sin\theta_o}{H} \tag{1.5}$$

$$k(B) \ge \frac{H}{\cos\theta_o}. \tag{1.6}$$

If one of the equality signs of (1.3)-(1.6) holds then Ω is a disk of radius $\frac{\cos\theta_o}{H}$ and

$$u(x) - u(C) \equiv \frac{1-\sin\theta_o}{H} \qquad \text{on} \quad \partial\Omega \tag{1.7}$$

$$k(x) \equiv \frac{H}{\cos\theta_o} \qquad \text{on} \quad \partial\Omega. \tag{1.8}$$

Conversely, (1.7)-(1.8) holds on $\partial\Omega$ if Ω is a disk of radius $\frac{\cos\theta_o}{H}$.

The proof of Theorem 1 is based on Hopf maximum principle [**9**] and the following minimum principle.

Theorem 2. *Let $u \in C^3(\bar\Omega)$ be a solution to (1.1)-(1.2), then the function*

$$P(x) = 2 - 2(1+|Du|^2)^{-\frac{1}{2}} - 2Hu$$

attains its minimum on the boundary $\partial\Omega$, unless $P(x)$ is a constant on $\bar\Omega$.

In [7], Payne and Philippin had proved a similar maximum principle for the above function $P(x)$ that under the same condition it also attains its maximum on $\partial\Omega$ unless $P(x)$ is a constant in Ω.

Since our theorems concern only qualitative property of the solution, so only under the hypothesis of the existence of the solution we prove theorems. For the existence of solution and background details we refer the reader to the sources [4].

According to the work Nirenberg [5] or [4] we conclude that u is an analytic function in Ω, a feature which will be used in this paper.

In Section 2 we will give the the proof of Theorem 2 which is based on the unique continuation of analytic function. Section 3 contains a proof of Theorem 1 and a Corollary, which give the estimates of capillary free surface area using the volume of a liquid, $|\Omega|$, θ_o, $u(A)$ and $u(B)$.

We conclude the introduction with some notations and an identity for Equation (1.1). Let Ω be a bounded convex smooth domain in the plane. We introduce curvilinear coordinate system (r, s), where s represents arc length along $\partial\Omega$ and $r(x_1, x_2)$ is the distance from a point $x = (x_1, x_2)$ in Ω to $\partial\Omega$. As in [11], we denotes $n = (n^1, n^2)$ the unit outward normal to $\partial\Omega$, $T = (T^1, T^2)$ is the unit tangent vector of $\partial\Omega$. The summation convention over repeated indices (from 1 to 2) will be employed. Assume that a function $u(x_1, x_2)$ is smooth in $\bar{\Omega}$, the following abbreviations will be adopted

$$u_1 = \frac{\partial u}{\partial x_1}, \qquad u_2 = \frac{\partial u}{\partial x_2}, \qquad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots, \qquad v = 1 + |Du|^2.$$

Following [11], we define the normal derivative $\frac{\partial u}{\partial n}$ of u by

$$u_n = \lim_{r\to 0} \frac{1}{r}(u(x) - u(x - rn)) = u_i n^i.$$

On $\partial\Omega$ we can also define a tagential derivative $\frac{\partial u}{\partial s}$ of u by

$$u_s = u_i T^i.$$

Then we have the following formulas on $\partial\Omega$

$$\begin{aligned} u_{ss} &= \frac{\partial^2 u}{\partial s^2} = u_{ij}T^iT^j - ku_in^i \\ u_{nn} &= \frac{\partial^2 u}{\partial n^2} = u_{ij}n^in^j \\ u_{ns} &= \frac{\partial}{\partial s}(\frac{\partial u}{\partial n}) = u_{ij}n^iT^j + ku_s \\ u_{sn} &= u_{ns} - ku_s. \end{aligned} \tag{1.9}$$

Using curvilinear coordinate system, Equation (1.1) implies the following formula on $\partial\Omega$

$$u_{nn}(1 + u_s^2) = 2Hv^{\frac{3}{2}} - (u_{ss} + ku_n)v + (u_{ss}u_s^2 + 2u_su_nu_{ns} - ku_nu_s^2), \tag{1.10}$$

which will be used in Section 3 to prove Theorem 1.

Remark. The formula (1.4) is implicit contained in [7].

2. A minimum principle.

We consider the boundary value problem (1.1)-(1.2) in a strict convex bounded domain Ω in R^2 with smooth boundary $\partial\Omega$, and define the following function:

$$P^{\alpha}(x) = 2 - 2(1 + |Du|^2)^{-\frac{1}{2}} - 2\alpha H u. \tag{2.1}$$

We know that $P^{\alpha}(x)$ takes its maximum value at the critical point for $\alpha \geq 2$ [6], and on the boundary $\partial\Omega$ for $\alpha \leq 1$ [7]. We concentrate now our attention on $\alpha \in [1, 2]$, and state the following:

Lemma 2.1 ([7, 8])**.** *The function $P^{\alpha}(x)$ defined in (2.1) satisfies the following elliptic differential equation:*

$$\begin{aligned}(\delta_{ij} - u_i u_j v^{-1})P^{\alpha}_{ij} &- [2Hv^{-\frac{1}{2}}u_i + 2v^{-1}|Du|^{-2}u_k u_{ki} \\ &+ 2(\alpha - 2)Hv^{-\frac{1}{2}}|Du|^{-2}u_i - 2v^{-2}|Du|^{-2}u_k u_l u_{kl} u_i]P^{\alpha}_i \\ &= 4(\alpha - 1)(\alpha - 2)H^2 v^{-\frac{1}{2}},\end{aligned} \tag{2.2}$$

where δ_{ij} is Kronecker symbol.

For the proof of Lemma 2.1, we make use of Definition (2.1) and of the following identity (valid in R^2 only):

$$u_{ij}u_i u_j |Du|^2 \equiv |Du|^2(\Delta u)^2 + 2u_i u_{ij} u_k u_{kj} - 2\Delta u u_i u_j u_{ij}.$$

The details of the computations are omitted here since they had been given in [7].

From Lemma 2.1 and Hopf maximum principle [9] we conclude that $P^{\alpha}(x)$ takes its minimum value either on the boundary of $\partial\Omega$, or at the unique critical point $C \in \Omega$ for $\alpha \in [1, 2]$. For $\alpha > 1$, the second alternative had been rejected by Philippin [7]. The purpose of this section is to show even $\alpha = 1$ the second alternative can also be rejected unless $P^{\alpha}(x)$ is a constant in Ω. This can be achieved as a consequence of the following:

Theorem 2.2. *Let $u \in C^3(\bar{\Omega})$ is a solution to (1.1)-(1.2), if*

$$P(x) = 2 - 2v^{-\frac{1}{2}} - 2Hu$$

attains its minimum at the unique critical point $C \in \Omega$, then $P(x)$ is a constant on $\bar{\Omega}$.

For the proof of the Theorem 2.2, we use the strong unique continuation of analytic function, so our program is to show all order derivatives of $P(x)$

are vanishing at $C \in \Omega$. To this end, we choose the origin of the coordinate axes at the critical point $C \in \Omega$, then

$$u_1(C) = u_2(C) = 0, \tag{2.3}$$

and orient the axes x_1 and x_2 in such a way that

$$u_{12}(C) = 0. \tag{2.4}$$

From Chen , Huang [**2**] and Sakaguchi [**10**], we know

$$u_{11}(C) > 0, \qquad u_{22}(C) > 0, \tag{2.5}$$

which will be used essential in the following proof.

Proof of Theorem 2.2. Our proof is divided four steps.

Step 1: We show the derivatives of $P(x)$ up to order 2 are vanishing at C.

First we compute the first derivaties of $P(x)$ at $C \in \Omega$. Since at any point $x \in \Omega$

$$P_1 = v^{-\frac{3}{2}} v_1 - 2Hu_1 = 2v^{-\frac{3}{2}} u_i u_{i1} - 2Hu_1, \tag{2.6}$$

$$P_2 = v^{-\frac{3}{2}} v_2 - 2Hu_2 = 2v^{-\frac{3}{2}} u_i u_{i2} - 2Hu_2, \tag{2.7}$$

then from (2.3), we have

$$P_1(C) = P_2(C) = 0. \tag{2.8}$$

Now we compute the second derivatives of $P(x)$ at C. From (2.3)-(2.6), we have at C

$$P_{11} = -\frac{3}{2} v^{-\frac{5}{2}} v_1^2 + v^{-\frac{3}{2}} v_{11} - 2Hu_{11} = 2u_{11}^2 - 2Hu_{11} \tag{2.9}$$

$$P_{12} = -\frac{3}{2} v^{-\frac{5}{2}} v_1 v_2 + v^{-\frac{3}{2}} v_{12} - 2Hu_{12} = 0 \tag{2.10}$$

$$P_{22} = -\frac{3}{2} v^{-\frac{5}{2}} v_2^2 + v^{-\frac{3}{2}} v_{22} - 2Hu_{22} = 2u_{22}^2 - 2Hu_{22}. \tag{2.11}$$

Use the fact that $P(x)$ attains its minimum at C, we have

$$P_{11}(C)P_{22}(C) - P_{12}^2(C) \geq 0. \tag{2.12}$$

From (2.5),(2.9) and (2.11) we know

$$u_{11}(C) = u_{22}(C) = H \tag{2.13}$$

$$P_{11}(C) = P_{22}(C) = 0. \tag{2.14}$$

Now we will use the induction to show that all order derivatives of $P(x)$ at C are vanishing.

Step 2: As a first step for induction, we will show the derivatives of $P(x)$ of order $3, 4$ at C are vanishing.

First we claim

$$\frac{\partial^3 P}{\partial x_1^k \partial x_2^{3-k}}(C) = 0, \quad k = 0, 1, 2, 3. \tag{2.15}$$

Using (2.9)-(2.11), (2.4) and (2.13) we have

$$P_{x_1^3}(C) = 4Hu_{x_1}^3(C) \tag{2.16}$$

$$P_{x_1^2 x_2}(C) = 4Hu_{x_1^2 x_2}(C) \tag{2.17}$$

$$P_{x_1 x_2^2}(C) = 4Hu_{x_1 x_2^2}(C) \tag{2.18}$$

$$P_{x_2^3}(C) = 4Hu_{x_2^3}(C). \tag{2.19}$$

Now, by differentiating (1.1), we obtain

$$u_{x_1^3}(C) = -u_{x_1 x_2^2}(C) \tag{2.20}$$

$$u_{x_1^2 x_2}(C) = -u_{x_2^3}(C). \tag{2.21}$$

To this end, use (2.8), (2.10), (2.14) and (2.20)-(2.21), we expand the function $P(x)$ in a Taylor series in a neighborhood of C:

(2.22)

$$P(x_1, x_2) - P(C) = \frac{r^3}{3!}\left\{\frac{\partial^3 P}{\partial x_1^3}(C) \times [\cos^3\varphi - 3\cos\varphi\sin^2\varphi] + \frac{\partial^3 P}{\partial x_1^2 \partial x_2}(C) \times [3\cos^2\varphi\sin\varphi - \sin^3\varphi]\right\} + O(r^4),$$

where (r, φ) are polar coordinates: $x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi$. Suppose

$$\sqrt{P^2_{x_1^3}(C) + P^2_{x_1^2 x_2}(C)} \neq 0,$$

then $P(x)$ is not a constant, so we are lead to the following representation of $P(x)$ in a neighborhood of the point C:

$$P(x) - P(C) = A_3\cos[3\varphi - \beta_3]r^3 + O(r^4), \tag{2.23}$$

with

$$A_3 = \frac{\sqrt{P^2_{x_1^3}(C) + P^2_{x_1^2 x_2}(C)}}{3!},$$

$$\cos\beta_3 = \frac{P_{x_1^3}(C)}{\sqrt{P^2_{x_1^3}(C) + P^2_{x_1^2 x_2}(C)}},$$

and

$$\sin\beta_3 = \frac{P_{x_1^2 x_2}(C)}{\sqrt{P^2_{x_1^3}(C) + P^2_{x_1^2 x_2}(C)}}.$$

From (2.23) we conclude that $P(x)$ has at least 3 nodal lines forming equal angles at the point C, using Lemma 2.1 we know that $P(x)$ attains its minimum only on $\partial\Omega$ or at the critical point C, which is a contradiction. Thus $A_3(C) = 0$ or

$$\frac{\partial^3 P}{\partial x_1^k \partial x_2^{3-k}}(C) = 0 \quad \text{and} \quad \frac{\partial^3 u}{\partial x_1^k \partial x_2^{3-k}}(C) = 0, \qquad k = 0, 1, 2, 3. \tag{2.24}$$

Use the similar argument we can show

$$\frac{\partial^4 P}{\partial x_1^k \partial x_2^{4-k}}(C) = 0, \quad k = 0, 1, 2, 3, 4 \tag{2.25}$$

and

$$u_{x_1^4}(C) = u_{x_2^4}(C) = 3H^3 \tag{2.26}$$

$$u_{x_1^2 x_2^2}(C) = H^3 \tag{2.27}$$

$$u_{x_1^3 x_2}(C) = u_{x_1 x_2^3}(C) = 0. \tag{2.28}$$

Step 3: Now we assume all order derivatives of $P(x)$ up to n are vanishing at C, where $n \geq 5$. Use similar argument as in Step 2 we have the following relations.

If $n = 2l, \quad l \geq 3$. Then

$$u_{x_1^m x_2^{k-m}}(C) = u_{x_1^{k-m} x_2^m}(C) \tag{2.29}$$
$$\text{for any} \quad m = 0, 1, 2, \ldots, k, \text{ if } \quad k = 5, 6, \ldots, 2l,$$

$$u_{x_1^m x_2^{k-m}}(C) = 0 \tag{2.30}$$
$$\text{for any} \quad m = 0, 1, 2, \ldots, k, \quad \text{if} \quad k = 5, 7, 9, \ldots, 2l-1,$$

$$u_{x_1^m x_2^{2p-m}}(C) = 0 \tag{2.31}$$
$$\text{for any} \quad m = 1, 3, 5, \ldots, 2p-1, \text{if} \quad p = 3, 4, 5, \ldots, l,$$

$$u_{x_1^{2p}}(C) = u_{x_2^{2p}}(C) = (2p-1)[(2p-3)(2p-5)\ldots 1]^2 H^{2p-1} \tag{2.32}$$
$$\text{for any} \quad p = 3, 4, \ldots, l.$$

When l is even, we obtain for any $p = 4, 6, \ldots l$

$$u_{x_1^{2p}}(C) \div u_{x_1^{2p-2} x_2^2}(C) = (2p-1) \div 1 \tag{2.33}$$

$$u_{x_1^{2p-2} x_2^2}(C) \div u_{x_1^{2p-4} x_2^4}(C) = (2p-3) \div 3 \tag{2.34}$$

$$\vdots$$

$$u_{x_1^{p+2} x_2^{p-2}}(C) \div u_{x_1^p x_2^p}(C) = (p+1) \div (p-1), \tag{2.35}$$

and for any $p = 3, 5, 7, \ldots l-1$, we have

$$u_{x_1^{2p}}(C) \div u_{x_1^{2p-2} x_2^2}(C) = (2p-1) \div 1 \tag{2.36}$$

$$u_{x_1^{2p-2}x_2^2}(C) \div u_{x_1^{2p-4}x_2^4}(C) = (2p-3) \div 3 \tag{2.37}$$

$$\vdots$$

$$u_{x_1^{p+3}x_2^{p-3}}(C) \div u_{x_1^{p+1}x_2^{p-1}}(C) = (p+2) \div (p-2). \tag{2.38}$$

When l is odd, we have the similar relations (2.36)-(2.38).

If $n = 2l+1, \quad l \geq 2$, a similar argument show (2.29)-(2.38) and

$$u_{x_1^m x_2^{2l+1-m}}(C) = 0, \quad \text{for any } \; m = 0, 1, 2, \dots, 2l+1 \tag{2.39}$$

hold.

Step 4: Now we show the derivatives of $P(x)$ of order $n+1$ are vanishing at C. We divided it two parts according to whether n is odd or even.

Part A: If $n = 2l+1, \quad l \geq 2$, so $n+1 = 2(l+1)$ is even, we first look for the relations among $P_{x_1^m x_2^{n+1-m}}(C), \quad$ where $\quad m = 0, 2, 4, \dots n+1$. Through calculating, we have

$$P_{x_1^{n+1}}(C) = 2nH\left\{u_{x_1^{n+1}}(C) - (2l+1)[(2l-1)(2l-3)\dots 1]^2 H^{2l+1}\right\} \tag{2.40}$$

$$P_{x_1^{n-1}x_2^2}(C) = 2nH\left\{u_{x_1^{n-1}x_2^2}(C) - [(2l-1)(2l-3)\dots 1]^2 H^{2l+1}\right\}. \tag{2.41}$$

Now, by differentiating (1.1), we obtain

$$\frac{\partial}{\partial x_1^{n-1}}(\Delta u - u_i u_j u_{ij} v^{-1})(C) = \frac{\partial}{\partial x_1^{n-1}}(2Hv^{\frac{1}{2}})(C), \tag{2.42}$$

and using the values of derivatives of u up to order n at C, this lead to

$$u_{x_1^{n+1}}(C) + u_{x_1^{n-1}x_2^2}(C) = (n+1)[(2l-1)(2l-3)\dots 1]^2 H^{2l+1}. \tag{2.43}$$

From (2.40)-(2.41) and (2.43) we obtain

$$P_{x_1^{n+1}}(C) = -P_{x_1^{n-1}x_2^2}(C). \tag{2.44}$$

A similar argument, it follows that

$$P_{x_1^{n-1}x_2^2}(C) = -P_{x_1^{n-3}x_2^4}(C) = \dots = (-1)^l P_{x_2^{n+1}}(C). \tag{2.45}$$

Now we will find the similar relations (2.44)-(2.45) among

$$P_{x_1^m x_2^{n+1-m}}(C), \quad \text{where} \qquad m = 1, 3, 5, \dots n.$$

Using the same argument, we have

$$P_{x_1^{n+1-m}x_2^m}(C) = 2nHu_{x_1^{n+1-m}x_2^m}(C), \quad \text{where } \; m = 1, 3, 5, \dots, n. \tag{2.46}$$

Now, by differentiating (1.1), we obtain as in (2.43) the following relations

$$u_{x_1^{n+1-m}x_2^m}(C) = -u_{x_1^{n-1-m}x_2^{m+2}}(C), \quad \text{for } \; m = 1, 3, 5, \dots, n-2. \tag{2.47}$$

From (2.46)-(2.47), it follows that

$$P_{x_1^n x_2}(C) = -P_{x_1^{n-2}x_2^3}(C) = \cdots = (-1)^l P_{x_1 x_2^n}(C). \tag{2.48}$$

Up to now we are able to show the derivatives of $P(x)$ of order $n+1$ are vanishing at C as Step 2. Using the induction assumption, (2.44)-(2.45) and (2.48), we expand $P(x)$ in a Tayor's series in a neighborhood of the point C:

(2.49)

$$\begin{aligned} &P(x) - P(C) \\ &= \frac{r^{n+1}}{(n+1)!}\Big\{P_{x_1^{n+1}}(C) \times \Big[\binom{n+1}{0}\cos^{n+1}\varphi \\ &- \binom{n+1}{2}\cos^{n-1}\varphi\sin^2\varphi + \cdots + (-1)^{l+1}\binom{n+1}{n+1}\sin^{n+1}\varphi\Big] \\ &+ P_{x_1^{n-1}x_2}(C) \times \Big[\binom{n+1}{1}\cos^n\varphi\sin\varphi - \binom{n+1}{3}\cos^{n-1}\varphi\sin^3\varphi \\ &+ \cdots + (-1)^l\binom{n+1}{n}\cos\varphi\sin^n\varphi\Big]\Big\} + O(r^{n+2}). \end{aligned}$$

As in Step 2, we can show the derivatives of $P(x)$ of order $n+1$ are vanishing at C.

Part B: If $n = 2l, \quad l \geq 3$, a similar argument as in Part A, we have

$$P_{x_1^{n+1-m}x_2^m}(C) = 2nHu_{x_1^{n+1-m}x_2^m}(C) \tag{2.50}$$

for $m = 0, 1, 2, 3, \ldots, n+1$. The same analysis as in Part A leads to imply the derivatives of $P(x)$ of order $n+1$ are vanishing at C.

According to the unique continuation of analytic function, we know if the function $P(x)$ attains its minimum at C, then it must be a constant, this establishes Theorem 2.2. □

Combination of Theorem 2.2 and Lemma 2.1 implies Theorem 2.

3. The proof of Theorem 1.

From Section 2, we know if $u \in C^3(\bar{\Omega})$ is a solution to Equations (1.1)-(1.2), then the function

$$P(x) = 2 - 2(1 + |Du|^2)^{-\frac{1}{2}} - 2Hu$$

attains its maximum [7] and minimum on $\partial\Omega$ unless $P(x)$ is a constant in $\bar{\Omega}$. As an application of these maximum and minimum principle of the function $P(x)$, in this section we get the size estimates of capillary free surface without gravity to complete the proof Theorem 1 and a Corollary. The proof of Theorem 1 will be divided two parts to show the different applications for maximum (minimum) principle.

Proof of Theorem 1. Part A: Use the fact that the function $P(x)$ attains its maximum on $\partial\Omega$, we first prove (1.3)-(1.4).

Assume $P(x)$ attains its maximum at $x_o \in \partial\Omega$. We must have at x_o:

$$\frac{1}{2}P_s = \frac{1}{2}(-2)\left(-\frac{1}{2}\right) v^{-\frac{3}{2}}(u_n^2 + u_s^2 + 1)_s - Hu_s \tag{3.1}$$

$$= v^{-\frac{3}{2}}(u_n u_{ns} + u_s u_{ss}) - Hu_s = 0,$$

from boundary condition (1.2), we have

$$\sin^2\theta_o u_n^2 = \cos^2\theta_o + \cos^2\theta_o u_s^2,$$

it follows that

$$u_n u_{ns} = \cot^2\theta_o u_s u_{ss}, \tag{3.2}$$

we conclude from (3.2) and (3.1) that

$$\frac{1}{2}P_s = v^{-\frac{3}{2}}(\cot^2\theta_o u_s u_{ss} + u_s u_{ss}) - Hu_s = u_s\left[\frac{u_{ss}}{\sin^2\theta_o v^{\frac{3}{2}}} - H\right] = 0. \tag{3.3}$$

According to Hopf maximum principle [9], we also have at x_o:

$$\frac{1}{2}P_n = v^{-\frac{3}{2}}[u_n u_{nn} + u_s u_{sn}] - Hu_n > 0, \tag{3.4}$$

unless $P(x)$ is a constant on $\bar{\Omega}$.

If $u_s(x_o) \neq 0$, then from (3.3)

$$u_{ss}(x_o) = H\sin^2\theta_o v^{\frac{3}{2}}(x_o). \tag{3.5}$$

Now we shall use (1.2), (1.9), (1.10), (3.4) and (3.5) to lead

$$-ku_n|Du|^2 > 0, \qquad \text{at} \quad x_o, \tag{3.6}$$

which is contradiction to the strictly convexity of $\partial\Omega$. The proof of (3.6) is a long calculation. Using (3.5), at x_o, we shall rewrite (1.10) as

(3.7)

$$u_{nn}(1+u_s^2) = 2Hv^{\frac{3}{2}} + u_{ss}u_s^2 + 2\cot^2\theta_o u_s^2 u_{ss} - ku_n u_s^2 - (u_{ss}+ku_n)(1+u_n^2+u_s^2).$$

Using (1.9) and (3.4) we have at x_o

$$u_n u_{nn} + u_s(u_{ns} - ku_s) - Hu_n v^{\frac{3}{2}} > 0,$$

thus

$$u_n^2(1+u_s^2)u_{nn} + u_n u_s(1+u_s^2)(u_{ns} - ku_s) - Hu_n^2(1+u_s^2)v^{\frac{3}{2}} > 0, \quad \text{at} \quad x_o. \tag{3.8}$$

Combining (3.7) with (3.8) yields at x_o

(3.9)

$$u_n^2\left[2Hv^{\frac{3}{2}} + u_{ss}u_s^2 + 2\cot^2\theta_o u_s^2 u_{ss} - ku_n u_s^2 - u_{ss}(1+u_n^2+u_s^2) - ku_n v\right]$$

$$+ u_n u_s(1+u_s^2)u_{ns} - ku_n u_s^2(1+u_s^2) - Hu_n^2(1+u_s^2)v^{\frac{3}{2}} > 0.$$

From (3.2), (3.9) can be rewritten as

$$\begin{aligned}\left[Hu_n^2v^{\frac{3}{2}} + u_{ss}(\cot^2\theta_o u_s^2u_n^2 - u_n^2 - u_n^4)\right] \\ + \left[u_{ss}u_s^2\cot^2\theta_o(1+u_n^2+u_s^2) - Hu_n^2u_s^2v^{\frac{3}{2}}\right] \\ - ku_n\left[u_n^2u_s^2 + u_n^2v + u_s^2 + u_s^4\right] > 0.\end{aligned} \tag{3.10}$$

Now we calculate (3.10), from (1.2) and (3.5), it follows that at x_o

$$\begin{aligned}&Hu_n^2v^{\frac{3}{2}} + u_{ss}u_n^2(\cot^2\theta_o u_s^2 - 1 - u_n^2) \\ &= Hu_n^2v^{\frac{3}{2}} - H\sin^2\theta_o v^{\frac{3}{2}}u_n^2\sin^2\theta_o = 0,\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&u_{ss}u_s^2\cot^2\theta_o(1+u_n^2+u_s^2) - Hu_n^2u_s^2v^{\frac{3}{2}} \\ &= H\sin^2\theta_o v^{\frac{3}{2}}\cot^2\theta_o u_s^2 v - Hu_n^2u_s^2v^{\frac{3}{2}} \\ &= Hu_s^2v^{\frac{3}{2}}(\cos^2\theta_o v - u_n^2) = 0,\end{aligned} \tag{3.12}$$

similarly we have

$$-ku_n(u_n^2u_s^2 + u_s^2 + u_s^4 + u_n^2v) = -ku_n(u_s^2v + u_n^2v) = -ku_n|Du|^2v. \tag{3.13}$$

Insertion (3.11)-(3.13) into (3.10) yields

$$-ku_n|Du|^2v > 0, \qquad \text{at} \quad x_o,$$

now we complete the proof (3.6).

Thus we must have $u_s(x_o) = 0$, from the expression for $P(x)$, x_o must be a point A where u attains its minimum on $\partial\Omega$ and we may use the fact that $P_{ss}(A) \le 0$ also. It follows that

$$0 \le u_{ss}(A) \le H\sin^2\theta_o v^{\frac{3}{2}}(A). \tag{3.14}$$

Using the similar calculation to get (3.6), we conclude from (1.10), (1.2), (1.9) and (3.4) that

$$k(A)\cos\theta_o < H - u_{ss}(A). \tag{3.15}$$

Insert (3.14) into (3.15) to find

$$k(A) < \frac{H}{\cos\theta_o}. \tag{3.16}$$

Moreover from the maximum principle we have $P(A) > P(C)$, it yields

$$u(A) - u(C) < \frac{1-\sin\theta_o}{H}. \tag{3.17}$$

If $P(x)$ is a constant on $\bar{\Omega}$ then a similar argument as (3.15) we have

$$k(x) \equiv \frac{H}{\cos\theta_o}, \qquad \text{for any} \quad x \in \partial\Omega, \tag{3.18}$$

$$u(x) - u(C) \equiv \frac{1-\sin\theta_o}{H}, \qquad \text{for any} \quad x \in \partial\Omega. \tag{3.19}$$

Which imply if $P(x)$ is a constant on $\bar{\Omega}$ then Ω is a disk with radius $\frac{\cos\theta_o}{H}$. Until now we complete the proof of (1.3)-(1.4).

Conversely, if at $A \in \partial\Omega$ we have $u(A) - u(C) = \frac{1-\sin\theta_o}{H}$ or $k(A) = \frac{H}{\cos\theta_o}$, from strong maximum principle it follows that $P(x)$ must be a constant on $\bar{\Omega}$, so (1.7)-(1.8) hold.

Part B: Similar to Part A, use the fact that $P(x)$ attains its minimum on $\partial\Omega$, we will prove (1.5)-(1.6).

From the boundary condition (1.2), we can see that the minimum of $P(x)$ on $\partial\Omega$ must be a point $B \in \partial\Omega$ where u itself is a maximum. It follows that

$$u_s(B) = 0, \qquad u_{ss}(B) \le 0. \tag{3.20}$$

A similar argument as in Part A, we have

$$u(B) - u(C) \ge \frac{1-\sin\theta_o}{H},$$
$$k(B) \ge \frac{H}{\cos\theta_o},$$

this is (1.5)-(1.6).

Conversely if $u(B)-u(C) = \frac{1-\sin\theta_o}{H}$ or $k(B) = \frac{H}{\cos\theta_o}$, then from strong maximum and Theorem 2.2 $P(x)$ must be a constant on $\bar{\Omega}$, as in Part A (1.7)-(1.8) holds and Ω is a disk with radius $\frac{\cos\theta_o}{H}$.

When Ω is a disk of radius $\frac{\cos\theta_o}{H}$, (1.7)-(1.8) hold obviously. Thus we have proved Theorem 1. □

As an another application of the minimum principle of Section 2, we prove the following Corollary for the capillary free surface area S defined as

$$S = \int_\Omega \sqrt{1+|Du|^2}\,dx.$$

Corollary. *Let A and B as in Theorem 1, $V = \int_\Omega u\,dx$ is the volume of a liquid in a vertical tube, then S satisfies the inequalities:*

$$[\sin\theta_o + 3Hu(A)]|\Omega| - 3HV \le S \le [\sin\theta_o + 3Hu(B)]|\Omega| - 3HV. \tag{3.21}$$

Proof. From the fact that $P(x)$ attains its maximum at $A \in \partial\Omega$. We must have

$$P(x) \le P(A) \qquad \text{for any} \quad x \in \Omega.$$

So we are actually have

$$\frac{1}{\sqrt{1+|Du|^2}} \geq \sin\theta_o + H[u(A)-u]. \tag{3.22}$$

Since

$$\frac{1}{\sqrt{1+|Du|^2}} = (1+|Du|^2)^{\frac{1}{2}} - \frac{|Du|^2}{\sqrt{1+|Du|^2}},$$

we obtain from (3.22)

$$H[u(A)-u] + \sin\theta_o \leq \sqrt{1+|Du|^2} - \frac{|Du|^2}{\sqrt{1+|Du|^2}}. \tag{3.23}$$

Using the fact that $\frac{|Du|^2}{\sqrt{1+|Du|^2}} = \frac{u_i u_i}{\sqrt{1+|Du|^2}}$ and the divergence theorem in conjuction with (1.1)-(1.2), we find from (3.23) after an intergration over Ω that

$$S \geq [3Hu(A) + \sin\theta_o]|\Omega| - 3HV. \tag{3.24}$$

Similar using the fact that $P(x)$ attains its minimum at $B \in \partial\Omega$, we know that

$$S \leq [3Hu(B) + \sin\theta_o]|\Omega| - 3HV. \tag{3.25}$$

Inequalities (3.24)-(3.25) are optimal in the sense that the equality signs in (3.24)-(3.25) holds if and only if Ω is a disk with radius $\frac{\cos\theta_o}{H}$. This establishes the Corollary. □

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