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On the Christoffel-Minkowski problem of Firey's p-sum

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1. Introduction

The classical Brunn-Minkowski theory for convex bodies was developed from a few basic concepts: support functions, Minkowski combinations, and mixed volumes. As a special case of mixed volumes, the Quermassintegrals are important geometrical quantities of a convex body, and surface area measures are local versions of Quermassintegrals. The Christoffel-Minkowski problem concerns with the existence of convex bodies with prescribed surface area measure, for details please refer to [17,3,8,18].

In 1962, Firey [5] generalized the Minkowski combination to p-sums from $p = 1$ to $p \geq 1$. Later, Lutwak [13,14] showed that Firey's p-sum also leads to a Brunn-Minkowski theory for each $p > 1$. This theory has found many geometry applications, see for example, [16] and its references. It was also shown in [13] that the classical surface area measures could be extended to the p-sum case. So it is natural to consider a generalization of the classical Christoffel-Minkowski problem for each $p > 1$. The generalized Minkowski problem has been treated in [13,15,7,4]. In this paper we study the remaining case, which may be called the Christoffel-Minkowski problem of p-sum. First we introduce some notations and relevant results.

Let $\mathcal{K}^{n+1}$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space R^{n+1}, and let $\mathcal{K}_0^{n+1}$ denote the set of convex bodies containing the origin in their interiors. For $K \in \mathcal{K}^{n+1}$, let $h_K = h(K, \cdot) : S^n \to R$ denote the support function of K, and let $W_0(K), W_1(K), ..., W_{n+1}(K)$ denote the Quermassintegrals of K (see, for example [18]). Thus $W_0(K) = V(K)$,

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the volume of K and $W_{n+1} = V(B) = \omega_{n+1}$, where B is the unit ball in R^{n+1}. For each Firey p-sum, Lutwak [13] defined the mixed p-Quermassintegrals by

$$\frac{n+1-k}{p} W_{p,k}(K,L) = \lim_{\epsilon \to 0^+} \frac{W_k(K +_p \epsilon \cdot L) - W_k(K)}{\epsilon}. \tag{1.1}$$

where $K, L \in \mathcal{K}_0^{n+1}$ and $k = 0, \cdots, n$. When $p = 1$, $W_{p,k}(K,L)$ is the usual mixed Quermassintegral and will be denoted as $W_k(K,L)$. According to [13], $W_{p,k}$ has the following integral representation:

$$W_{p,k}(K,L) = \frac{1}{n+1} \int_{S^n} h(L,u)^p h(K,u)^{1-p} \mathrm{d}S_{n-k}(K,u) \tag{1.2}$$

for all $L \in \mathcal{K}_0^{n+1}$, where $S_{n-k}(K,u)$ is the $(n-k)$-th surface area measure of K. We say that K is of class C_+^2 if ∂K is a C^2 hypersurface with (everywhere) positive principal curvatures. From the theory of convex bodies and differential geometry (see for example [18] and [19]), we see that in this case

$$dS_{n-k}(K, \cdot) = S_{n-k}(h_{ij} + \delta_{ij} h) d\omega. \tag{1.3}$$

where $d\omega$ is the Lebesgue measure on S^n, h is the support function of K, h_{ij} is the second covariant derivative with respect to local orthonormal frame $\{e_1, e_2, ..., e_n\}$ on S^n and $S_{n-k}(h_{ij} + \delta_{ij} h)$ is the $(n-k)$-th elementary symmetric function of the eigenvalues of $(h_{ij} + \delta_{ij} h)$. This leads to

Definition 1. *For each* $p \geq 1$ *and* $K \in \mathcal{K}_o^{n+1}$ *of class* C_+^2, *we call*

$$h^{1-p} S_k(h_{ij} + \delta_{ij} h)$$

the k*-th p-area function of* K.

For each Firey's p-sum, one can consider the problem of prescribing the k-th p-area function of a convex body. In the smooth category, it reduces to the problem of finding the convex solutions to the nonlinear elliptic equation

$$S_k(u_{ij} + \delta_{ij} u) = u^{p-1} f \qquad on \quad S^n \tag{1.4}$$

where the convexity means

$$(u_{ij} + \delta_{ij} u) > 0 \qquad on \quad S^n. \tag{1.5}$$

When $p = 1$, this corresponds to the classical Christoffel-Minkowski problem [18].

Lutwak [13] is the first to study the generalized Minkowski problem for $p > 1$. He proved existence of a unique solution $K \in \mathcal{K}_0^{n+1}$ with $h(K,u)^{1-p} dS_n(K,u) = d\mu (p \neq n+1)$, where μ is a given even positive Borel measure on S^n which does not concentrated on a great sphere of S^n. The regularity of the solution was proved in [15]. Recently, K.S.Chou and X.J.Wang [4] and P.F.Guan and C.S.Lin [7] dropped the evenness assumption for $p \geq n+1$ in the smooth category. In [7] they got partial smoothly results for $\frac{n-1}{2} < p < n+1$. Moreover the weak solution for $-n-1 < p < n+1$ was obtained in [4].

In the present paper we study the existence of convex bodies of prescribed the k-th p-area functions for $1 \leq k < n$. For a positive function $f \in C^m(S^n) \quad (m \geq 2)$, we say that f is *spherical convex* on S^n if the spherical hessian $(f_{ij} + \delta_{ij} f)$ is positive semidefinite on S^n, where subscript denotes covariant derivative in a local orthonormal frame field on S^n.

Theorem 1. *Let $1 \leq k < n$ and $p \geq k+1$.*

(i) When $p > k+1$, for any $0 < f \in C^m(S^n)(m \geq 2)$, if $f^{-\frac{1}{p+k+1}}$ is spherical convex on S^n, then there exists a unique convex body $K \in \mathcal{K}_0^{n+1}$ with $C^{m+1,\alpha}(0 < \alpha < 1)$ boundary ∂K, such that its k-th p-area function equal to f.

(ii) When $p = k+1$, for any $0 < f \in C^m(S^n)(m \geq 2)$, if $f^{-\frac{1}{2k+1}}$ is spherical convex on S^n, then there exists a unique positive constant γ and a unique (up to dilation) convex body $K \in \mathcal{K}_0^{n+1}$ with $C^{m+1,\alpha}(0 < \alpha < 1)$ boundary ∂K, such that its k-th p-area function equal to γf.

Theorem 1 is main result in this paper. To prove theorem 1, we write equation (1.4) as

$$S_k(u_{ij} + \delta_{ij} u) = u^{p_0} f \qquad on \quad S^n, \tag{1.6}$$

where $p_0 = p - 1$. This is a Hessian equation, so it is more natural to consider the existence of admissible solutions (see Definition 3 below). The following two theorems deal with the existence of admissible solutions of (1.6). First we treat the case $p_0 > k$

Theorem 2. *Let $1 \leq k < n$. If $p_0 > k$, then for any positive function $f \in C^m(S^n)(m \geq 2)$, there is a unique positive admissible solution of (1.6) which satisfies*

$$\|u\|_{C^{m+1,\alpha}(S^n)} \leq C. \tag{1.7}$$

where $0 < \alpha < 1$ and the constant C depends only on $n, k, \alpha,\ p_0, \min_{S^n} f$, $\|f\|_{C^m(S^n)}$.

For the case $p_0 = k$, we adapt the technique in [7] to obtain

Theorem 3. *Let $1 \leq k < n$ and $p_0 = k$, then for any positive function $f \in C^m(S^n)(m \geq 2)$, there is a unique positive constant γ such that the equation*

$$S_k(u_{ij} + \delta_{ij} u) = u^k \gamma f \qquad on \quad S^n. \tag{1.8}$$

has a positive admissible solution, which is unique upto a dilation.

We remark if f is analytic, then the solutions above are also analytic. The admissible solutions above are in general not convex. In [8] P.F.Guan and X.N.Ma used ideas in [1] and [11] to prove a *full rank theorem* for the classical Christoffel-Minkowski problem. The corresponding version for the p-sum case is the following

Theorem 4. *Consider the equation*

$$S_k(u_{ij} + \delta_{ij}u) = u^{p_0} f \qquad on \quad S^n, \tag{1.9}$$

where $p_o \geq 0$. *Suppose a positive* C^4 *admissible solution* u *satisfies*

$$(u_{ij} + \delta_{ij}u) \geq 0. \tag{1.10}$$

Then if $f^{-\frac{1}{k+p_0}}$ *is spherical convex, i.e.*

$$\left(\left(f^{-\frac{1}{k+p_0}}\right)_{ij} + \delta_{ij} f^{-\frac{1}{k+p_0}}\right) \geq 0,$$

the matrix $(u_{ij} + \delta_{ij}u)$ *is positive definite.*

With the help of the full rank theorem, we can prove the admissible solutions obtained in Theorems 2 and 3 are strictly convex (i.e. $(u_{ij} + \delta_{ij}u)$ is positive definition) under the assumptions in Theorem 1.

This paper is organized as follows: in Section 2 we state some notations and preliminary results needed in this paper. We prove the a priori estimates for a positive admissible solutions of equation (1.6) in Section 3. Theorems 2 and 3 will be proved in Section 4. Finally in Section 5 we sketch the proof of our full rank Theorem and prove Theorem 1.

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2. Preliminary

In this section we state some definitions and basic properties of elementary symmetric functions, which are needed in this paper.

Definition 2. *For* $1 \leq k \leq n$ *and* $\lambda = (\lambda_1, \cdots, \lambda_n) \in R^n$, *the k-th elementary symmetric function is*

$$S_k(\lambda) = \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n}. \tag{2.1}$$

where $i_1, \cdots, i_k \in \{1, \cdots, n\}$. *Let* $W = (W_{ij})$ *be a symmetric* $n \times n$ *matrix. We set* $S_k(W) = S_k(\lambda(W))$, *where* $\lambda(W) = (\lambda_1(W), \cdots, \lambda_n(W))$ *are the eigenvalues of* W. *We also set* $S_0 = 1$ *and* $S_k = 0$ *for* $k > n$.

Definition 3. *Let* $\mathcal{S}$ *be the set of all* $n \times n$ *symmetric matrices. For* $1 \leq k \leq n$, *denote*

$$\Gamma_k = \{W \in \mathcal{S} : S_1(W) > 0, \cdots, S_k(W) > 0\}. \tag{2.2}$$

If $u \in C^2(S^n)$ *is the solution of (1.6), we say* u *is an* admissible solution *if* $W(x) = (u_{ij}(x) + \delta_{ij}u(x))$ *is in* Γ_k *for each* $x \in S^n$.

The sets Γ_k are open convex cones and satisfy $\Gamma_{k+1} \subset \Gamma_k$ for any $k = 1, \cdots, n-1$.

Proposition 1. *Let $W = (W_{ij})$ be a $n \times n$ symmetric matrix, let $G(W) = S_k(W)$ for some $1 \leq k \leq n$. Then the following relations hold.*

$$S_k(W) = \frac{1}{k!} \sum_{\substack{i_1,\cdots,i_k=1 \\ j_1,\cdots,j_k=1}}^{n} \delta(i_1,\cdots,i_k; j_1,\cdots,j_k) W_{i_1 j_1} \cdots W_{i_k j_k},$$

$$G^{\alpha\beta} := \frac{\partial G}{\partial W_{\alpha\beta}}(W)$$

$$= \frac{1}{(k-1)!} \sum_{\substack{i_1,\cdots,i_{k-1}=1 \\ j_1,\cdots,j_{k-1}=1}}^{n} \delta(\alpha, i_1, \cdots, i_{k-1}; \beta, j_1, \cdots, j_{k-1}) W_{i_1 j_1} \cdots W_{i_{k-1} j_{k-1}},$$

$$G^{ij,rs} := \frac{\partial^2 G}{\partial W_{ij} \partial W_{rs}}$$

$$= \frac{1}{(k-2)!} \sum_{\substack{i_1,\cdots,i_{k-2}=1 \\ j_1,\cdots,j_{k-2}=1}}^{n} \delta(i,r,i_1,\cdots,i_{k-2}; j,s,j_1,\cdots,j_{k-2}) W_{i_1 j_1} \cdots W_{i_{k-2} j_{k-2}},$$

where the Kronecker symbol $\delta(I;J)$ for indices $I = (i_1, \cdots, i_m)$ and $J = (j_1, \cdots, j_m)$ is defined by

$$\delta(I;J) = \begin{cases} 1, & \text{if I is an even permutation of J ;} \\ -1, & \text{if I is an odd permutation of J;} \\ 0, & \text{otherwise.} \end{cases}$$

Let $S_k(\lambda|i)$ denote $S_k(\lambda)$ in which $\lambda_i = 0$. We have the following identities in:

Proposition 2. *For any $k = 0, \cdots, n$, $i = 1, \cdots, n$ and $\lambda \in R^n$*

$$\frac{\partial S_{k+1}(\lambda)}{\partial \lambda_i} = S_k(\lambda|i),$$
$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i),$$
$$\sum_{i=1}^{n} S_k(\lambda|i) = (n-k) S_k(\lambda),$$
$$\sum_{i=1}^{n} \lambda_i S_k(\lambda|i) = (k+1) S_{k+1}(\lambda).$$

Also we need the following

Proposition 3. *Let $\lambda \in \Gamma_k$ for some $k \in \{2, \cdots, n\}$, then for any $0 \leq h \leq k-1$, and $1 \leq i \leq n$, $S_h(\lambda|i) > 0$. Therefore $\lambda_i \leq S_1(\lambda)$ for $1 \leq i \leq n$.*

The refined Newton-Maclaurin inequality is crucial to our a priori C^2 estimates.

Proposition 4. *For $\forall \lambda \in \Gamma_k$, and $\quad k \geq r, \quad l \geq s, \quad k - l \geq r - s \geq 0,$ we have*

$$\left[\frac{S_k C_n^l}{S_l C_n^k} \right]^{\frac{1}{k-l}} \leq \left[\frac{S_r C_n^s}{S_s C_n^r} \right]^{\frac{1}{r-s}}. \tag{2.3}$$

Proposition 1 and 2 are standard. We refer the readers to [10,7] for Proposition 3 and 4.

3. A priori estimates for positive admissible solutions of (1.6)

In this section we get the a priori estimates for positive admissible solutions of the Hessian equation (1.6).

The C^0 estimate is easy.

Lemma 1. *Suppose $p_0 \geq k$. If $u > 0$ is an admissible solution of (1.6), then*

$$\frac{C_n^k}{max_{S^n} f} \leq u^{p_0-k} \leq \frac{C_n^k}{min_{S^n} f}. \tag{3.1}$$

When $p_0 > k$, (3.1) gives a C^0 estimate for admissible solutions of (1.6).

Proof. Let x_0 be a maximum point of u. Choose a local orthonormal frame field $\{e_i\}$ such that (u_{ij}) is diagonal at x_0, namely,

$$(u_{ij} + \delta_{ij}u) = diag(u + u_{11}, ..., u + u_{nn})$$

then at x_0, $u_{ii} \leq 0$,$i = 1, ..., n$. Since u is admissible, by the Newton - Maclaurin inequality (2.3) we have

$$\frac{S_k}{C_n^k} \leq \left(\frac{S_1}{C_n^1}\right)^k = \left[\frac{1}{n}(nu + \sum_{i=1}^n u_{ii})\right]^k \leq u^k.$$

So

$$u^{p_0} f \leq u^k C_n^k.$$

which asserts the second inequality. The first inequality is obvious. □

Now we utilize the C^0 - estimate to get an uniform C^1 estimate for the case $p_0 \geq k$.

Lemma 2. *Suppose $p_0 \geq k$. If $u > 0$ is an admissible solution of (1.6), then there exists a positive constant C which depends only on n, k, and $\max_{S^n} \frac{|\nabla f|}{f}$ (independent of p_0) such that*

$$\max_{S^n} \frac{|\nabla u|}{u} \leq C.$$

From Lemma 2 we get a Harnack inequality for positive admissible solutions of equation (1.6) when $p_0 \geq k$.

Corollary 1. *Let $p_o \geq k$. There is a constant $C > 0$ depending only on n, k, and $\max_{S^n} \frac{|\nabla f|}{f}$ (independent of p_0) such that*

$$\frac{max_{S^n} u}{min_{S^n} u} \leq C.$$

Now we begin the proof Lemma 2.

Proof. Write equation (1.6) as

$$S_k(W_{ij}) = u^{p_0} f \qquad on \quad S^n, \tag{3.2}$$

where $p_0 \geq k$ and $W_{ij} = u_{ij} + \delta_{ij} u$. For any positive admissible solution u, denote $v = logu$, then (3.2) becomes

$$S_k(v_{ij} + v_i v_j + \delta_{ij}) = e^{(p_0 - k)v} f \qquad on \quad S^n. \tag{3.3}$$

Let $P = |\nabla v|^2$, and suppose x_0 is a maximum point of P, choose a local orthonormal frame field such that at x_o,

$$\nabla v = v_1 e_1, v_1 > 0.$$

Since at x_0, $P_i = 0$ and $P_{ii} \leq 0 \quad (i = 1, \cdots, n)$. It follows that

$$\sum_{j=1}^{n} v_j v_{ji} = 0 \quad i.e. \quad v_{1i} = 0, \quad \forall i, \tag{3.4}$$

and

$$\frac{1}{2} P_{ii} = \sum_{j=1}^{n} {v_{ji}}^2 + \sum_{j=1}^{n} v_j v_{jii} \leq 0. \tag{3.5}$$

From (3.4) we may rotate $e_2, \cdots, e_n$ such that (v_{ij}) is diagonal at x_0, therefore $(a_{ij}) := v_{ij} + v_i v_j + \delta_{ij} = diag(1 + v_1^2, 1 + v_{22}, \cdots, 1 + v_{nn}) := diag(\lambda_1, \cdots, \lambda_n)$.

In what follows our calculations will be done at x_0. Since the solution u is admissible so the matrix $(a_{ij}) \in \Gamma_k$. Denote $F^{ij} = \frac{\partial S_k(a_{ij})}{\partial a_{ij}}$, then $\sum\limits_{i,j=1}^{n} F^{ij} P_{ij} \leq 0$, from (3.5) and the following Ricci identity

$$v_{jii} = v_{iji} = v_{iij} + v_s R_{siji} = v_{iij} + v_s(\delta_{sj}\delta_{ii} - \delta_{si}\delta_{ij}) = v_{iij} + v_j - v_i \delta_{ij},$$

we have

$$\sum_{i=1}^{n} F^{ii} v_{ii}^2 + \sum_{i,s=1}^{n} F^{ii} v_{iis} v_s + v_1^2 \sum_{i=2}^{n} F^{ii} \leq 0. \tag{3.6}$$

Differentiating (3.3) in direction e_s and contracting with v_s, we get

$$\sum_{i,s=1}^{n} F^{ii} v_{iis} v_s = [(p_0 - k) v_1^2 f + v_1 f_1] e^{(p_0 - k)v}. \tag{3.7}$$

From (3.6)–(3.7) and the fact $p_0 \geq k$, we obtain

$$v_1 \sum_{i=2}^{n} F^{ii} + f_1 e^{(p_0 - k)v} \leq 0. \tag{3.8}$$

By Proposition 2 and 3,

$$\sum_{i=2}^{n} F^{ii} = \sum_{i=2}^{n} S_{k-1}(\lambda|i) = (n-k+1)S_{k-1}(\lambda) - S_{k-1}(\lambda|1) \geq (n-k)S_{k-1}(\lambda).$$

Furthermore, by the Newton-Maclaurin inequality

$$S_{k-1}(\lambda) \geq C_n^{k-1}\left[\frac{S_k(\lambda)}{C_n^k}\right]^{\frac{k-1}{k}} = C_n^{k-1}\left[\frac{e^{(p_0-k)v}}{C_n^k}f\right]^{\frac{k-1}{k}}.$$

So (3.8) implies

$$v_1\frac{(n-k)k}{n-k+1}\left[\frac{e^{(p_0-k)v}}{C_n^k}f\right]^{-\frac{1}{k}} + \frac{f_1}{f} \leq 0. \tag{3.9}$$

Noting that $e^v = u$ is an admissible solution of equation (1.6), by (3.1) we get

$$v_1\frac{k(n-k)}{n-k+1} \leq \max_{S^n}\frac{|\nabla f|}{f}\left(\frac{maxf}{minf}\right)^{\frac{1}{k}}.$$

Since $\frac{maxf}{minf}$ can be controlled by $\max_{S^n}\frac{|\nabla f|}{f}$, so

$$|\nabla v| \leq C(n,k)\left[\max_{S^n}\frac{|\nabla f|}{f}\right]^{1+\frac{1}{k}}.$$

The proof is complete. □

In the following we will get C^2 estimates for positive admissible solutions of equation (1.6) under the assumption $p_0 > k \geq 2$. Since for $k=1$ (1.6) is semilinear elliptic equation, the C^0 and C^1 estimates implies high order derivative estimates.

Let $u > 0$ be an admissible solution of equation (1.6) for $p_0 > k$, let

$$\tilde{u} = \frac{u}{l}, \qquad where \qquad l = min_{S^n}u.$$

Then (1.6) can be written as

$$S_k(\tilde{u}_{ij} + \delta_{ij}\tilde{u}) = \tilde{u}^{p_0}\tilde{f}, \tag{3.10}$$

where $\tilde{f} = l^{p_0-k}f$. By (3.1) we have

$$\frac{f}{maxf}C_n^k \leq \tilde{f} \leq \frac{f}{minf}C_n^k,$$

$$min\tilde{u} = 1,$$

$$\tilde{u}^{p_0-k} \leq \frac{maxf}{minf}.$$

Define

$$G = \frac{\tilde{H}}{\tilde{u}},$$

where $\tilde{H} = tr(\tilde{u}_{ij} + \delta_{ij}\tilde{u})$. Then $G = \frac{H}{u}$ with $H = tr(u_{ij} + \delta_{ij}u)$.

Lemma 3. *Suppose $p_0 > k \geq 2$. There is a positive constant C which depends only on n, k, $\max_{S^n} \frac{|\nabla f|}{f}$, $\max_{S^n} \frac{|\triangle f|}{f}$, and p_0 such that for any positive admissible solution u of (1.6)*

$$max_{S^n} G \leq C.$$

Furthermore, if $k < p_0 \leq k+1$, C is independent of p_0.

Proof. Write equation (3.10) as

$$F(\tilde{W}) = \tilde{u}^{\sigma} \tilde{\varphi}, \tag{3.11}$$

where $F = S_k^{\frac{1}{k}}$, $\tilde{W} = (\tilde{W}_{ij})$, $\tilde{W}_{ij} = \tilde{u}_{ij} + \delta_{ij}\tilde{u}$, $\sigma = \frac{p_0}{k}$, and $\tilde{\varphi} = \tilde{f}^{\frac{1}{k}}$.

At a maximum point x_o of G,

$$\frac{\nabla \tilde{H}}{\tilde{H}} = \frac{\nabla \tilde{u}}{\tilde{u}}, \qquad and \qquad \tilde{H}_{ij} \leq G\tilde{u}_{ij}. \tag{3.12}$$

Choose a local orthonormal frame field $\{e_i\}$ such that $\tilde{W} = (\tilde{W}_{ij})$ is diagonal at x_0. Denote $F^{ij} = \dfrac{\partial F}{\partial \tilde{W}_{ij}}$, then (F^{ij}) is also diagonal at x_0 and

$$F^{ii} = \frac{1}{k} S_k^{\frac{1}{k}-1}(\lambda) S_{k-1}(\lambda|i), \qquad (i = 1, \cdots, n),$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ are the eigenvalues of $\tilde{W}$. In what follows our calculations will be done at x_o. By (3.12)

$$\sum_{ij} F^{ij} \tilde{H}_{ij} \leq G \sum_{ij} F^{ij} \tilde{u}_{ij}. \tag{3.13}$$

Commuting the second derivatives,

$$\sum_{ij} F^{ij} \tilde{H}_{ij} = \sum_{ik} F^{ii} \tilde{W}_{iikk} + \tilde{H} \sum_{i} F^{ii} - n\tilde{u}^{\sigma} \tilde{\varphi}. \tag{3.14}$$

Differentiating (3.10) along e_k twice, and by the concavity of F, we get

$$\begin{aligned} \sum_{ik} F^{ii} \tilde{W}_{iikk} \geq \; & \sigma(\sigma - 1)\tilde{u}^{\sigma-2} |\nabla \tilde{u}|^2 \tilde{\varphi} + \sigma \tilde{u}^{\sigma-1} \tilde{\varphi} \tilde{H} - n\sigma \tilde{u}^{\sigma} \tilde{\varphi} \\ & + 2\sigma \tilde{u}^{\sigma-1} \nabla \tilde{u} \cdot \nabla \tilde{\varphi} + \tilde{u}^{\sigma} \triangle \tilde{\varphi}. \end{aligned} \tag{3.15}$$

Combining (3.13)–(3.15) we have

$$\begin{aligned} G \sum_{ij} F^{ij} \tilde{u}_{ij} \geq \; & \tilde{H} \sum_{i} F^{ii} - n(\sigma + 1)\tilde{u}^{\sigma} \tilde{\varphi} + \sigma(\sigma - 1)\tilde{u}^{\sigma-2} |\nabla \tilde{u}|^2 \tilde{\varphi} \\ & + \sigma \tilde{u}^{\sigma-1} \tilde{\varphi} \tilde{H} + 2\sigma \tilde{u}^{\sigma-1} \nabla \tilde{u} \cdot \nabla \tilde{\varphi} + \tilde{u}^{\sigma} \triangle \tilde{\varphi}. \end{aligned} \tag{3.16}$$

On the other hand

$$\begin{aligned} G\sum_{ij} F^{ij}\tilde{u}_{ij} &= G\sum_{ij} F^{ij}(\tilde{W}_{ij}-\delta_{ij}\tilde{u}) \\ &= GF-\tilde{u}G\sum_i F^{ii} \\ &= \frac{\tilde{H}}{\tilde{u}}\tilde{u}^{\sigma}\tilde{\varphi}-\tilde{H}\sum_i F^{ii}. \end{aligned}$$

Inserting this into (3.16), we get

$$\begin{aligned} 0 \geq\ & 2\tilde{H}\sum_i F^{ii} + (\sigma-1)\tilde{u}^{\sigma-1}\tilde{\varphi}\tilde{H} - n(1+\sigma)\tilde{u}^{\sigma}\tilde{\varphi} \\ & + \sigma(\sigma-1)\tilde{u}^{\sigma-2}\tilde{\varphi}|\ \nabla\tilde{u}\ |^2 + 2\sigma\tilde{u}^{\sigma-1}\nabla\tilde{u}\cdot\nabla\tilde{\varphi} + \tilde{u}^{\sigma}\triangle\tilde{\varphi}. \end{aligned} \tag{3.17}$$

Notice that

$$\begin{aligned} \sum_i F^{ii} &= \frac{1}{k}S_k^{\frac{1}{k}-1}(\lambda)\sum_i S_{k-1}(\lambda|i) \\ &= \frac{n-k+1}{k}S_k^{\frac{1}{k}}(\lambda)\frac{S_{k-1}(\lambda)}{S_k(\lambda)} \\ &= \frac{n-k+1}{k}\tilde{u}^{\sigma}\tilde{\varphi}\frac{S_{k-1}(\lambda)}{S_k(\lambda)}. \end{aligned}$$

Inserting the above relation into (3.17) and then divided by $\tilde{u}^{\sigma}\tilde{\varphi}$, we obtain

$$\begin{aligned} 0 \geq\ & 2\frac{(n-k+1)}{k}\frac{S_{k-1}(\lambda)}{S_k(\lambda)}\tilde{H} + (\sigma-1)\frac{\tilde{H}}{\tilde{u}} - n(1+\sigma) \\ & + \sigma(\sigma-1)\frac{|\ \nabla\tilde{u}\ |^2}{\tilde{u}^2} + 2\sigma\frac{\nabla\tilde{u}}{\tilde{u}}\cdot\frac{\nabla\tilde{\varphi}}{\tilde{\varphi}} + \frac{\triangle\tilde{\varphi}}{\tilde{\varphi}}. \end{aligned} \tag{3.18}$$

By Lemma 2 and notice the relations:

$$\frac{\nabla\tilde{u}}{\tilde{u}} = \frac{\nabla u}{u}, \qquad \frac{\nabla\tilde{\varphi}}{\tilde{\varphi}} = \frac{\nabla f^{1/k}}{f^{1/k}} \qquad and \qquad \frac{\triangle\tilde{\varphi}}{\tilde{\varphi}} = \frac{\triangle f^{1/k}}{f^{1/k}},$$

we can write (3.18) in the following form:

$$2\frac{(n-k+1)}{k}\frac{S_{k-1}(\lambda)}{S_k(\lambda)}\tilde{H} + (\sigma-1)\frac{\tilde{H}}{\tilde{u}} \leq \sigma c_1 + c_2, \tag{3.19}$$

where $c_1 \sim (n, k, \max_{S^n}\frac{|\nabla f|}{f})$ and $c_2 \sim (n, k, \max_{S^n}\frac{|\nabla f|}{f}, \max_{S^n}\frac{|\triangle f|}{f})$.

Now we treat the first term in (3.19). First we have

$$\tilde{u}^{\sigma-1}\tilde{\varphi} = (\frac{u}{l})^{\frac{p_0-k}{k}}(l^{p_0-k}f)^{\frac{1}{k}} = u^{\frac{p_0-k}{k}}f^{1/k}.$$

By equation (3.11) and the above formula,

$$\frac{\tilde{H}}{S_k^{\frac{1}{k}}} = \frac{\tilde{u}G}{\tilde{u}^{\sigma}\tilde{\varphi}} = \frac{G}{\tilde{u}^{\sigma-1}\tilde{\varphi}} = \frac{G}{u^{\frac{p_0-k}{k}} f^{1/k}}. \tag{3.20}$$

Taking $l = 1, r = k-1$, and $s = 1$ in the refined Newton-Maclaurin inequality (2.3) gives

$$\frac{S_{k-1}}{S_k} \geq c(n,k)\frac{\tilde{H}^{\frac{1}{k-1}}}{S_k^{\frac{1}{k-1}}}.$$

Hence from (3.20), we obtain

$$\begin{aligned} \frac{S_{k-1}}{S_k}\tilde{H} \geq c(n,k)\frac{\tilde{H}^{\frac{k}{k-1}}}{S_k^{\frac{1}{k-1}}} &= c(n,k)\left(\frac{\tilde{H}}{S_k^{\frac{1}{k}}}\right)^{\frac{k}{k-1}} \\ &= c(n,k)\left[\frac{G}{u^{\frac{p_0-k}{k}} f^{1/k}}\right]^{\frac{k}{k-1}}. \end{aligned} \tag{3.21}$$

By Lemma 1

$$\frac{minf}{maxf}C_n^k \leq f u^{p_0-k} \leq \frac{maxf}{minf}C_n^k.$$

It follows that

$$\frac{S_{k-1}(\lambda)}{S_k(\lambda)}\tilde{H} \geq c_0 G^{\frac{k}{k-1}}. \tag{3.22}$$

where $c_0 \sim (n, k, \max_{S^n}\frac{|\nabla f|}{f})$. Now (3.19) and (3.22) imply

$$c_0 G^{\frac{k}{k-1}} - (\sigma-1)G \leq \sigma c_1 + c_2.$$

So we finish the proof of this Lemma. □

From the above Lemmas we have the following a priori estimates.

Corollary 2. *Suppose $p_0 > k$. Then there is a constant $C > 0$ depending only on $n, k,$ $\min_{S^n} f, \max_{S^n}\frac{|\nabla f|}{f}$, $\max_{S^n}\frac{|\triangle f|}{f}$, and p_0 such that for any positive admissible solution u of (1.6),*

$$\|u\|_{C^2} \leq C.$$

If $k \leq p_0 \leq k+1$, then there exists constant C depending only on $n, k,$ $\max_{S^n}\frac{|\nabla f|}{f}$, $\max_{S^n}\frac{|\triangle f|}{f}$ (independent of p_0) such that for any positive admissible solution $\tilde{u}$ of equation (3.10),

$$\|\tilde{u}\|_{C^2} \leq C.$$

Proof. We prove the first formula since the other is similar. When $k = 1$, equation (1.6) is semilinear so the a priori estimates follow from standard elliptic theory. For $k \geq 2$, it is a consequence of Proposition 3 and Lemma 3. □

4. Proof of Theorem 2 and Theorem 3

We want to solve equation (1.6) in the class of positive admissible solutions. For the case $p_0 > k$, the following proposition shows that the linearized operator of (1.6) is invertible at any positive admissible solution, so we can use the continuity method to obtain Theorem 2. For $p_0 = k$ we shall use approximation to get the existence.

Lemma 4. *Let $u > 0$ be an admissible solution of (1.6) and $p_0 > k$, then the linearized operator L_u has trivial kernel.*

Proof. Let $W_{ij} = u_{ij} + \delta_{ij}u$ and $F = S_k^{\frac{1}{k}}$, then F is homogeneous of degree one about W_{ij}. Write equation (1.6) as

$$F(W_{ij}) = u^\sigma \varphi, \tag{4.1}$$

where $\sigma = \frac{p_0}{k}$, $\varphi = f^{\frac{1}{k}}$. The linearized operator of (4.1) is:

$$L_u(v) = \sum_{i,j=1}^{n} F^{ij} \cdot (v_{ij} + \delta_{ij}v) - \sigma u^{\sigma-1}\varphi v, \tag{4.2}$$

where $F^{ij} = \frac{\partial F}{\partial W_{ij}}$. Since u is an admissible solution, the matrix $(F^{ij}) > 0$. As in [2], let $v = uw$, then (4.2) can be written as

$$L_u(v) = (1-\sigma)u^\sigma \varphi w + 2F^{ij}u_i w_j + uF^{ij}w_{ij}.$$

Suppose $L_u(v) = 0$, then

$$uF^{ij}w_{ij} + 2F^{ij}u_i w_j + (1-\sigma)u^\sigma \varphi w = 0,$$

where $\sigma > 1$. At a maximum point of w, $(w_{ij}) \le 0$, $(F^{ij}) > 0$ and $w_j = 0$, we get

$$0 \ge uF^{ij}w_{ij} = (\sigma - 1)u^\sigma \varphi w,$$

so $\max_{S^n} w \le 0$. At a minimum point of w, similar argument shows that $\min_{S^n} w \ge 0$. Therefore $w \equiv 0 \quad on \quad S^n$ i.e. $v \equiv 0$. This proves $KerL_u = 0$. □

Now we begin the proof of Theorem 2 via the continuity method.

Proof. For $0 \le t \le 1$, consider the family of equations

$$S_k(u_{ij} + \delta_{ij}u) = u^{p_0} f_t \tag{$*_t$}$$

where $f_t = (1-t)C_n^k + tf$. Let

$$\mathrm{I} = \{t | 0 \le t \le 1 \quad s.t. (*_t) \ \textit{has positive admissible solution} \ \}$$

Obviously, $u \equiv 1$ is the solution of $(*_0)$, so $I \neq \emptyset$. The openness of I comes from implicit function theorem, Lemma 4, Schauder theory and Fredholm's alternative. From Corollary 2 in Section 3 we can get an uniform C^2 estimate (with p_0 fixed) for positive admissible solutions for $(*_t)$, so it is uniform elliptic for positive admissible solutions. From the Evans – Krylov's theory [9], we can get higher order estimates. Therefore, I is closed. So we get the existence part of Theorem 2.

Now we prove the uniqueness of positive admissible solutions of (1.6). Suppose u and v are two positive admissible solutions. Write (1.6) in the form (4.1). If $u \not\equiv v$, then we may assume that $u < v$ in a neighborhood O of some point x_0. Choose a positive constant $\rho > 1$ such that $\rho u \geq v$ in O and $\rho u(x_0) = v(x_0)$. Let $w = \rho u$, then $w - v \geq 0$ in O and $(w - v)(x_0) = 0$. Define

$$u^t = (1 - t)v + tu$$

and denote

$$U^t = (u^t_{ij} + \delta_{ij} u^t) = (1 - t)V + tW,$$

where $V = (v_{ij} + \delta_{ij} v)$ and $W = (w_{ij} + \delta_{ij} w)$. Since u and v are admissible solutions, we have $V \in \Gamma_k$ and $W \in \Gamma_k$. Since Γ_k is a convex cone, we have $U^t \in \Gamma_k$. Thus

$$(F^{ij}(U^t)) > 0$$

where $(F^{ij}(U^t)) = \dfrac{\partial S_k^{\frac{1}{k}}}{\partial U^t_{ij}}$.

Let $f(t) = F(U^t)$, then

$$\text{(4.3)} \quad F(W) - F(V) = \int_0^1 f\prime(t)\mathrm{d}t$$
$$= \sum_{ij=1}^{n} \left(\int_0^1 F^{ij}(U^t)\mathrm{d}t \right) [(w_{ij} - v_{ij}) + \delta_{ij}(w - v)].$$

On the other hand, by equation (4.1)

$$\text{(4.4)} \qquad F(W) - F(V) = \rho^{1-\sigma} w^\sigma \varphi - v^\sigma \varphi.$$

Denote $a_{ij} = \int_0^1 F^{ij}(U^t)\mathrm{d}t$, then $(a_{ij}) > 0$. From (4.3) and (4.4) we get

$$\text{(4.5)} \quad \left(\rho^{1-\sigma} w^\sigma - v^\sigma\right) \varphi = \sum_{ij} a_{ij} \cdot (w_{ij} - v_{ij}) + \left(\sum_i a_{ii} \right) \cdot (w - v).$$

Since x_0 is a local minimum point of $w - v \geq 0$ in O,

$$(w_{ij} - v_{ij}) \geq 0 \qquad at \quad x_0,$$

and the right side of (4.5) is nonnegative at x_0. But at x_o

$$(\rho^{1-\sigma} - 1)v(x_0)^\sigma \varphi(x_0) < 0,$$

a contradiction. □

When $p_0 = k$, equation (1.6) is dilation invariant and we can not get the C^0 estimate as in the case $p_0 > k$. Furthermore, the linearized operator of (1.6) is not invertible and the openness is false in this case. Therefore we can not use the continuity method. It turns out that the method in [7] is effective for our purpose.

Proof of Theorem 3. Now we consider the equation

$$S_k(u_{ij} + \delta_{ij}u) = u^k f. \tag{4.6}$$

For $\forall r \in Z^+$, let u^r be the solution of

$$S_k(u_{ij} + \delta_{ij}u) = u^{k+\frac{1}{r}} f. \tag{4.7}$$

We normalize u^r by setting :

$$\tilde{u}^r = \frac{u^r}{l_r},$$

where $l_r = \min u^r$, $L_r = \max u^r$. Now $\tilde{u}^r$ satisfies the equation

$$S_k(\tilde{u}_{ij} + \delta_{ij}\tilde{u}) = \{\tilde{u}\}^{k+\frac{1}{r}} \tilde{f}_r, \tag{4.8}$$

with $\tilde{f}_r = l_r^{1/r} f$. From (3.1) we have

$$\frac{C_n^k}{maxf} \le l_r^{1/r} \le L_r^{1/r} \le \frac{C_n^k}{minf}, \tag{4.9}$$

and

$$\frac{f}{maxf} C_n^k \le \tilde{f}_r \le \frac{f}{minf} C_n^k,$$

$$\frac{max\tilde{f}_r}{min\tilde{f}_r} = \frac{maxf}{minf},$$

$$\frac{\triangle \tilde{f}_r}{\tilde{f}_r} = \frac{\triangle f}{f}.$$

By Corollary 1 and Corollary 2, there is a constant C independent of r such that

$$||\tilde{u}_r||_{C^2} \le C.$$

By the Evans-Krylov and Schauder theory,

$$\|\tilde{u}^r\|_{C^{m+1,\alpha}} \le C_{m+1,\alpha},$$

with $C_{m+1,\alpha}$ independent of r. So there is a subsequence $r_j \longrightarrow \infty$ such that

$$\tilde{u}^{r_j} \longrightarrow u \qquad in \qquad C^{m+1,\alpha}(S^n),$$

and

$$l_{r_j}^{\frac{1}{r_j}} \longrightarrow \gamma,$$

for some positive constant γ. By (4.9)

$$\frac{C_n^k}{maxf} \leq \gamma \leq \frac{C_n^k}{minf}.$$

Therefore u satisfies the equation

$$S_k(u_{ij} + \delta_{ij}u) = u^k \gamma f \qquad on \qquad S^n.$$

This proves the existence part of Theorem 3.

The uniqueness upto a dilation is similar to the uniqueness part in Theorem 2.

Finally we prove the uniqueness of γ. Let $M(u) = \dfrac{S_k(u_{ij} + \delta_{ij}u)}{u^k}$. Suppose $\exists\, \gamma_1, \gamma_2$ and $u^1 > 0$, $u^2 > 0$, such that

$$S_k(u^s_{ij} + \delta_{ij}u^s) = (u^s)^k \gamma_s f, \qquad s = 1, 2.$$

We may assume that $\gamma_1 > \gamma_2$. So

$$M(u^1) - M(u^2) = (\gamma_1 - \gamma_2) f > 0. \tag{4.10}$$

Since M is invariant under scaling, we may assume $u^1 \leq u^2$, and $u^1(x_0) = u^2(x_0)$ at some point $x_0 \in S^n$. Let

$$u^t = tu^1 + (1-t)u^2, \quad and \quad M_t = M(u^t).$$

Then

$$(u^t_{ij} + \delta_{ij}u^t) = t \cdot (u^1_{ij} + \delta_{ij}u^1) + (1-t) \cdot (u^2_{ij} + \delta_{ij}u^2).$$

We have

$$M(u^1) - M(u^2) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} M_t \mathrm{d}t. \tag{4.11}$$

On the other hand,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M_t &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{S_k(u^t_{ij} + \delta_{ij}u^t)}{(u^t)^k}\right] \\
&= \frac{G^{ij} \cdot [(u^1 - u^2)_{ij} + \delta_{ij}(u^1 - u^2)]}{(u^t)^k} - \frac{kS_k({u^t}_{ij} + \delta_{ij}u^t)}{(u^t)^{k+1}}(u^1 - u^2) \\
&= \frac{G^{ij} \cdot (u^1 - u^2)_{ij}}{(u^t)^k} + \frac{(\sum G^{ii})u^t - kS_k}{(u^t)^{k+1}} \cdot (u^1 - u^2).
\end{aligned}$$

where $G^{ij} = \frac{\partial S_k}{\partial W_{ij}}$. Since u^s is admissible and Γ_k is a convex cone, we have

$$(G^{ij}) > 0 \qquad for \quad 0 \leq t \leq 1.$$

Denote

$$a_{ij} = \int_0^1 \frac{G^{ij}}{(u^t)^k} \mathrm{d}t,$$

$$c = \int_0^1 \frac{(\sum G^{ii})u^t - kS_k}{(u^t)^{k+1}} \mathrm{d}t.$$

Then $(a_{ij}) > 0$ and (4.11) can be written as

$$M(u^1) - M(u^2) = \sum_{ij} a_{ij}(u^1, u^2) \cdot (u^1 - u^2)_{ij} + c(u^1, u^2) \cdot (u^1 - u^2).$$

At x_0, $((u^1 - u^2)_{ij}) \leq 0$, thus $\sum_{ij} a_{ij}(u^1 - u^2)_{ij} \leq 0$. This means that at x_0,

$$M(u^1) - M(u^2) \leq 0,$$

which contradicts to (4.10). Thus we finish the proof of Theorem 3.

5. Proof of Theorems 1 and 4

In this section we will generalize Guan-Ma's deformation Lemma [8] to equation (1.6), then similarly to their proof we can get Theorem 4. Combining Theorems 2, 3 and 4 we shall prove Theorem 1. The idea of Theorem 4 comes from [1,11] and [8]. We follow the notations [8] with the exception that we denote the symbols $F^{\alpha\beta}$ in [8] as $G^{\alpha\beta}$.

Lemma 5. *Let $O \subset S^n$ be an open subset. Suppose $u \in C^4(O)$ is a solution of (1.6) in O, and the matrix $W = (W_{ij}) = (u_{ij} + \delta_{ij}u)$ is positive semidefinite. Suppose there is a positive constant $C_0 > 0$, such that for a fixed integer $(n-1) \geq l \geq k$, $S_l(W(x)) \geq C_0$ for all $x_0 \in O$. Let $\phi(x) = S_{l+1}(W(x))$ and let $\tau(x)$ be the largest eigenvalue of $\{-(f^{-\frac{1}{k+p}}) - \delta_{ij} f^{-\frac{1}{k+p_0}}\}$. Then there are constants C_1, C_2 depending only on $\|u\|_{C^3}$, $\|f\|_{C^{1,1}}$, n, k and C_0, such that differential inequality*

$$\text{(5.1)} \quad \sum_{\alpha,\beta}^{n} G^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \lesssim (p_0 + k)(n - l)u^{p_0} f^{\frac{p_0+k+1}{p_0+k}}(x) S_l(W(x))\tau(x),$$

holds in O, where the $G^{\alpha\beta}$ as in Proposition 1 .

Remark 1. When $p_0 = 0$, (1.6) is just the equation considered in [8] and Lemma 5 is the deformation Lemma proved there.

Sketch of Proof. Following the notation of Caffarelli and Friedman [1], for two functions defined in an open set $O \subset S^n$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y).$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constant c_1, and c_2 depending only on $||u||_{C^3}$, $||\varphi||_{C^2}$, n and C_0 (independent of y and O). Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We note that we use $G^{\alpha\beta}$to stand for $F^{\alpha\beta}$ in [8]. Following the arguments of [8] step by step, except that replacing φ by $u^{p_0}f$, we can get the following relation (which corresponds to (4.29) in [8]):

$$\sum_{\alpha,\beta}^{n} G^{\alpha\beta}\phi_{\alpha\beta} \lesssim S_l(G)\sum_{i\in B}\left\{(u^{p_0}f)_{ii} - \frac{k+1}{k}\frac{(u^{p_0}f)_i^2}{u^{p_0}f} - ku^{p_0}f\right\}. \tag{5.2}$$

Computing the right hand side of (5.2), noticing that for $i \in B, W_{ii} \sim 0$ implies $u_{ii} = W_{ii} - u \sim -u, \forall i \in B$, we have

$$(u^{p_0}f)_{ii} - \frac{k+1}{k}\frac{(u^{p_0}f)_i^2}{u^{p_0}f} - ku^{p_0}f \lesssim u^{p_0}\left[-\left(1+\frac{1}{p_0+k}\right)\frac{{f_i}^2}{f} + f_{ii} - (p_0+k)f\right].$$

Here we have used Schwartz inequality to part the "cross term":

$$2p_0\frac{1}{k}u^{p_0-1}u_if_i \le p_0\left(1+\frac{p_0}{k}\right)u^{p_0-2}{u_i}^2f + \frac{p_0}{k^2(1+\frac{p_0}{k})}u^{p_0}\frac{f_i^2}{f}.$$

Notice that

$$(f^\alpha)_{ij} = \alpha f^{\alpha-1}\left[\frac{(\alpha-1)f_if_j}{f} + f_{ij}\right]. \tag{5.3}$$

Let $\alpha = -\frac{1}{p_0+k}$, then (5.3) reads

$$\left[f^{-\left(1+\frac{1}{p_0+k}\right)}\right]_{ii} = -\frac{1}{p_0+k}f^{-\left(1+\frac{1}{p_0+k}\right)}\left[-\left(1+\frac{1}{p_0+k}\right)\frac{f_i^2}{f} + f_{ii}\right].$$

So

$$\begin{aligned} &u^{p_0}\left[-\left(1+\frac{1}{p_0+k}\right)\frac{f_i^2}{f} + f_{ii} - (p_0+k)f\right] \\ &= -(p_0+k)u^{p_0}f^{1+\frac{1}{p_0+k}}\left[\left(f^{-\frac{1}{p_0+k}}\right)_{ii} + f^{-\frac{1}{p_0+k}}\right], \end{aligned}$$

combining these formulas we get (5.1). □

The same argument as in [8] gives the following

Proof of Theorem 4. Suppose $u > 0$ is an admissible solution of equation (1.6) with positive semidefinite spherical hessian $W = (u_{ij} + \delta_{ij}u)$ on S^n, if $f^{-\frac{1}{p_0+k}}$ is spherical convex on S^n, from Lemma 5, strong minimum principle and Minkowski integral formula [18], we know W is positive definite on S^n.

Now let's prove the main theorem in this paper.

Proof of Theorem 1. Let's begin the case $p > k+1$, i.e. $p_o = p-1 > k$. For any positive $f \in C^m(S^n)$, $m \ge 2$ we consider the equation

$$S_k(u_{ij}^t + \delta_{ij}u^t) = (u^t)^{p_0-k}f_t, \qquad (*_t) \tag{5.4}$$

where $p_0 = p - 1$, $0 \le t \le 1$ and

$$f_t = \left[(1-t)C_n^k + tf^{-\frac{1}{p_0+k}}\right]^{-(p_0+k)}.$$

It is obvious equation $(*_1)$ is equation (1.6). Suppose that $f^{-\frac{1}{p_0+k}}$ is spherical convex , then $f_t^{-\frac{1}{p_0+k}}$ is spherical convex for every $0 \le t \le 1$. Theorem 2 assures the existence of positive admissible solution u^t of equation (5.4). Notice that the solution of $(*_0)$, $u \equiv 1$ is strictly spherical convex. We claim that for all $0 \le t \le 1$, the solution u^t is strictly spherical convex. In fact, if the claim is not true, there would be some $t_0 \in (0,1]$ such that u^t is strictly spherical convex for $0 \le t < t_0$ but u^{t_0} is not strictly spherical convex. By the a priori estimate for u^t, u^{t_0} is positive semidefinite , but the Theorem 4 asserts that u^{t_0} is positive definite. This contradiction proves the first part of Theorem 1.

Now we prove treat the $p = k$.

Our method is similar to the proof of Theorem 3. First suppose $f^{-\frac{1}{2k}}$ is strictly spherical convex. For any natural number $r \in N$, consider the equation

$$S_k(u_{ij} + \delta_{ij}u) = u^{k+\frac{1}{r}}f \tag{5.5}$$

It is easy to see that when r is large, $f^{-\frac{1}{2k+\frac{1}{r}}}$ is also strictly spherical convex. According to (i), equation (5.5) has strictly spherical convex positive solution $u^{(r)}$. As in the proof of Theorem 2, let

$$\tilde{u}^{(r)} = \frac{u^{(r)}}{l_r}, \qquad with \quad l_r = min_{S^n} u^{(r)}.$$

Then $\tilde{u}^{(r)}$ is strictly spherical convex and satisfies the equation

$$S_k(\tilde{u} + \delta_{ij}\tilde{u}) = \tilde{u}^{k+\frac{1}{r}}\tilde{f}_r,$$

where $\tilde{f}_r = l_r^{\frac{1}{r}} f$. By similar argument as in Theorem 2, we can get a subsequence of $\{\tilde{u}^{(r)}\}$ converges to a function u (in $C^{m+1,\alpha}$) which satisfies equation

$$S_k(u_{ij} + \delta_{ij}u) = u^k\gamma f \qquad on \qquad S^n.$$

Furthermore, u is spherical convex. Since $f^{-\frac{1}{2k}}$ is spherical convex, then as in (i) it follows that u is strictly spherical convex and we get the result.

If $f^{-\frac{1}{2k}}$ is just spherical convex, we can construct a sequence of positive functions $f_s (s = 1, 2, \cdots)$ such that it converges to f in $C^m(S^n)$ and $f_s^{-\frac{1}{2k}}$ is strictly spherical convex. For every f_s, consider equation (5.5), and repeat the above argument, we can get a sequence of positive solutions $v^{(s)}$ which are strictly spherical convex, and satisfies the equation

$$S_k(v_{ij} + \delta_{ij}v) = v^k\gamma_s f_s.$$

By Corollary 2, we can get a subsequence such that $v^{(s)} \longrightarrow v$ and $\gamma_s \longrightarrow \gamma$, with v being spherical convex and

$$S_k(v_{ij} + \delta_{ij}v) = v^k\gamma f.$$

Now the same process in (i) implies that v is strictly spherical convex, and we get the existence part of (ii). The uniqueness has been proved in Theorems 2 and 3.

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