Invent. math. 151, 553–577 (2003)
DOI: 10.1007/s00222-002-0259-2

Inventiones mathematicae

The Christoffel-Minkowski problem I: Convexity of solutions of a Hessian equation

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Oblatum 28-IX-2001 & 7-VIII-2002
Published online: 8 November 2002 –

1. Introduction

Surface area measures are local versions of quermassintegrals in the theory of convex bodies. If the boundary of the convex body is smooth, the corresponding surface area function is a symmetric function of the principal radii of its boundary. The general problem of finding a convex hypersurface with the k-th symmetric function of the principal radii prescribed on its outer normals is often called the Christoffel-Minkowski problem. It corresponds to finding *convex* solutions of the nonlinear elliptic Hessian equation (see next section for the derivation):

$$S_k((u_{ij} + u\delta_{ij})) = \varphi \quad \text{on} \quad \mathbb{S}^n, \tag{1.1}$$

with the positive definite condition

$$(u_{ij} + u\delta_{ij}) > 0, \quad \text{on} \quad \mathbb{S}^n, \tag{1.2}$$

where u_{ij} are the second order covariant derivatives with respect to any orthonormal frame $\{e_1, e_2, \dots, e_n\}$ on $\mathbb{S}^n$, δ_{ij} is the standard Kronecker symbol and S_k is the k-th elementary symmetric function.

We will call a function $u \in C^2(\mathbb{S}^n)$ *convex* if u satisfies (1.2). The natural class of solutions of (1.1) consists of k-convex functions (see Definition 2.1), which in general are not *convex*. For the Christoffel-Minkowski problem, one needs to find the *convex* solutions. The main objective of this paper is to

Research of the first author was supported in part by NSERC Grant OGP-0046732. Research of the second author was supported by in part by grants of Foundation for University Key Teacher of the Ministry of Education of China, Shanghai Priority Academic Discipline and NSFC No.10001011.

study the existence of the solutions of equation (1.1) satisfying the *convexity* condition (1.2).

The Alexandrov-Fenchel-Jessen Theorem ([2] and [10]) asserts the uniqueness of the convex solutions of equation (1.1). In the case $k = 1$, (1.1) is the equation for the Christoffel problem. The early treatments in this case were given in Christoffel [8], Hurwitz [19], Hilbert [17], Süss [30] and others; the final solution was obtained in Firey [11], [12] and Berg [3].

The other extreme case of (1.1) is $k = n$, which corresponds to the Minkowski problem. This case has been settled by the works of Minkowski [25], Alexandrov [1], Lewy [24], Nirenberg [26], Pogorelov [28] and Cheng-Yau [7]. The intermediate problems still remain open; very little is known though there is an extensive literature devoted to them (e.g., see [4], [29] and the references there).

It is known that for (1.1) to be solvable, the function φ has to satisfy (e.g., see [28])

$$\int_{\mathbb{S}^n} x_i \varphi(x)\, dx = 0, \qquad i = 1, \dots, n+1. \tag{1.3}$$

For the Minkowski problem, (1.3) is also sufficient. But it is not sufficient for the cases $1 \le k < n$ as pointed out by Alexandrov in [2]. For both the Minkowski problem and the Christoffel problem ($k = n$ and $k = 1$), the sum of k-th surface area functions of two convex bodies is again a k-th surface area function of a third convex body. These k-th surface area sums are related to the Blaschke and Minkowski sums of convex bodies respectively. For the intermediate cases $2 \le k \le n-1$, this is no longer true in general. There exist two strictly convex bodies with analytic boundary such that the sum of their k-th surface area functions is not a k-th surface area function of any convex body (see [9] and [15], also discussion on p. 396 in [29]). This type of example suggests that the intermediate problems are much more complicated.

The intermediate Christoffel-Minkowski problems raise the following fundamental question in PDE:

Question: for what functions φ on the right hand side of the equation (1.1), is there a regular *convex* solution?

In order to establish the strict *convexity* of a solution, it is necessary to prove the positivity of the eigenvalues of the hessian of the solution. If $k = n$, we have a Monge-Ampère equation which is the product of the eigenvalues of the spherical hessian. The works of Cheng-Yau [7] and Pogorelov [28] give upper bounds for the eigenvalues and so automatically also give a lower positive bound on the eigenvalues since the product is positive. Thus the continuity method builds *convexity* into the solution class. When $k < n$, the matter is more delicate. For example, in the case $k = 1$, one may have the trace of the hessian positive while some eigenvalues might be non-positive. Hence, the major issue here is to find conditions for the existence of *convex* solutions of (1.1). In the case of the Christoffel problem, the equation (1.1) is linear. The necessary and sufficient conditions in [11] were derived from the linear representation formula of Green's function. For the intermediate cases ($2 \le k \le n-1$), (1.1) is a fully nonlinear equation. We have to take a different approach. We deal with the problem using the continuity method as a deformation process together with the strong minimum principle to force *convexity*. This approach has been successfully used previously by Caffarelli-Friedman [6] and Korevaar-Lewis [22] (it appears that Yau also suggested a similar approach, see [21]) for semilinear equations in domains of $\mathbb{R}^n$. A crucial deformation lemma (Lemma 4.1) will be established in this paper for the fully nonlinear Hessian equation (1.1).

We introduce some notation.

Definition 1.1. *For $s \in \mathbb{R}$, we define $\mathcal{C}_s$ to be the cone of positive $C^{1,1}$ functions on $\mathbb{S}^n$ satisfying (1.3) and such that $(f^s_{ij} + \delta_{ij} f^s)$ is positive semi-definite almost everywhere in $\mathbb{S}^n$, where $f^s(x) = (f(x))^s$. Moreover we say f is connected to g in $\mathcal{C}_s$ if there is a continuous (relative to the $C^{1,1}$-norm) path $h(t, .) \in \mathcal{C}_s$, such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$, $\forall x \in \mathbb{S}^n$.*

We note that the definition is independent of the choice of orthonormal frame. Moreover, note that a positive $C^{1,1}$ function f is in $\mathcal{C}_s$ if and only if f satisfies (1.3) and $\tilde{f^s}$ is a convex function in $\mathbb{R}^{n+1}$, where $\tilde{f^s}(x) = |x| f^s(\frac{x}{|x|})$ is the homogeneous extension of f^s of order one to $\mathbb{R}^{n+1}$. One of the main thrusts of this paper is that $\mathcal{C}_{-\frac{1}{k}}$ turns out to be related to the existence of *convex* solutions of equation (1.1). We refer to Remark 5.2 below for a heuristic discussion.

We now state our main results.

Theorem 1.2. (Full Rank Theorem) *Suppose u is an admissible solution (Definition 2.1) of equation (1.1) with positive semi-definite spherical hessian $W = (u_{ij} + u\delta_{ij})$ on $\mathbb{S}^n$. If $\varphi \in \mathcal{C}_{-\frac{1}{k}}$, then W is positive definite on $\mathbb{S}^n$.*

The following is the existence theorem.

Theorem 1.3. (Existence Theorem) *Let $\varphi(x) \in \mathcal{C}_{-\frac{1}{k}}$ and suppose φ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$. Then the Christoffel-Minkowski problem (1.1) has a unique solution up to translations. More precisely, there exists a closed strictly convex hypersurface M in $\mathbb{R}^{n+1}$ of class $C^{3,\alpha}$ (for all $0 < \alpha < 1$) whose principal radii of curvature function of order k is $\varphi(x)$. M is unique up to translations. Furthermore, if $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$ ($l \ge 2, \gamma > 0$), then M is $C^{2+l,\gamma}$. If φ is analytic, M is analytic.*

In subsequent joint work with B. Andrews, we will study a curvature flow equation associated to Christoffel-Minkowski problems. The condition on φ in Theorem 1.3 will be replaced by the simpler condition $\varphi \in \mathcal{C}_{-\frac{1}{k}}$

alone via a curvature flow approach with the assistance of the Full Rank Theorem (Theorem 1.2).

The organization of the paper is as follows. We derive equation (1.1) together with some basic facts about elementary symmetric functions in the next section. In Sect. 3, we establish C^2 a priori estimates for convex solutions. The key deformation lemma will be proved in Sect. 4 for the Hessian equation on $\mathbb{S}^n$. In Sect. 5, we will use the a priori estimates in Sect. 3 and the deformation lemma in Sect. 4 to prove Theorem 1.2 and Theorem 1.3.

Acknowledgements. This work was done while the second author was visiting McMaster University. He would like to thank the Department of Mathematics at McMaster University for its warm hospitality. We would like to thank Professor D. Jerison and the referee for valuable comments and suggestions. We would also like to thank Professors C. Riehm and E. Sawyer for the help in the exposition of the paper.

2. Preliminaries

We recall the definition of k-symmetric functions: For $1 \le k \le n$, and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum \lambda_{i_1} \dots \lambda_{i_k}, \tag{2.1}$$

where the sum is taken over all strictly increasing sequences $i_1, \dots, i_k$ of the indices from the set $\{1, \dots, n\}$. The definition can be extended to symmetric matrices by letting $S_k(W) = S_k(\lambda(W))$, where $\lambda(W) = (\lambda_1(W), \dots, \lambda_n(W))$ are the eigenvalues of the symmetric matrix W. We also set $S_0 = 1$ and $S_k = 0$ for $k > n$.

For a strictly convex body K in $\mathbb{R}^{n+1}$ with smooth boundary M, the Gauss map $\vec{n}$ is a diffeomorphism from M to $\mathbb{S}^n$. For $x \in \mathbb{S}^n$, let $\lambda(x) = (\lambda_1(x), \dots, \lambda_n(x))$ be the principal radii of curvature of M at the point $\vec{n}^{-1}(x)$. Then

$$S_k(x) = S_k(\lambda(x)) \tag{2.2}$$

is the k-th surface area function over the unit sphere $\mathbb{S}^n$ at the point x. The support function of the convex body K, defined by $u(x) = \max_{y \in K} x \cdot y$, can in this case be written as $u(x) = x \cdot \vec{n}^{-1}(x)$. Let $e_1, \dots, e_n$ be any orthonormal frame on $\mathbb{S}^n$, and let u_{ij} be the covariant derivatives with respect to this frame. The Hessian matrix $W(x) = (u_{ij}(x) + u(x)\delta_{ij})$ is the reverse second fundamental form of the hypersurface. The eigenvalues $\lambda(W(x)) = (\lambda_1(x), \dots, \lambda_n(x))$ of $W(x)$ (with respect to the standard metric on $\mathbb{S}^n$) are the principal radii of M at $\vec{n}^{-1}(x)$ (see [7], [28] and also Sect. 2.5 in [29]). Hence, the function u satisfies equation (1.1).

On the other hand, suppose u is a solution of (1.1) satisfying the *convex* condition (1.2). After extending it to in $\mathbb{R}^{n+1}$ as a homogeneous function of degree 1, u is convex in $\mathbb{R}^{n+1}$. So u satisfies the sublinear relation $u(x+y) \le u(x) + u(y), \forall x, y \in \mathbb{R}^{n+1}$. In turn, u is the support function of some convex body in $\mathbb{R}^{n+1}$ (e.g., Theorem 1.6.5 in [29]). Since W is positive definite and $u \in C^2$, the boundary M of the convex body is C^2 (see Appendix in [33]). In fact, if we view u as a homogeneous function of degree 1 in $\mathbb{R}^{n+1}$, M can be recovered from u explicitly as the image of the gradient map $\nabla_{\mathbb{R}^{n+1}} u$ (e.g., p. 106 in [29]):

$$\nabla_{\mathbb{R}^{n+1}} u(x) = \vec{n}^{-1}\left(\frac{x}{|x|}\right), \quad \forall x \neq 0 \in \mathbb{R}^{n+1}. \tag{2.3}$$

It follows from this identity that for $0 \le \alpha < 1$ and $l \ge 2$, the inverse Gauss map $\vec{n}^{-1}$ is $C^{l-1,\alpha}$ if and only if u is in $C^{l,\alpha}$. It is easy to see that M is in $C^{l,\alpha}$ if and only if $\vec{n}$ is in $C^{l-1,\alpha}$ as it can be expressed as a local graph. Therefore, we have a precise regularity relationship ($l \ge 2$): $M \in C^{l,\alpha}$ if and only if $u \in C^{l,\alpha}$.

We now turn to the definition of admissible solutions of equation (1.1). The structure of this type of equations has been investigated in [5], [20], [31], [32], [23]. The natural solution class for this type of equations is the class of k-convex functions are defined as follows.

Definition 2.1. *Let $\mathcal{S}$ be the space consisting all $n \times n$ symmetric matrices. For $1 \le k \le n$, let Γ_k be the connected cone in $\mathcal{S}$ containing the identity matrix determined by*

$$\Gamma_k = \{W \in \mathcal{S} : \quad S_1(W) > 0, \dots, S_k(W) > 0\}.$$

If $u \in C^2(\mathbb{S}^n)$, we say u is **k-convex** *if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in \mathbb{S}^n$. We observe that u is convex (i.e., satisfying (1.2)) on $\mathbb{S}^n$ if and only u is n-convex. Furthermore, u is called an* **admissible** *solution of (1.1) if u is k-convex and satisfies (1.1).*

It will become clear that the algebraic properties of the elementary symmetric functions are indispensable in our proofs. We also refer to the recent work [18] for the important role of elementary symmetric functions in other contexts. The following are some basic results for the elementary symmetric functions that we will use in later sections.

Proposition 2.2. *If $W = (W_{ij})$ is an $n \times n$ symmetric matrix, let $F(W) = S_k(W)$ for $1 \le k \le n$. Then the following relations hold.*

$$S_k(W) = \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ j_1,\dots,j_k=1}}^{n} \delta(i_1, \dots, i_k; j_1, \dots, j_k) W_{i_1 j_1} \cdots W_{i_k j_k},$$

$$\begin{aligned} F^{\alpha\beta} &:= \frac{\partial F}{\partial W_{\alpha\beta}}(W) \\ &= \frac{1}{(k-1)!} \sum_{\substack{i_1,\dots,i_{k-1}=1 \\ j_1,\dots,j_{k-1}=1}}^{n} \delta(\alpha, i_1, \dots, i_{k-1}; \beta, j_1, \dots, j_{k-1}) W_{i_1 j_1} \cdots W_{i_{k-1} j_{k-1}} \\ F^{ij,rs} &:= \frac{\partial^2 F}{\partial W_{ij} \partial W_{rs}}(W) \\ &= \frac{1}{(k-2)!} \sum_{\substack{i_1,\dots,i_{k-2}=1 \\ j_1,\dots,j_{k-2}=1}}^{n} \delta(i, r, i_1, \dots, i_{k-2}; j, s, j_1, \dots, j_{k-2}) W_{i_1 j_1} \cdots W_{i_{k-2} j_{k-2}}, \end{aligned}$$

where the Kronecker symbol $\delta(I; J)$ for indices $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_m)$ is defined as

$$\delta(I; J) = \begin{cases} 1, & \text{if } I \text{ is an even permutation of } J; \\ -1, & \text{if } I \text{ is an odd permutation of } J; \\ 0, & \text{otherwise.} \end{cases}$$

We will need the next two lemmas in later sections.

Lemma 2.3. *For $1 \le k \le l$, $G = (\lambda_1, \dots, \lambda_l)$, $1 \le i, j \le l, i \ne j$, we denote by $S_k(G|i)$ the symmetric function with $\lambda_i = 0$ and $S_k(G|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$. Then the following hold,*

$$\begin{aligned} S_k(G) S_{l-1}(G|\alpha) S_{k-1}(G|\alpha) - S_l(G) S_{k-1}^2(G|\alpha) \\ = S_k(G|\alpha) S_{l-1}(G|\alpha) S_{k-1}(G|\alpha). \end{aligned}$$

If $1 \le k \le l$, and $\alpha \ne \beta$,

$$\begin{aligned} S_k(G) S_{k-2}(G|\alpha\beta) - S_{k-1}(G|\alpha) S_{k-1}(G|\beta) \\ = S_k(G|\alpha\beta) S_{k-2}(G|\alpha\beta) - S_{k-1}^2(G|\alpha\beta). \end{aligned}$$

Proof. We first make a simple observation on $S_l(G)$ which will also be used repeatedly in the rest of the paper. As l is equal to the size of G, $S_l(G) = \lambda_1 \dots \lambda_l$, and we have for $\alpha \ne \beta$ fixed,

(2.4)
$$S_l(G|\alpha) = 0, \quad S_l(G) = \lambda_\alpha S_{l-1}(G|\alpha), \quad S_l(G) = \lambda_\alpha \lambda_\beta S_{l-2}(G|\alpha\beta).$$

From the definition of $S_k(\lambda)$, we have the following identities:

$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i), \tag{2.5}$$

$$S_k(\lambda) = S_k(\lambda|ij) + \lambda_i S_{k-1}(\lambda|ij) + \lambda_j S_{k-1}(\lambda|ij) + \lambda_i\lambda_j S_{k-2}(\lambda|ij). \tag{2.6}$$

Now for any fixed $\alpha \in G$,

$$\begin{aligned}
&S_k(G)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)\\
&\quad = [\lambda_\alpha S_{k-1}(G|\alpha) + S_k(G|\alpha)]S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)\\
&\quad = S_l(G)S_{k-1}(G|\alpha)^2 + S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha) - S_l(G)S_{k-1}^2(G|\alpha)\\
&\quad = S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha).
\end{aligned}$$

The second identity in the lemma follows directly from the identities (2.5) and (2.6). □

Lemma 2.4. *For $1 \le k \le l$, $G = (\lambda_1, \dots, \lambda_l)$ and with $\lambda_i \ge 0$, for $1 \le i \le l$, $\forall \alpha \ne \beta$ and for all real numbers $\gamma_1, \dots, \gamma_l$,*

$$\begin{aligned}
&\sum_{\alpha\in G} S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)\gamma_\alpha^2\\
&\quad \ge S_l(G)\sum_{\alpha\ne\beta}\big(S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\big)\gamma_\alpha\gamma_\beta.
\end{aligned} \tag{2.7}$$

Proof. For convenience in notation, we write $\alpha \in G$ for $\lambda_\alpha \in G$. We first prove the following equality: for $1 \le \alpha \le l$,

$$\sum_{\beta\in G,\beta\ne\alpha} \lambda_\beta\big[S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\big] = S_k(G|\alpha)S_{k-1}(G|\alpha). \tag{2.8}$$

By counting the terms in the definition of S_k, for $\alpha \in G, 0 \le m \le l-1$ fixed,

$$\sum_{\beta\in G,\beta\ne\alpha} S_m(G|\alpha\beta) = (l-m-1)S_m(G|\alpha).$$

It follows that

$$\sum_{\beta\in G,\beta\ne\alpha} S_{k-1}(G|\alpha\beta)S_k(G|\alpha) = (l-k)S_k(G|\alpha)S_{k-1}(G|\alpha),$$

and

$$\sum_{\beta\in G,\beta\ne\alpha} S_k(G|\alpha\beta)S_{k-1}(G|\alpha) = (l-k-1)S_{k-1}(G|\alpha)S_k(G|\alpha).$$

From (2.4),

$$
\begin{aligned}
&\sum_{\substack{\beta\in G\\ \beta\neq\alpha}}\left\{\lambda_\beta S_{k-1}^2(G|\alpha\beta)-\lambda_\beta S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right\}\\
&=\sum_{\substack{\beta\in G\\ \beta\neq\alpha}}[S_{k-1}(G|\alpha\beta)S_k(G|\alpha)-S_k(G|\alpha\beta)(S_{k-1}(G|\alpha\beta)+\lambda_\beta S_{k-2}(G|\alpha\beta))]\\
&=\sum_{\substack{\beta\in G\\ \beta\neq\alpha}}[S_{k-1}(G|\alpha\beta)S_k(G|\alpha)-S_k(G|\alpha\beta)S_{k-1}(G|\alpha)]\\
&=S_k(G|\alpha)[(l-k)S_{k-1}(G|\alpha)-(l-k-1)S_{k-1}(G|\alpha)]\\
&=S_k(G|\alpha)S_{k-1}(G|\alpha).
\end{aligned}
$$

This proves (2.8). Now we use the Cauchy inequality and (2.8) to prove (2.7).

For any $\alpha\neq\beta$, $S_l(G)=\lambda_\alpha\lambda_\beta S_{l-2}(G|\alpha\beta)$ by (2.4), and by the Newton-MacLaurin inequality, $S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\geq 0$. Therefore,

$$
\begin{aligned}
&S_l(G)\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\left[S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right]\gamma_\alpha\gamma_\beta\\
&=\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\left\{S_{l-2}(G|\alpha\beta)\left[S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right]\right\}(\lambda_\beta\gamma_\alpha)(\lambda_\alpha\gamma_\beta)\\
&\leq\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\left\{S_{l-2}(G|\alpha\beta)\left[S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right]\right\}\frac{\lambda_\beta^2\gamma_\alpha^2+\lambda_\alpha^2\gamma_\beta^2}{2}\\
&=\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}S_{l-2}(G|\alpha\beta)\lambda_\beta\left[S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right]\lambda_\beta\gamma_\alpha^2\\
&=\sum_{\alpha\in G}S_{l-1}(G|\alpha)\sum_{\beta\in G,\beta\neq\alpha}\lambda_\beta\left[S_{k-1}^2(G|\alpha\beta)-S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)\right]\gamma_\alpha^2\\
&=\sum_{\alpha\in G}S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)\gamma_\alpha^2.
\end{aligned}
$$

This completes the proof of (2.7). □

3. A priori estimates

We establish a priori estimates for solutions of equation (1.1) in this section. The equation (1.1) will be uniformly elliptic once C^2 estimates are established for u (see [5]). By the Evans-Krylov theorem and Schauder theory, one can obtain higher derivative estimates for u. Therefore, we only need to get C^2 estimates for u. We first establish a C^0 bound for u. In the case $k = n$, Cheng-Yau in [7] obtained a C^0 bound using the isoperimetric inequality. We modify their approach by making use of a quermassintegral inequality.

For a solution u of equation (1.1), $u + f$ is also a solution for any linear function f. In order to have C^0 estimates, we restrict u to satisfy the following orthogonality condition:

$$\int_{\mathbb{S}^n} x_i u \, dx = 0, \qquad \forall i = 1, 2, \dots, n+1. \tag{3.1}$$

If u is a support function of some convex body Ω, condition (3.1) implies that the Steiner point of Ω coincides with the origin.

Lemma 3.1. *Suppose $M \in C^2$, M is a compact convex hypersurface in $\mathbb{R}^{n+1}$, and let φ be the k-th surface area function of M. If L is the extrinsic diameter of M, then*

$$L \le c_{n,k} \left(\int_{\mathbb{S}^n} \varphi \right)^{\frac{k+1}{k}} \left\{ \inf_{y \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max(0, \langle y, x \rangle) \varphi(x) \right\}^{-1},$$

where $c_{n,k}$ is a constant depending only on n and k. In particular, if u is a support function of M satisfying (1.1) and (3.1), then

$$0 \le \min u \le \max u \le c_{n,k} \left(\int_{\mathbb{S}^n} \varphi \right)^{\frac{k+1}{k}} \left\{ \inf_{y \in \mathbb{S}^n} \int_{\mathbb{S}^n} \max(0, \langle y, x \rangle) \varphi(x) \right\}^{-1}.$$

Proof. The argument follows mainly that in [7]. Here we will make use of a quermassintegral inequality and the Minkowski formula.

Let $p, q \in M$, such that the line segment joining p and q has length L. We may assume 0 is in the middle of the line segment. Let $\vec{y}$ be a unit vector in the direction of this line. Let v be the support function, and $W = \{v_{ij} + v\delta_{ij}\}$. We have $S_k(W) = \varphi$. Now, for $x \in \mathbb{S}^n$, we get

$$v(x) = \sup_{Z \in M} \langle Z, x \rangle \ge \frac{1}{2} L \max(0, \langle y, x \rangle).$$

If we multiply by φ and integrate over $\mathbb{S}^n$, we get

$$L \le 2 \left(\int_{\mathbb{S}^n} v\varphi \right) \left(\int_{\mathbb{S}^n} \max(0, \langle y, x \rangle) \varphi \right)^{-1}.$$

By the Quermassintegral inequality,

$$\left(\int_{\mathbb{S}^n} v S_k(W) \right)^{\frac{1}{k+1}} \le C_{n,k} \left(\int_{\mathbb{S}^n} v S_{k-1}(W) \right)^{\frac{1}{k}}.$$

On the other hand, from a Minkowski type formula (e.g., (5.3.14) in p. 291 in [29], note that we have a different normalization of S_k), we have

$$(n-k+1)\int_{\mathbb{S}^n} vS_{k-1}(W) = k\int_{\mathbb{S}^n} S_k(W) = k\int_{\mathbb{S}^n}\varphi.$$

In turn, we get

$$L \le c_{n,k}\left(\int_{\mathbb{S}^n}\varphi\right)^{\frac{k+1}{k}}\left(\inf_{y\in\mathbb{S}^n}\int_{\mathbb{S}^n}\max(0,(y,x))\varphi\right)^{-1}.$$

Since u satisfies (3.1), the Steiner point of M is the origin. The last inequality is a consequence of the above inequality. $\square$

In the case of $k=1$, equation (1.1) is a linear elliptic equation on the sphere. That C^2 a priori estimates hold for a solution u satisfying (3.1) in this case follows from standard linear elliptic theory. Therefore, we will restrict ourselves to the case $k\ge 2$.

Proposition 3.2. *There is a constant $C>0$ depending only on n, k, $\|\varphi\|_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n}\varphi$, such that if u satisfies (3.1) and u is an admissible solution of (1.1), then $\|u\|_{C^2(\mathbb{S}^n)}\le C$. There is an explicit bound for the function $H := trace(u_{ij}+\delta_{ij}u) = \triangle u + nu$,*

$$\min_{x\in\mathbb{S}^n}(n\tilde\varphi(x)) \le \max_{x\in\mathbb{S}^n} H(x) \le \max_{x\in\mathbb{S}^n}(n\tilde\varphi(x)-\triangle\tilde\varphi(x)), \tag{3.2}$$

where $\tilde\varphi := (\frac{\varphi}{C_n^k})^{\frac{1}{k}}$, $C_n^k = \frac{n!}{k!(n-k)!}$.

Proof. Since u is k-convex ($k\ge 2$), the entries $|u_{ij}+\delta_{ij}u|$ are controlled by H. By Lemma 3.1, we have a C^0 bound on u. So the $|u_{ij}|$ are controlled by H. C^1 estimates follows from interpolation if we have bounds on the second derivatives. Therefore, we only need to bound H. The first inequality follows from the Newton-MacLaurin inequality. Assume the maximum value of H is attained at a point $x_0\in\mathbb{S}^n$. We choose an orthonormal local frame $e_1, e_2, \ldots, e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal. If $W=(u_{ij}+\delta_{ij}u)$, we define $G(W) := (\frac{S_k}{C_n^k})^{\frac{1}{k}}(W)$. Then equation (1.1) becomes

$$G(W) = \tilde\varphi. \tag{3.3}$$

For the standard metric on $\mathbb{S}^n$, one may easily check the commutator identity $H_{ii} = \triangle W_{ii} - nW_{ii} + H$. By assumption the matrix $W\in\Gamma_k$, so (G^{ij}) is positive definite. Since $(H_{ij})\le 0$, and (G^{ij}) is diagonal, by the above commutator identity, it follows that at x_0,

$$0 \ge G^{ij}H_{ij} = G^{ii}(\triangle W_{ii}) - nG^{ii}W_{ii} + H\sum_i^n G^{ii}. \tag{3.4}$$

As G is homogeneous of degree one, we have

$$G^{ii}W_{ii} = \tilde{\varphi}. \tag{3.5}$$

Next we apply the Laplace operator to equation (3.3) to obtain

$$G^{ij}W_{ijk} = \nabla_k\tilde{\varphi}, \quad G^{ij,rs}W_{ijk}W_{rsk} + G^{ij}\triangle W_{ij} = \triangle\tilde{\varphi}.$$

By the concavity of G, at x_o we have

$$G^{ii}\triangle(W_{ii}) \ge \triangle\tilde{\varphi}. \tag{3.6}$$

Combining (3.5), (3.6) and (3.4), we see that

$$0 \ge \triangle\tilde{\varphi} - n\tilde{\varphi} + H\sum_{i=1}^{n} G^{ii}. \tag{3.7}$$

As W is diagonal at the point, we may write $W = (W_{11}, \dots, W_{nn})$ as a vector in $\mathbb{R}^n$. A simple calculation yields

$$G^{ii} = \frac{S_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}\frac{\partial S_k(W)}{\partial W_{ii}} = \frac{S_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}S_{k-1}(W|i),$$

where $(W|i)$ is the vector given by W with W_{ii} deleted. It follows from the Newton-MacLaurin inequality that

$$\sum_{i=1}^{n} G^{ii} = (n-k+1)\frac{S_k(W)^{\frac{1}{k}-1}}{(C_n^k)^{\frac{1}{k}}}S_{k-1}(W) \ge 1.$$

By (3.7), $H \le n\tilde{\varphi} - \triangle\tilde{\varphi}$. □

By the Evans-Krylov theorem and Schauder theory (e.g, see [14]), together with Proposition 3.2, we have the following a priori estimates.

Theorem 3.3. *For each integer $l \ge 1$ and $0 < \alpha < 1$, there exist a constant C depending only on $n, l, \alpha, \min\varphi$, and $||\varphi||_{C^{l,1}}(\mathbb{S}^n)$ such that*

$$||u||_{C^{l+1,\alpha}}(\mathbb{S}^n) \le C, \tag{3.8}$$

for all admissible solution of (1.1) satisfying the condition (3.1).

So far, we have obtained an upper bound for the principal radii of the Christoffel-Minkowski problem. For the Minkowski problem, a lower bound for the principal radii then follows directly from equation (1.1). But when $k < n$, there is no such lower bound in general. In the next section, we will show that the principal radii of the general Christoffel-Minkowski problem are bounded from below if φ satisfies the condition in Theorem 1.3. In the case of the Christoffel problem, Firey's conditions [11] are necessary and sufficient. But they are very cumbersome and difficulty to verify. It is desirable to have some simple sufficient conditions. Pogorelov in [27] established a lower bound of the principal radii on S^2 under the condition

$$\varphi(x) - \varphi_{ss}(x) > 0, \quad \text{on} \quad S^2, \tag{3.9}$$

where $\varphi(x)$ is differentiated at the point x with respect to arc length of the great circle on S^2.

To conclude this section, we derive a simple estimate which drops the dimensionality restriction in (3.9). For the Christoffel problem, equation (1.1) can be written in the simple form $\sum_{i=1}^n W_{ii} = \varphi$.

We may assume that the smallest eigenvalue of the matrix (W_{ij}) is attained at some point $x_o \in \mathbb{S}^n$ and along the e_1 direction. Then we have

$$\nabla_i W_{11}(x_o) = 0, \quad i = 1, 2, \ldots, n; \quad \Delta W_{11}(x_o) \geq 0.$$

As $W_{11ii} = W_{ii11} + W_{11} - W_{ii}$, at the point x_o,

$$0 \leq \sum_{i=1}^n W_{11ii} = (\Delta W)_{11} + nW_{11} - \sum_{i=1}^n W_{ii} = \varphi_{11} + nW_{11} - \varphi.$$

Therefore at x_o, $nW_{11} \geq \varphi - \varphi_{11}$.

4. A Deformation Lemma

In this section, we establish the key deformation lemma which will set the stage for the strong minimum principle. As in the previous section, we let $W = (u_{ij} + \delta_{ij} u)$.

Lemma 4.1. (Deformation Lemma) *Let $O \subset \mathbb{S}^n$ be an open subset, suppose $u \in C^4(O)$ is a solution of (1.1) in O, and that the matrix $W = (W_{ij})$ is positive semi-definite. Suppose there is a positive constant $C_0 > 0$, such that for a fixed integer $(n-1) \geq l \geq k$, $S_l(W(x)) \geq C_0$ for all $x \in O$, . Let $\phi(x) = S_{l+1}(W(x))$ and let $\tau(x)$ be the largest eigenvalue of $\{-(\varphi^{-\frac{1}{k}})_{ij}(x) - \delta_{ij}\varphi^{-\frac{1}{k}}(x)\}$. Then there are constants C_1, C_2 depending only on $||u||_{C^3}$, $||\varphi||_{C^{1,1}}$, n, k and C_0, such that differential inequality*

$$\sum_{\alpha,\beta}^n F^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq k(n-l)\varphi^{\frac{k+1}{k}}(x) S_l(W(x))\tau(x) + C_1|\nabla\phi(x)| + C_2\phi(x) \tag{4.1}$$

holds in O, where the $F^{\alpha\beta}$ are defined in Proposition 2.2.

Remark 4.2. The lemma is a fully nonlinear version of the corresponding results of Caffarelli-Friedman [6] and Korevaar-Lewis [22] for the Lapalace equation in $\mathbb{R}^n$. We note that the a priori estimates established in the previous section are not used here since we are working on the assumption $u \in C^4$. Also, we make no assumption on the size of S_{l+1}, and the constants C_1, C_2 in Lemma 4.1 depend only on $||u||_{C^3}$, $||\varphi||_{C^{1,1}}$, n, k and C_0. This dependence is crucial in establishing Theorem 1.3 for $\varphi \in C^{1,1}$.

The rest of this section will be devoted to the proof of the Deformation Lemma. As the proof is technically complicated, we would like to sketch some of the main lines first. Let

$$F^{\alpha\beta} = \frac{\partial S_k(W)}{\partial W_{\alpha\beta}}, \qquad F^{ij,rs} = \frac{\partial S_k(W)}{\partial W_{ij}\partial W_{rs}} \tag{4.2}$$

as defined in Proposition 2.2. We set

$$S^{ij} = \frac{\partial S_{l+1}(W)}{\partial W_{ij}}, \qquad S^{ij,rs} = \frac{\partial^2 S_{l+1}(W)}{\partial W_{ij}\partial W_{rs}}. \tag{4.3}$$

Recall that $\varphi(x) = S_k(W(x))$, and $\phi(x) = S_{l+1}(W(x))$. Since W is positive semi-definite and u is k-convex, $(F^{\alpha\beta})$ is positive definite and (S^{ij}) is positive semi-definite. We observe that there are at least l positive eigenvalues of W with a controlled lower bound by the assumption $S_l(W) \geq C_0$. Let B be that part of the index set so arranged such that the W_{ii} might be small (controlled by ϕ) for $i \in B$ (see the proof below for the precise definition). In view of this observation, W_{ii} is negligible for each $i \in B$. The basic idea in the proof of Deformation Lemma is to to explore the relationship between $\sum_{\alpha,\beta}^n F^{\alpha\beta}\phi_{\alpha\beta}$ and $\varphi^{\frac{k+1}{k}} S_l(W)\sum_i\{(\varphi^{-\frac{1}{k}})_{ii} + \delta_{ii}\varphi^{-\frac{1}{k}}\}$. One of the key terms to be handled will be $\sum_{i,\alpha} S^{ii}F^{\alpha\alpha}W_{ii\alpha\alpha}$. With the help of some basic properties of elementary symmetric functions, it turns out that some algebraic cancellations will occur after commuting covariant derivatives and re-arranging the terms to fit the right algebraic formats! Almost all of the computations in the proof are algebraic and the inequality in Lemma 2.4 will be used in a crucial way in the last step of the proof.

Proof of the Deformation Lemma. Following the notation of Caffarelli and Friedman [6], for two functions defined in an open set $O \subset \mathbb{S}^n$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{4.4}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constants c_1, and c_2 depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We shall show that

$$\sum_{\alpha,\beta=1}^n F^{\alpha\beta}\phi_{\alpha\beta} \lesssim k(n-l)\varphi^{\frac{k+1}{k}} S_l(W)\tau. \tag{4.5}$$

For any $z \in O$, let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ be the eigenvalues of W at z. Since $S_l(W) \geq C_0 > 0$ and $u \in C^3$, for any $z \in \mathbb{S}^n$, there is a positive constant $C > 0$ depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0, such that $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_l \geq C$. Let $G = \{1, 2, \ldots, l\}$ and $B = \{l+1, \ldots, n\}$ be the "good" and "bad" sets of indices respectively, and define $S_k(W|i) = S_k((W|i))$ where $(W|i)$ means that the matrix W excluding the i-column and i-row, and $(W|ij)$ means that the matrix W excluding the i, j columns and i, j rows. Let $\Lambda_G = (\lambda_1, \ldots, \lambda_l)$ be the "good" eigenvalues of W at z; for convenience in notation, we also write $G = \Lambda_G$ if there is no confusion. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (4.4) are under control.

For each fixed $z \in O$ fixed, we choose a local orthonormal frame $e_1, \ldots, e_n$ so that W is diagonal at z, and $W_{ii} = \lambda_i, \forall i = 1, \ldots, n$. Now we compute ϕ and its first and second derivatives in the direction e_α.

We note that S^{ij} in (4.3) is diagonal at the point since W is diagonal. As $\phi = S_{l+1}(W)$ and $\phi_\alpha = \sum_{i,j} S^{ij} W_{ij\alpha}$, we find that (as W is diagonal at z),

(4.6)
$$0 \sim \phi(z) \sim \Big(\sum_{i\in B} W_{ii}\Big) S_l(G) \sim \sum_{i\in B} W_{ii}, \quad (\text{so} \quad W_{ii} \sim 0, \quad i \in B),$$

This relation yields that, for $1 \le m \le l$,

$$(4.7) \quad S_m(W) \sim S_m(G), \quad S_m(W|j) \sim \begin{cases} S_m(G|j), & \text{if } j \in G; \\ S_m(G), & \text{if } j \in B. \end{cases}$$

$$S_m(W|ij) \sim \begin{cases} S_m(G|ij), & \text{if } i, j \in G; \\ S_m(G|j), & \text{if } i \in B, j \in G; \\ S_m(G), & \text{if } i, j \in B, i \neq j. \end{cases}$$

Also,

$$(4.8) \qquad 0 \sim \phi_\alpha \sim S_l(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i\in B} W_{ii\alpha}$$

By Proposition 2.2,

$$(4.9) \qquad S^{ij} \sim \begin{cases} S_l(G), & \text{if } i = j \in B, \\ 0, & \text{otherwise.} \end{cases}$$

$$(4.10) \qquad S^{ij,rs} = \begin{cases} S_{l-1}(W|ir), & \text{if } i = j, r = s, i \neq r; \\ -S_{l-1}(W|ij), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we will use the relations (4.6)–(4.10) to single out the main terms in the calculation.

Since $\phi_{\alpha\alpha} = \sum_{i,j} (S^{ij,rs} W_{rs\alpha} W_{ij\alpha} + S^{ij} W_{ij\alpha\alpha})$, it follows from (4.10) that for any $\alpha \in \{1, 2, \dots, n\}$

$$
\begin{aligned}
\phi_{\alpha\alpha} &= \sum_{i\neq j} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} - \sum_{i\neq j} S_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i S^{ii} W_{ii\alpha\alpha} \\
&= \Big(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}} \Big) S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \\
&\quad - \Big(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}} \Big) S_{l-1}(W|ij) W_{ij\alpha}^2 + \sum_i S^{ii} W_{ii\alpha\alpha}.
\end{aligned} \tag{4.11}
$$

We want to simplify the above expression. From (4.8) and (4.7), we have

$$
\sum_{\substack{i\in B\\ j\in G}} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim \Big(\sum_{j\in G} S_{l-1}(G|j) W_{jj\alpha} \Big) \sum_{i\in B} W_{ii\alpha} \sim 0. \tag{4.12}
$$

By (4.8), $\forall i \in B$ fixed and $\forall \alpha$, $-W_{ii\alpha} \sim \sum_{\substack{j\in B\\ j\neq i}} W_{jj\alpha}$. Then (4.7) yields,

$$
\sum_{\substack{i,j\in B\\ i\neq j}} S_{l-1}(W|ij) W_{ii\alpha} W_{jj\alpha} \sim -S_{l-1}(G) \sum_{i\in B} W_{ii\alpha}^2. \tag{4.13}
$$

and

$$
\sum_{j\in G, i\in B} S_{l-1}(W|ij) W_{ij\alpha}^2 \sim \sum_{i\in B, j\in G} S_{l-1}(G|j) W_{ij\alpha}^2. \tag{4.14}
$$

Inserting (4.12)–(4.14) into (4.11), by (4.7) we obtain

$$
\phi_{\alpha\alpha} \sim \sum_i S^{ii} W_{ii\alpha\alpha} - 2 \sum_{\substack{i\in B\\ j\in G}} S_{l-1}(G|j) W_{ij\alpha}^2 - S_{l-1}(G) \sum_{i,j\in B} W_{ij\alpha}^2. \tag{4.15}
$$

Now we would like to construct a contraction of $\phi_{\alpha\alpha}$ with $F^{\alpha\alpha}$ in (4.2) (note that $F^{\alpha\beta}$ is diagonal at the point). By Proposition 2.2 and (4.6), we have for any $\alpha \in \{1, 2, \dots, n\}$

$$
F^{\alpha\beta} \sim \begin{cases} S_{k-1}(G|\alpha), & \text{if } \alpha \in G, \alpha = \beta; \\ S_{k-1}(G), & \text{if } \alpha \in B, \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta, \end{cases} \tag{4.16}
$$

and

$$
F^{ij,rs} = \begin{cases} S_{k-2}(W|ir), & \text{if } i = j, r = s, i \neq r; \\ -S_{k-2}(W|ij), & \text{if } i \neq j, r = j, s = i; \\ 0, & \text{otherwise.} \end{cases} \tag{4.17}
$$

From (4.15) and (4.16) we obtain the contraction

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} = \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim \sum_{\alpha=1}^{n}\sum_{i} S^{ii}F^{\alpha\alpha}W_{ii\alpha\alpha}$$

$$-2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} S_{l-1}(G|j)F^{\alpha\alpha}W_{ij\alpha}^2 - S_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2. \tag{4.18}$$

To put the above in a useful form, we will start to commute the covariant derivatives and make use of some basic properties of the elementary symmetric functions. For example, homogeneity of S_k, identities (2.4)–(2.6) and the Newton-MacLaurin inequality will be used repeatedly.

By (4.6), (4.9) and homogeneity of S_k and S_{l+1} (since $|B| = n-l$)

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}(W_{ii}-W_{\alpha\alpha}) = (l+1)\phi\sum_{\alpha=1}^{n}F^{\alpha\alpha} - k\varphi\sum_{i=1}^{n}S^{ii}$$

$$\sim -k\varphi\sum_{i\in B}S^{ii} \sim -(n-l)k\varphi S_l(G).$$

Commuting the covariant derivatives, it follows that

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}W_{ii\alpha\alpha} = \sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}(W_{\alpha\alpha ii}+W_{ii}-W_{\alpha\alpha}) \tag{4.19}$$

$$\sim \sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii} - (n-l)k\varphi S_l(G).$$

Differentiating equation (1.1), we get

$$\varphi_{ii} = \sum_{\alpha,\beta,r,s} F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + \sum_{\alpha,\beta}F^{\alpha\beta}W_{\alpha\beta ii}.$$

Propositon 2.2, together with (4.17) and (4.9) yield,

$$\sum_{\alpha}\sum_{i} S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii} = \sum_{i} S^{ii}\Big\{\varphi_{ii} - \sum_{\alpha,\beta,r,s}F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi}\Big\}$$

$$\sim \sum_{i\in B}\Big\{-\Big(\sum_{\substack{\alpha\in G\\ \beta\in B}} + \sum_{\substack{\alpha\in B\\ \beta\in G}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}\Big)S_{k-2}(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}$$

$$+\varphi_{ii} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} S_{k-2}(W|\alpha\beta)W_{\alpha\beta i}^2\Big\}S_l(G). \tag{4.20}$$

It follows from (4.7) and (4.8) that for $1 \le m \le n$,

$$\sum_{\substack{\alpha\in B\\ \beta\in G}} S_m(W|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i} \sim \Big[\sum_{\beta\in G} S_m(G|\beta)W_{\beta\beta i}\Big]\sum_{\alpha\in B} W_{\alpha\alpha i} \sim 0. \tag{4.21}$$

In turn,

(4.22)

$$\sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}W_{\alpha\alpha ii} \sim S_l(G)\sum_{i\in B}\Big\{\varphi_{ii} - \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\beta\beta i}W_{\alpha\alpha i}$$

$$- \sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)W_{\beta\beta i}W_{\alpha\alpha i} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i}\Big\}.$$

We note that $|B| = n-l$, so $\sum_{i\in B} k\varphi = (n-l)k\varphi$. Now inserting (4.22) and (4.19) to (4.18), by (4.7) and (4.16) we have

(4.23)

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \sim S_l(G)\sum_{i\in B}(\varphi_{ii} - k\varphi) - S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}$$

$$- S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)W_{\alpha\alpha i}W_{\beta\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G} S_{l-1}(G|\beta)S_{k-1}(W|\alpha)W^2_{i\beta\alpha}$$

$$+ S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta} S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - \sum_{\alpha=1}^{n} S_{l-1}(G)\sum_{i,\beta\in B} S_{k-1}(W|\alpha)W^2_{i\beta\alpha}.$$

We need to further simplify the terms in (4.23). We first deal with the fourth and fifth terms on the right hand side of (4.23). For $i \in B$, we regroup the summations in these terms as

$$\sum_{\alpha\neq\beta} = 2\sum_{\substack{\alpha\in B\\ \beta\in G}} + \sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}, \quad \sum_{\alpha=1}^{n}\sum_{\beta\in G} = \sum_{\substack{\alpha\in B\\ \beta\in G}} + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} + \sum_{\alpha=\beta\in G}. \tag{4.24}$$

Since W is positive semi-definite, by (2.5), $\forall\beta \in G$ fixed, $W_{\beta\beta}S_{k-2}(G|\beta) \le S_{k-1}(G)$. For any $\alpha \in B$, $\beta \in G$, $S_l(G) = S_{l-1}(G|\beta)W_{\beta\beta}$ by (2.4), and $S_{k-2}(W|\alpha\beta) \sim S_{k-2}(G|\beta)$ by (4.7). So we have,

$$\sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} \sim \sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_{l-1}(G|\beta)W_{\beta\beta}S_{k-2}(G|\beta)W^2_{\alpha\beta i}$$

$$\le \sum_{\substack{i,\alpha\in B\\ \beta\in G}} S_{l-1}(G|\beta)S_{k-1}(G)W^2_{\alpha\beta i}. \tag{4.25}$$

Also, when $\alpha, \beta \in G, \alpha \neq \beta$, as W is diagonal, by (2.4) and (2.5),

(4.26)
$$\begin{aligned} S_{l-1}(G|\beta)S_{k-1}(G|\alpha) &= S_{l-1}(G|\beta)[S_{k-1}(G|\alpha\beta) + W_{\beta\beta}S_{k-2}(G|\alpha\beta)] \\ &\geq S_{l-1}(G|\beta)W_{\beta\beta}S_{k-2}(G|\alpha\beta) = S_l(G)S_{k-2}(G|\alpha\beta). \end{aligned}$$

From (4.26), we get

(4.27)
$$\begin{aligned} &\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-1}(G|\beta)S_{k-1}(G|\alpha)W^2_{\alpha\beta i} \\ &\qquad \lesssim -\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{l-1}(G|\beta)S_{k-1}(G|\alpha)W^2_{\alpha\beta i} \leq 0. \end{aligned}$$

It is easy to check that $W_{i\beta\alpha} = W_{\alpha\beta i}$ on the standard $\mathbb{S}^n$ (recall that $W_{\alpha\beta} = u_{\alpha\beta} + \delta_{\alpha\beta}u$). Combining (4.25) and (4.27), taking into the account of the regroup identity (4.24), we obtain the inequality

$$\begin{aligned} &S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta} S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G} S_{l-1}(G|\beta)S_{k-1}(W|\alpha)W^2_{i\beta\alpha} \\ &\lesssim \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(W|\alpha\beta)W^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i}. \end{aligned}$$

We note that $S_m(W|\alpha\beta) \sim S_m(G), \forall \alpha, \beta \in B$ by (4.7). Putting the previous inequality into (4.23),

$$\begin{aligned} \sum_{\alpha,\beta}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim S_l(G)\Big[&\sum_{i\in B}(\varphi_{ii} - k\varphi) - \sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\Big] \\ &- 2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} \\ &- S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)W_{\alpha\alpha i}W_{\beta\beta i} \\ &- \sum_{i=1}^{n} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(W|i)W^2_{\alpha\beta i} \\ &+ \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(G)W^2_{\alpha\beta i} \end{aligned}$$

$$= S_l(G)\sum_{i\in B}\left[\varphi_{ii} - \frac{k+1}{k}\frac{\varphi_i^2}{\varphi} - k\varphi\right] + I_1 + I_2 + I_3, \tag{4.28}$$

where

$$I_1 = S_l(G)S_{k-2}(G) \sum_{\substack{i,\alpha,\beta\in B\\ \alpha\neq\beta}} \left[W^2_{\alpha\beta i} - W_{\alpha\alpha i}W_{\beta\beta i}\right]$$
$$- \sum_{i=1}^{n} S_{l-1}(G) \sum_{\alpha,\beta\in B} S_{k-1}(W|i)W^2_{\alpha\beta i},$$

$$I_2 = \sum_{i\in B}\left(\frac{S_l(G)\varphi_i^2}{k\varphi} - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i}\right),$$

and

$$I_3 = \sum_{i\in B}\left\{S_l(G)\left[\frac{\varphi_i^2}{\varphi} - \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G|\alpha\beta)W_{\alpha\alpha i}W_{\beta\beta i}\right]\right.$$
$$\left. - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i}\right\}.$$

Claim. $I_1 \lesssim 0$, $I_2 \lesssim 0$ and $I_3 \lesssim 0$.

If the *Claim* is true, it follows from (4.28) that

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim S_l(G)\sum_{i\in B}\left[\varphi_{ii} - \frac{k+1}{k}\frac{\varphi_i^2}{\varphi} - k\varphi\right]. \tag{4.29}$$

Then (4.5) follows from (4.29).

Proof of the Claim. Since $W_{i\beta\alpha} = W_{\alpha\beta i}$, we observe that by (4.7),

$$-\sum_{i=1}^{n} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(W|i)W^2_{\alpha\beta i}$$
$$\leq -\sum_{i\in B} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(W|i)W^2_{\alpha\beta i}$$
$$= -\sum_{i\in B} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(W|\alpha)W^2_{\alpha\beta i}$$
$$\lesssim -S_{l-1}(G)S_{k-1}(G)\left\{\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} W^2_{\alpha\beta i} + \sum_{i\in B}\sum_{\alpha\in B} W^2_{\alpha\alpha i}\right\}. \tag{4.30}$$

If we put (4.30) into I_1, by (4.8), (4.7) and the Newton-MacLaurin inequality, we get

$$
\begin{aligned}
I_1 \lesssim & -\Big\{S_l(G)S_{k-2}(G)\sum_{i,\alpha\in B} W_{\alpha\alpha i}\Big(\sum_{\beta\in B,\beta\neq\alpha} W_{\beta\beta i}\Big) \\
& + S_{l-1}(G)S_{k-1}(G)\sum_{i,\alpha\in B} W^2_{\alpha\alpha i}\Big\} \\
& + \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} [S_l(G)S_{k-2}(G) - S_{l-1}(G)S_{k-1}(G)]W^2_{\alpha\beta i} \\
\sim & [S_l(G)S_{k-2}(G) - S_{l-1}(G)S_{k-1}(G)]\Big[\sum_{i,\alpha\in B} W^2_{\alpha\alpha i} + \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} W^2_{\alpha\beta i}\Big] \le 0.
\end{aligned}
$$

To treat I_2, by (4.8) and Proposition 2.2, for $i\in B$,

$$
\varphi_i = \Big(\sum_{\alpha\in B} + \sum_{\alpha\in G}\Big) S_{k-1}(W|\alpha)W_{\alpha\alpha i} \sim \sum_{\alpha\in G} S_{k-1}(G|\alpha)W_{\alpha\alpha i}. \tag{4.31}
$$

By homogeneity of $S_k(W)$, (4.31) and (2.4),

$$
\begin{aligned}
I_2 &\sim \frac{1}{k\varphi}\Big(\sum_{\alpha\in G} S_l^{\frac12}(G)S_{k-1}(G|\alpha)W_{\alpha\alpha i}\Big)^2 - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} \\
&= \frac{1}{k\varphi}\Big[\sum_{\alpha\in G} S_{l-1}^{\frac12}(G|\alpha)W_{\alpha\alpha}^{\frac12}S_{k-1}(G|\alpha)W_{\alpha\alpha i}\Big]^2 - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} \\
&\le \frac{1}{k\varphi}\sum_{\alpha,\beta\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i}W_{\beta\beta}S_{k-1}(G|\beta) \\
&\quad - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} \\
&\sim \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} - \sum_{\alpha\in G} S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W^2_{\alpha\alpha i} \\
&= 0.
\end{aligned}
$$

Now we deal with I_3. It follows from (4.31) that for any $i\in B$,

$$
{\varphi_i}^2 \sim \sum_{\alpha\in G} S^2_{k-1}(G|\alpha)W^2_{\alpha\alpha i} + \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-1}(G|\alpha)S_{k-1}(G|\beta)W_{\alpha\alpha i}W_{\beta\beta i}.
$$

By Lemma 2.3 and Lemma 2.4, as $\varphi \sim S_k(G)$ by (4.7), we have

$$\begin{aligned}\varphi I_3 \sim \sum_{i\in B}\Big\{&\sum_{\alpha\in G}[S_l(G)S_{k-1}^2(G|\alpha) - S_k(G)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)]W_{\alpha\alpha i}^2\\
&+ S_l(G)\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}[S_{k-1}(G|\alpha)S_{k-1}(G|\beta) - S_k(G)S_{k-2}(G|\alpha\beta)]W_{\alpha\alpha i}W_{\beta\beta i}\Big\}\\
= \sum_{i\in B}\Big\{&-\sum_{\alpha\in G}S_k(G|\alpha)S_{l-1}(G|\alpha)S_{k-1}(G|\alpha)W_{\alpha\alpha i}^2\\
&+ S_l(G)\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}}[S_{k-1}^2(G|\alpha\beta) - S_k(G|\alpha\beta)S_{k-2}(G|\alpha\beta)]W_{\alpha\alpha i}W_{\beta\beta i}\Big\} \leq 0.\end{aligned}$$

The *Claim* is verified. The proof of the Deformation Lemma is complete. □

5. The existence and convexity

First, we prove Theorem 1.2.

Proof of Theorem 1.2. By the a priori estimates in Theorem 3.3, $u \in C^{3,\alpha}(\mathbb{S}^n)$, for $0 < \alpha < 1$. If W is not of full rank at some point x_0, then there is $n-1 \geq l \geq k$ such that $S_l(W(x)) > 0, \forall x \in \mathbb{S}^n$ and $\phi(x_0) = S_{l+1}(W(x_0)) = 0$. By (4.1) in the Deformation Lemma 4.1, as $\varphi \in \mathcal{C}_{-\frac{1}{k}}$,

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq C_1|\nabla\phi(x)| + C_2\phi(x).$$

The strong minimum principle implies $\phi = S_{l+1}(W) \equiv 0$. On the other hand, we may assume u satisfies (3.1), so u is nonnegative on $\mathbb{S}^n$. By the Minkowski type formula (e.g., page 291 in [29]), $(n-l)\int_{\mathbb{S}^n} uS_l(W) = (l+1)\int_{\mathbb{S}^n} S_{l+1}(W)$. We conclude that $u \equiv 0$. This is a contradiction to (1.1). □

Now we proceed to prove Theorem 1.3.

Since φ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$, there is a continuous function $h(t, x)$ in $[0, 1] \times \mathbb{S}^n$ such that $h(0, x) = 1$, $h(1, x) = \varphi(x)$ and h is in $\mathcal{C}_{-\frac{1}{k}}$ for each fixed t. Now, we approximate h by a sequence of positive functions h^m satisfying

Properties:

(i) h^m is continuous in $[0, 1] \times \mathbb{S}^n$, and $h^m(0, x) = 1$;
(ii) for each t fixed, h^m is smooth in the x variables and satisfies (1.3);
(iii) for each fixed m and l, h^m is in $C([0, 1] \times C^l(\mathbb{S}^n))$;
(iv) $h^m \longrightarrow h$ uniformly in $C([0, 1] \times C^{1,1}(\mathbb{S}^n))$.

Such a sequence can be easily obtained by the operations of smoothing and projecting in the x variables (to satisfy (1.3)). We point out that we do not require $h^m(t,.)$ to be in $\mathcal{C}_{-\frac{1}{k}}$.

We consider the following equation:

$$S_k\big(u^{t,m}_{ij}(x) + \delta_{ij}u^{t,m}(x)\big) = h^m(t,x), \qquad \forall x \in \mathbb{S}^n. \tag{5.1}$$

Proposition 5.1. *For sufficient large m, the equation (5.1) has a unique smooth strictly convex solution $u^{t,m}$ satisfying (3.1) for all $t \in [0,1]$.*

Proof. Uniqueness follows from the Alexandrov-Fenchel-Jessen Theorem, regularity follows from Theorem 3.3. We use the continuity method for existence.

For each m fixed, let $I_m = \{t \in [0,1] | (5.1)\text{has a strictly convex solution}\}$.

Since u is strictly convex and satisfies (1.1), the linearized operator L_u at u is self-adjoint and $Span\{x_1, \ldots, x_{n+1}\}$ is the exact kernel. By the standard implicit function theorem, I_m is open and non-empty (as $0 \in I_m$).

We claim I_m is closed when m is sufficiently large. Suppose this is not true. Then by Theorem 3.3 and the continuity method, there is a sequence of smooth functions $\{u^{t_m}\}$, and $t_m > 0$, $x_m \in \mathbb{S}^n$ such that $W_t = (u^{t,m}_{ij} + \delta_{ij}u^{t,m})$ is positive definite for $t < t_m$, u_{t_m} satisfying (5.1) with $\det(W_{t_m}(x_m)) = 0$. Since $h^m \longrightarrow h$ uniformly in $C^{1,1}$, by Theorem 3.3 there is a subsequence $\{t_{m_j}\}$ which converges to t_0, so $h^{m_j}(t_{m_j}, x)$ converges to $h(t_0, x)$ in $C^{1,1}$, and $u^{t_{m_j}}$ converges to a function u in $C^{3,\alpha}$ for every $0 \le \alpha < 1$. The Hessian matrix $W = \{u_{ij} + \delta_{ij}u\}$ is positive semi-definite on $\mathbb{S}^n$ and is degenerate at some point. On the other hand, u satisfies (1.1) with $\varphi = h(t_0, x)$. This is a contradiction to Theorem 1.2 as $h(t_0,.) \in \mathcal{C}_{-\frac{1}{k}}$. □

Proof of Theorem 1.3. As mentioned before, uniqueness is given by the Alexandrov-Fenchel-Jessen theorem. By Proposition 5.1, there is a sequence of strictly convex functions $\{u^m\}$ satisfying

$$S_k(W_m(x)) = h^m(1,x), \qquad \text{on} \quad \mathbb{S}^n.$$

By Theorem 3.3, there is a subsequence of smooth strictly convex functions $\{u^{m_j}\}$ which converges to u in $C^{3,\alpha}$ for every $0 \le \alpha < 1$. And u satisfies (1.1). By Theorem 1.2, u is strictly convex. The higher regularity and the analyticity of u follows from standard elliptic theory. □

We conclude this paper with some remarks regarding the sufficient condition in Theorem 1.3.

Remark 5.2. As mentioned in the introduction, $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ is equivalent to $\varphi^{-\frac{1}{k}}$ being convex in usual sense in $\mathbb{R}^{n+1}$ if we view it as a homogeneous function of degree 1. Equation (1.1) can be rewritten as

$$\varphi^{-\frac{1}{k}} S_k^{\frac{1}{k}}(u_{ij} + u\delta_{ij}) = 1.$$

We note that the differential operator $S_k^{\frac{1}{k}}$ is concave. The condition $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ may be interpreted as a dual condition to the concavity of $S_k^{\frac{1}{k}}$. If we consider the standard hessian equation on a domain in $\mathbb{R}^n$

$$S_k(u_{ij}(x)) = \varphi(x),$$

where u_{ij} are the second derivatives of u with respect to the standard flat metric on $\mathbb{R}^n$. There is a natural technical explanation of the convex condition on φ. The matrix (u_{ij}) is strictly positive if and only if there is a positive lower bound on its eigenvalues. This is equivalent to having an upper bound on the eigenvalues of $(\tilde{u}_{ij})$, where $\tilde{u}$ is the Legendre transform of u. Note that $\tilde{u}$ satisfies

$$\frac{S_n(\tilde{u}_{ij})}{S_{n-k}(\tilde{u}_{ij})} = \varphi^{-1}(\nabla\tilde{u}). \tag{5.2}$$

As the hessian quotient operator $(\frac{S_n}{S_{n-k}})^{1/k}$ is concave, the strict convexity of $\varphi^{-1/k}$ is a natural condition to ensure C^2 estimates for equation (5.2).

Remark 5.3. From the proof, the condition on φ in Theorem 1.3 can also be replaced by the condition that $\varphi > 0$ is connected to $S_k(v_{ij} + v\delta_{ij}) \in \mathcal{C}_{-\frac{1}{k}}$ for some arbitrary smooth strictly convex body with support function v in Theorem 1.3. If $\varphi > 0$ satisfies (1.3), it is easy to check that the following condition

$$\{-\varphi_{ij} + k\delta_{ij}\varphi\} \geq 0, \tag{5.3}$$

implies the condition in Theorem 1.3. So condition (5.3) is stronger than the condition Theorem 1.3. When $k = 1$, the Pogorelov's condition (3.9) for the Christoffel problem implies condition (5.3). Hence even in the case of $k = 1$, the condition in Theorem 1.3 is weaker than condition (3.9).

Remark 5.4. If G is an automorphic group of $\mathbb{S}^n$ which has no fixed point (e.g., G a symmetry action with respect to the origin) and if $\varphi_1 > 0$ and $\varphi_2 > 0$ are invariant under G, one may connect φ_1 and φ_2 in $\mathcal{C}_{-\frac{1}{k}}$ by the function $h(t,x) = (t\varphi(x)_1{}^{-\frac{1}{k}} + (1-t)\varphi_2{}^{-\frac{1}{k}})^{-k}$. Then $h(t,x)$ satisfies (1.3) automatically as it is invariant under G (see [16]). In particular, every G-invariant function $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ is connected to 1 in $\mathcal{C}_{-\frac{1}{k}}$. In this special situation, $\varphi \in \mathcal{C}_{-\frac{1}{k}}$ if and only if $\varphi = v^{-k}$ for some positive support function v of a G-invariant convex body. By Theorem 1.3, for any G-invariant convex bodies K_1 and K_2 with support functions v_1 and v_2 respectively, for $\lambda \in [0,1]$, there is a unique G-invariant convex body $\tilde{K}_\lambda$ with support function u such that $S_k(\{u_{ij} + \delta_{ij}u\}) = (\lambda v_1 + (1-\lambda)v_2)^{-k}$. The relation defines an operation for such G-invariant convex bodies. This observation shows that the class of functions which satisfy the condition in Theorem 1.3 is quite large.

Remark 5.5. In the case of figures of revolution, the intermediate Christoffel-Minkowski problems were solved in Firey [13]. Pogorelov in [28] obtained a sufficient condition for the intermediate Christoffel-Minkowski problems. Set $g_t = (t\varphi(x) + 1 - t)^{\frac{1}{k}}$, and let $\eta_1(x), \ldots, \eta_n(x)$ be the eigenvalues of the matrix $\{\delta_{ij} g_t - (g_t)_{ij}\}$ at the point x. Set $\tau = \max_{i,x}(\eta_i(x))$. Pogorelov's condition can be stated as $(\frac{n-1}{n})^{\frac{1}{2(k-1)}}\tau < \min_{\mathbb{S}^n} g_t$. Note that at any maximum point of g_t, it yields $(\frac{n-1}{n})^{\frac{1}{2(k-1)}} \max_{\mathbb{S}^n} g_t < \min_{\mathbb{S}^n} g_t$. This puts the restriction on g_t that φ is close to a constant.

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