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The Christoffel-Minkowski Problem II: Weingarten Curvature Equations****

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Abstract In this paper the authors discuss the existence and convexity of hypersurfaces with prescribed Weingarten curvature.

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1 Introduction

Convex hypersurfaces are among the most important subjects in Euclidean geometry. One of the basic problem in classical geometry is to find a convex hypersurface with prescribed curvature function. This in turn poses a fundamental question in nonlinear partial differential equations. In [11], we treated the Christoffel-Minkowski problem as a convexity problem of a spherical hessian equation on $\mathbb{S}^n$ via Gauss map. In this paper, we study such convexity problem for curvature equations. Our main concern is the existence and convexity of starshaped hypersurface with prescribed Weingarten curvature. More specifically, we treat two type of problems: the convexity of hypersurfaces with prescribed Weingarten curvature in the work of Caffarelli-Nirenberg-Spruck [5], and the existence and convexity of solution to homogeneous Weingarten curvature equations.

For a compact hypersurface M in $\mathbb{R}^{n+1}$, the kth Weingarten curvature at $x \in M$ is defined as

$$W_k(x) = S_k(\kappa_1(x), \kappa_2(x), \cdots, \kappa_n(x)),$$

where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ the principal curvatures of M, and S_k is the kth elementary symmetry function. If the surface is starshaped about the origin, it follows that the surface can be parametrized as a graph over $\mathbb{S}^n$:

$$X = \rho(x)x, \quad x \in \mathbb{S}^n, \tag{1.1}$$

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where ρ is the radial function. In this correspondence, the Weingarten curvature can be considered as a function on $\mathbb{S}^n$ or in $\mathbb{R}^{n+1}$. There is an extensive literature on the problem of prescribing curvature functions. For example, given a positive function F in $\mathbb{R}^{n+1} \setminus \{0\}$, one would like to find a starshaped hypersurface M about the origin such that its kth Weingarten curvature is F. The problem is equivalent to solve the following equation

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(X) = F(X) \quad \text{for any } X \in M. \tag{1.2}$$

Alexandrov [2] and Aeppli [1] studied the uniqueness question of starshaped hypersurfaces with prescribed curvature. The prescribing Weingarten curvature problem and similar problems have been studied by various authors, we refer to [3, 5, 6, 8, 9, 13, 16–18] and references therein for related works.

We will use notations in [11]. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, $S_k(\lambda)$ is defined as

$$S_k(\lambda) = \sum \lambda_{i_1} \cdots \lambda_{i_k},$$

where the sum is taken over for all increasing sequences $i_1, \cdots, i_k$ of the indices chosen from the set $\{1, \cdots, n\}$. The definition can be extended to symmetric matrices.

Definition 1.1 *For $1 \le k \le n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by*

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_1(\lambda) > 0, \cdots, S_k(\lambda) > 0\}.$$

A C^2 surface M is called k-admissible if at every point $X \in M$, $(\kappa_1, \kappa_2, \cdots, \kappa_n) \in \Gamma_k$.

The following existence result was proved by Caffarelli-Nirenberg-Spruck in [5] (in the case $k = 1$, by Bakelman-Kantor [3] and Treibergs-Wei [16]).

Theorem 1.1 *Let $F(X)$ be a smooth positive function in $r_1 \le |X| \le r_2$, $r_1 < 1 < r_2$, satisfying*

$$F(X)^{\frac{1}{k}} \ge (C_n^k)^{\frac{1}{k}} \frac{1}{r_1} \quad \text{for } |X| = r_1, \quad F(X)^{\frac{1}{k}} \le (C_n^k)^{\frac{1}{k}} \frac{1}{r_2} \quad \text{for } |X| = r_2, \tag{1.3}$$

$$\frac{\partial}{\partial \rho}(\rho^k F(X)) \le 0, \quad \text{where } \rho = |X|. \tag{1.4}$$

Then there is a C^∞ k-admissible hypersurface M satisfying

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(X) = F(X). \tag{1.5}$$

Any two solutions are endpoints of a one-parameter family of homothetic dilations, all of which are solutions.

The solution of the problem in Theorem 1.1 in general is not convex if $k < n$. Unlike in the case $k = n$ where the convexity is natural in the solution class of equation (1.2), in the case $k < n$ a k-admissible solution is not necessarily convex. The question of convexity of solution in Theorem 1.1 was treated by Chou [6] (see also [18]) for the mean curvature case under concavity assumption on F, and by Gerhardt [8] for general Weingarten curvature case under concavity assumption on $\log F$. The convexity question for solution of PDEs also appears in many other problems, where the deformation lemma plays an important role. We refer to [4, 11, 12, 14] and references therein for related works.

Our first result is the following general principle for strict convexity.

Theorem 1.2 *Let M be a C^3 oriented immersed connect hypersurface in $\mathbb{R}^{n+1}$ with a nonnegative definite second fundamental form. Suppose $\Sigma \subset \mathbb{R}^{n+1} \times \mathbb{S}^n$ is a bounded open set, positive function $F \in C^{1,1}(\Gamma)$ and $F^{-\frac{1}{k}}(X, y)$ is locally convex in X variable for any $y \in \mathbb{S}^n$. If $(X, \vec{n}(X)) \in \Sigma$ for each $X \in M$ and the principal curvatures of M satisfies the following curvature equation*

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(X) = F(X, e_{n+1}), \quad \forall\, X \in M, \tag{1.6}$$

then the second fundamental form of M is of constant rank. If in addition M is compact, then M is strictly convex.

Theorem 1.2 is a constant rank theorem, and it is very useful in the proof of existence of convex solutions when combined with homotopy deformation argument. The nonnegativity assumption on the second fundamental form looks quite strong at first appearance, but it is natural and automatic in the deformation process as one usually starts a strictly convex solution in such process. Theorem 1.2 guarantees that the strict convexity will be preserved in such deformation process. This type of deformation argument was first used in the important works of Singer-Wong-Yau-Yau [14] and Caffarelli-Friedman [4].

As a consequence of Theorem 1.2, we have the following existence of convex hypersurface with respect to Caffarelli-Nirenberg-Spruck solutions.

Corollary 1.1 *Suppose $F(X)$ is a smooth positive function in $r_1 \le |X| \le r_2$, $r_1 < 1 < r_2$, satisfying* (1.3), *and in addition $F(X)^{-\frac{1}{k}}$ is a convex function in the region $r_1 < |X| < r_2$. Then there is a C^∞ convex hypersurface M satisfying* (1.5). *If $F(X)$ satisfies the monotonicity condition* (1.4), *then any two solutions are endpoints of a one-parameter family of homothetic dilations, all of which are solutions.*

Remark 1.1 If we only concern the existence of convexity surface, the condition (1.4) in Theorem 1.1 can be removed in Corollary 1.1. It was used to get gradient estimates in [5]. We do not need it there as we treat convex hypersurfaces, since in the convex hypersurface case, it is standard that the gradient estimate comes from the C^0 estimate (see [13]).

Remark 1.2 In fact Theorem 1.2 implies the stronger result, which even allows the eigenvalues of Hessian of $F^{-\frac{1}{k}}(X)$ to be negative (see Theorem 2.1).

We also consider homogeneous Weingarten curvature problem. If M is a starshaped hypersurface about the origin in $\mathbb{R}^{n+1}$, by dilation property of the curvature function, the kth Weingarten curvature can be considered as a function of homogeneous degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$. The homogeneous Weingarten curvature problem is: given a homogeneous function F of degree $-k$ in $\mathbb{R}^{n+1} \setminus \{0\}$, does there exist a starshaped hypersurface M such that its kth Weingarten curvature is equal to $F(x)$ at $x \in M$? If F is of homogeneous degree $-k$, then the barrier condition (1.3) will never be valid unless the function is constant. Therefore Theorem 1.1 is not applicable, and the problem needs a different treatment. In fact, the problem is a nonlinear eigenvalue problem for the curvature equation. When $k = n$, equation (1.2) can be expressed as a Monge-Ampère equation of radial function ρ on $\mathbb{S}^n$, the problem was studied by Delanoë [7]. The case $k = 1$ was considered by Treibergs in [15]. Here we give a uniform treatment for $1 \leq k \leq n$. We also discuss the existence of convex solutions.

Theorem 1.3 *Suppose $n \geq 2$, $\quad 1 \leq k \leq n$ and f is a positive smooth function on $\mathbb{S}^n$. If $k < n$, assume further that f satisfies*

$$\sup_{\mathbb{S}^n} \frac{|\nabla f|}{f} < 2k. \tag{1.7}$$

Then there exist a unique constant $\gamma > 0$ with

$$\frac{C_n^k}{\max\limits_{\mathbb{S}^n} f} \leq \gamma \leq \frac{C_n^k}{\min\limits_{\mathbb{S}^n} f} \tag{1.8}$$

and a smooth k-admissible hypersurface M satisfying

$$S_k(k_1, k_2, \cdots, k_n)(X) = \gamma f\Big(\frac{X}{|X|}\Big)|X|^{-k}, \quad \forall\, X \in M, \tag{1.9}$$

and the solution is unique up to homothetic dilations. Furthermore, for $1 \leq k < n$, if in addition $|X| f(\frac{X}{|X|})^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$, then M is strictly convex.

Remark 1.3 Condition (1.7) in Theorem 1.3 can be weakened, we refer to Proposition 3.2 for the precise statement. When $k = n$, the above result was proved by Delanoë [7]. In this case, the solution is convex automatically. Our treatment here is different from [7]. When $k = 1$, the existence part of Theorem 1.3 was proved in [15], along with a sufficient condition for convexity.

The paper is organized as follows. In Section 2, we treat the convexity issue by establishing a deformation lemma for the curvature equation. The corresponding deformation lemma for spherical hessian equation was proved in [11]. The homogeneous Weingarten curvature problem is considered in Section 3, the main part (Lemma 3.2) is to obtain a Harnack type inequality for the curvature equations. Theorem 1.3 will be proved there.

2 The Issue of Convexity

In this section we establish a deformation lemma for prescribing intermediate curvature equations, the prescribed curvature function may depend on the position X and its outer unit normal. This type of deformation argument was initially used in the works of Singer-Wong-Yau-Yau [14] and Caffarelli-Friedman [4]. The main argument of the proof follows the one in [11], which in turn was motivated by [4, 12]. For the simplicity of notations, the summation convention is always used. Covariant differentiation will simply be indicated by indices. We will make use of some properties of elementary symmetric functions as in [11].

We first recall some identities for the relevant geometric quantities of a smooth closed compact starshaped hypersurfaces $M \subset \mathbb{R}^{n+1}$ about the origin. We assume the origin is not on M.

Since M is starshaped with respect to origin, the position vector X of M can be written as in (1.1). For any local orthonormal frame on $\mathbb{S}^n$, let ∇ be the gradient on $\mathbb{S}^n$ and covariant differentiation will simply be indicated by indices. Then in term of ρ the metric g_{ij} and its inverse g^{ij} on M are given by

$$g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j, \quad g^{ij} = \rho^{-2}\Big(\delta_{ij} - \frac{\rho_i\rho_j}{\rho^2 + |\nabla\rho|^2}\Big).$$

The second fundamental form of M is

$$h_{ij} = (\rho^2 + |\nabla\rho|^2)^{-\frac{1}{2}}(\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}),$$

and the unit outer normal of the hypersurface M in $\mathbb{R}^{n+1}$ is $\mathbf{N} = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2+|\nabla\rho|^2}}$. The principal curvatures of M are the eigenvalues of the second fundamental form with respect to the metric and therefore are the solutions of

$$\det(h_{ij} - kg_{ij}) = 0.$$

Equivalently they satisfy

$$\det(A_{ij} - k\delta_{ij}) = 0,$$

where the symmetric matrix $\{A_{ij}\}$ is given by

$$\{A_{ij}\} = \{g^{ik}\}^{\frac{1}{2}} h_{kl} \{g^{lj}\}^{\frac{1}{2}}. \tag{2.1}$$

Let $\{g^{ij}\}^{\frac{1}{2}}$ be the positive square root of $\{g^{ij}\}$ and be given by

$$[g^{ij}]^{\frac{1}{2}} = \rho^{-1}\Big[\delta_{ij} - \frac{\rho_i\rho_j}{\sqrt{\rho^2+|\nabla\rho|^2}\,(1+\sqrt{\rho^2+|\nabla\rho|^2}\,)}\Big].$$

We may also work on orthonormal frame on M directly. We choose an orthonormal frame $\{e_A\}$ such that $\{e_1, e_2, \cdots, e_n\}$ are tangent to M and e_{n+1} is normal. Let the corresponding coframe be denoted by $\{\omega_A\}$ and the connection forms by $\{\omega_{A,B}\}$. The pull-backs of those through the immersion will still be denoted by $\{\omega_A\}$,$\{\omega_{A,B}\}$ if there is no confusion. Therefore $\omega_{n+1} = 0$ on M. The second fundamental form is defined by the symmetric matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{2.2}$$

The following fundamental formulas are well known for hypersurfaces in $\mathbb{R}^{n+1}$.

$$X_{ij} = -h_{ij}e_{n+1} \qquad \text{(Gauss formula)}, \tag{2.3}$$

$$(e_{n+1})_i = h_{ij}e_j \qquad \text{(Weingarten equation)}, \tag{2.4}$$

$$h_{ijk} = h_{ikj} \qquad \text{(Codazzi formula)}, \tag{2.5}$$

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)}, \tag{2.6}$$

where R_{ijkl} is the curvature tensor. Using (2.5), (2.6) and the rule for interchanging the orders of derivatives, we observe the following commutation formula

$$h_{ijkl} = h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \tag{2.7}$$

From (2.3)–(2.4),

$$(e_{n+1})_{ii} = h_{iij}e_j - h_{ij}^2 e_{n+1}. \tag{2.8}$$

Then $S_k(\kappa_1, \kappa_2, \cdots, \kappa_n) = S_k(\lambda\{h_{ij}\})$. We consider the following curvature equation

$$S_k(\lambda\{h_{ij}\})(X) = f(X, e_{n+1}), \quad \forall\, X \in M, \tag{2.9}$$

where f is a positive function defined in $U \times \mathbb{S}^n$ for some neighborhood of M in $\mathbb{R}^{n+1}$.

Lemma 2.1 (Deformation Lemma) *Assume that M_o is a piece of C^4 hypersurface M, M is the solution of equation* (2.9) *and the matrix $W = \{h_{ij}\}$ is semi-positive definite. Suppose there is a positive constant $C_o > 0$, such that for a fixed integer $(n-1) \geq l \geq k$, $\forall\, X \in M_o$, $S_l(W(X)) \geq C_o$. Let $\phi(X) = S_{l+1}(W(X))$ and let $\tau(X)$ be the largest eigenvalue of $\{-(f^{-\frac{1}{k}})_{X_A X_B}(X, e_{n+1})\}$, where the differential is the standard differential in $\mathbb{R}^{n+1}$. Let $F^{\alpha\beta} = \frac{\partial S_k(W)}{\partial h_{\alpha\beta}}$. Then there are constants C_1, C_2 depending only on $\|M_o\|_{C^3}, \|f\|_{C^2}$ and C_o such that the following differential inequality holds in M_o,*

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta}\phi_{\alpha\beta}(X) \leq k(n-l) f^{\frac{k+1}{k}}(X) S_l(W(X))\tau(X) + C_1|\nabla\phi(X)| + C_2\phi(X). \tag{2.10}$$

Remark 2.1 We would like to point out that from our proof (see (2.22)) $\tau(x)$ in the lemma can be replaced by $\tilde{\tau}(x) = \sup\left[f_{AC}(X) - \frac{k+1}{k}\frac{f_A f_C}{f}(X)\right]\eta^A\eta^C$, where superium is taken over all unit tangential vector fields $\eta = (\eta^1, \cdots, \eta^{n+1})$ at X. In turn, the condition in Theorem 1.2 can also be replaced by the condition that $\tilde{\tau}$ is nonnegative.

Proof As in [11], for any two functions g and h defined in an open set $M_o \subset M$, we write $h \lesssim g$ if there exist positive constants c_1 and c_2 depending only on $\|X\|_{C^3}, \|f\|_{C^2}, n$ and C_o (independent of y and M_o), such that

$$(h - g)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y), \quad \forall\, y \in M_o.$$

We also write $h \backsim g$ if $h \lesssim g$ and $g \lesssim h$. We shall show that

$$\sum_{\alpha,\beta}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim k(n-l) f^{\frac{k+1}{k}} S_l(W)\tau.$$

For any $z \in M_o$, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalue of W at z. Since $S_l(W) \geq C_o > 0$ and $M \in C^3$, for any $z \in M$, there is a positive constant $C > 0$ depending only on $\|X\|_{C^3}$, $\|f\|_{C^2}$, n and C_o, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq C$. Let $G = \{1, 2, \cdots, l\}$ and $B = \{l+1, \cdots, n\}$ be the "good" and "bad" sets of indices, and define $S_k(W \mid i) = S_k((W \mid i))$ where $(W \mid i)$ means that the matrix W excludes the i-column and i-row, and $(W \mid ij)$ means that the matrix W excludes the i, j columns and i, j rows. All the calculations will be at the point z using the relation "$\lesssim$".

For each $z \in M_o$ fixed, we choose a local orthonormal frame $\{e_A\}$ in the neighborhood of z in M_o with $\{e_1, e_2, \cdots, e_n\}$ tangent to M_o and e_{n+1} is the normal so that the second fundamental form $\{h_{ij}\}$ is diagonal at z, and $h_{ii} = \lambda_i$, $\forall\, i = 1, 2, \cdots, n$. Now we compute ϕ and its first and second derivatives in the direction e_α. Let

$$S^{ij} = \frac{\partial S_{l+1}(W)}{\partial h_{ij}}, \quad S^{ij,rs} = \frac{\partial^2 S_{l+1}(W)}{\partial h_{ij}\partial h_{rs}}.$$

As $\phi = S_{l+1}(W)$ and $\phi_\alpha = \sum\limits_{i,j} S^{ij} h_{ij\alpha}$, we find that (as W is diagonal at z),

$$0 \backsim \phi(z) \backsim \Big(\sum_{i\in B} h_{ii}\Big) S_l(G) \backsim \sum_{i\in B} h_{ii}, \quad \text{so } h_{ii} \backsim 0,\ i \in B, \tag{2.11}$$

$$0 \backsim \phi_\alpha \backsim S_l(G)\sum_{i\in B} h_{ii\alpha} \backsim \sum_{i\in B} h_{ii\alpha}, \tag{2.12}$$

and

$$S_{l-1}(W \mid ij) \sim \begin{cases} 0, & \text{if } i, j \in G; \\ S_{l-1}(G \mid j), & \text{if } i \in B,\ j \in G; \\ S_{l-1}(G), & \text{if } i, j \in B,\ i \neq j. \end{cases} \tag{2.13}$$

Since $\phi_{\alpha\alpha} = \sum\limits_{i,j}[S^{ij,rs} h_{rs\alpha} h_{ij\alpha} + S^{ij} h_{ij\alpha\alpha}]$, from (2.11) and (2.12), it follows that for any $\alpha \in \{1, 2, \cdots, n\}$,

$$\begin{aligned} \phi_{\alpha\alpha} &= \sum_{i\neq j} S_{l-1}(W \mid ij) h_{ii\alpha} h_{jj\alpha} - \sum_{i\neq j} S_{l-1}(W \mid ij) h_{ij\alpha}^2 + \sum_i S^{ii} h_{ii\alpha\alpha} \\ &= \Big(\sum_{i\in G, j\in B} + \sum_{i\in B, j\in G} + \sum_{i,j\in B, i\neq j} + \sum_{i,j\in G, i\neq j}\Big) S_{l-1}(W \mid ij) h_{ii\alpha} h_{jj\alpha} \\ &\quad - \Big(\sum_{i\in G, j\in B} + \sum_{i\in B, j\in G} + \sum_{i,j\in B, i\neq j} + \sum_{i,j\in G, i\neq j}\Big) S_{l-1}(W \mid ij) h_{ij\alpha}^2 + \sum_i S^{ii} h_{ii\alpha\alpha}. \end{aligned} \tag{2.14}$$

Using (2.13), we obtain from (2.14),

$$\phi_{\alpha\alpha} \sim \sum_i S^{ii} h_{ii\alpha\alpha} - 2\sum_{i\in B, j\in G} S_{l-1}(G \mid j) h_{ij\alpha}^2 - S_{l-1}(G)\sum_{i,j\in B} h_{ij\alpha}^2. \tag{2.15}$$

By (2.11), and from (2.15) we obtain

$$\begin{aligned} \sum_{\alpha=1}^n F^{\alpha\alpha}\phi_{\alpha\alpha} \sim{}& \sum_{\alpha=1}^n \sum_i S^{ii} F^{\alpha\alpha} h_{ii\alpha\alpha} - 2\sum_{\alpha=1}^n \sum_{i\in B, j\in G} S_{l-1}(G \mid j) F^{\alpha\alpha} h_{ij\alpha}^2 \\ & - S_{l-1}(G)\sum_{\alpha=1}^n \sum_{i,j\in B} F^{\alpha\alpha} h_{ij\alpha}^2. \end{aligned} \tag{2.16}$$

By (2.7) and (2.11),

$$\begin{aligned} \sum_{\alpha=1}^n \sum_{i=1}^n S^{ii} F^{\alpha\alpha} h_{ii\alpha\alpha} &= \sum_{\alpha=1}^n \sum_{i=1}^n S^{ii} F^{\alpha\alpha}(h_{\alpha\alpha ii} + h_{\alpha\alpha} h_{ii}^2 - h_{\alpha\alpha}^2 h_{ii}) \\ &\lesssim \sum_{\alpha=1}^n \sum_{i=1}^n S^{ii} F^{\alpha\alpha} h_{\alpha\alpha ii} + kS_k(W)\sum_{i=1}^n S^{ii} h_{ii}^2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}h_{\alpha\alpha ii} + kS_k(W)(S_1(W)S_{l+1}(W) - (l+2)S_{l+2}(W)) \\
&\lesssim \sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}h_{\alpha\alpha ii}. \qquad (2.17)
\end{aligned}
$$

This is the main difference to the calculation in [11]. Here we make use of the commutation formula (2.7).

Differentiating the equation (2.9), we get

$$
f_{ii} = \sum_{\alpha,\beta,r,s} F^{\alpha\beta,rs}h_{\alpha\beta i}h_{rsi} + \sum_{\alpha,\beta} F^{\alpha\beta}h_{\alpha\beta ii}.
$$

From (2.11)–(2.13), we have

$$
\begin{aligned}
\sum_{\alpha=1}^{n}\sum_{i=1}^{n} S^{ii}F^{\alpha\alpha}h_{\alpha\alpha ii} \sim S_l(G)\sum_{i\in B}\Big\{ f_{ii} &- \sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G\mid\alpha\beta)h_{\beta\beta i}h_{\alpha\alpha i} \\
&- \sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)h_{\beta\beta i}h_{\alpha\alpha i} + \sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{n} S_{k-2}(W\mid\alpha\beta)h^2_{\alpha\beta i}\Big\}. \qquad (2.18)
\end{aligned}
$$

Inserting (2.18) and (2.17) to (2.16) yields

$$
\begin{aligned}
\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \sim{}& S_l(G)\sum_{i\in B} f_{ii} - S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G\mid\alpha\beta)h_{\alpha\alpha i}h_{\beta\beta i} \\
&- S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)h_{\alpha\alpha i}h_{\beta\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G} S_{l-1}(G\mid\beta)S_{k-1}(W\mid\alpha)h^2_{i\beta\alpha} \\
&+ S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta} S_{k-2}(W\mid\alpha\beta)h^2_{\alpha\beta i} - \sum_{\alpha=1}^{n} S_{l-1}(G)\sum_{i,\beta\in B} S_{k-1}(W\mid\alpha)h^2_{i\beta\alpha}. \qquad (2.19)
\end{aligned}
$$

Since $W = \{h_{ij}\}$ is semi-positive and $S_l(G)$ is a monomial, we obtain

$$
\begin{aligned}
&S_l(G)\sum_{i\in B}\sum_{\alpha\neq\beta} S_{k-2}(W\mid\alpha\beta)h^2_{\alpha\beta i} - 2\sum_{\alpha=1}^{n}\sum_{i\in B,\beta\in G} S_{l-1}(G\mid\beta)S_{k-1}(W\mid\alpha)h^2_{i\beta\alpha} \\
&\lesssim \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(W\mid\alpha\beta)h^2_{\alpha\beta i} - 2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G\mid\alpha)S_{k-1}(G\mid\alpha)h^2_{\alpha\alpha i}.
\end{aligned}
$$

Putting the above into (2.19), we have

$$
\begin{aligned}
\sum_{\alpha,\beta}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim{}& S_l(G)\Big[\sum_{i\in B} f_{ii} - \sum_{i\in B}\sum_{\substack{\alpha,\beta\in G\\ \alpha\neq\beta}} S_{k-2}(G\mid\alpha\beta)h_{\alpha\alpha i}h_{\beta\beta i}\Big] \\
&- 2\sum_{i\in B}\sum_{\alpha\in G} S_{l-1}(G\mid\alpha)S_{k-1}(G\mid\alpha)h^2_{\alpha\alpha i} - S_l(G)\sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_{k-2}(G)h_{\alpha\alpha i}h_{\beta\beta i} \\
&- \sum_{i=1}^{n} S_{l-1}(G)\sum_{\alpha,\beta\in B} S_{k-1}(G\mid\alpha)h^2_{\alpha\beta i} + \sum_{i\in B}\sum_{\substack{\alpha,\beta\in B\\ \alpha\neq\beta}} S_l(G)S_{k-2}(G\mid\alpha\beta)h^2_{\alpha\beta i}. \qquad (2.20)
\end{aligned}
$$

This is exactly the same form of formula (4.28) in [11]. The same proof in [11] yields that

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim S_l(G)\sum_{i\in B}\Big[f_{ii} - \frac{k+1}{k}\frac{f_i^2}{f}\Big]. \tag{2.21}$$

Since $\forall\, i \in \{1,2,\cdots,n\}$,

$$\begin{aligned} f_i &= \sum_{A=1}^{n+1} f_{X_A} e_i^A + f_{e_{n+1}}(e_{n+1})_i, \\ f_{ii} &= \sum_{A,C=1}^{n+1} f_{X_A X_C} e_i^A e_i^C + \sum_{A=1}^{n+1} f_{X_A} X_{ii}^A + 2\sum_{A=1}^{n+1} f_{X_A e_{n+1}} e_i^A (e_{n+1})_i \\ &\quad + f_{e_{n+1},e_{n+1}}(e_{n+1})_i(e_{n+1})_i + f_{e_{n+1}}(e_{n+1})_{ii}, \end{aligned}$$

from (2.3)–(2.4), (2.8)–(2.12), it follows that

$$\sum_{i\in B}\Big[f_{ii} - \frac{k+1}{k}\frac{f_i^2}{f}\Big] \backsim \sum_{i\in B}\sum_{A,C=1}^{n+1}\Big[f_{AC} - \frac{k+1}{k}\frac{f_A f_C}{f}\Big]e_i^A e_i^C. \tag{2.22}$$

Finally (2.10) follows from (2.21) and (2.22). The proof is complete.

Proof of Theorem 1.2 If $W = \{h_{ij}\}$ is not of full rank at some point x_o, then there is $n-1 \ge l \ge k$ such that $S_l(W(x)) > 0$, $\forall\, x \in M$ and $\phi(x_o) = S_{l+1}(W(x_o)) = 0$. By our assumption on $F(X,y)$, we conclude from (2.10) in the Deformation Lemma,

$$\frac{1}{\sigma_l(G)}\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim 0. \tag{2.23}$$

The strong minimum principle implies that W is of constant rank l. If M is compact, there is at least one point where the second fundamental form of M is positive definite. Therefore it is positive definite everywhere and M is the boundary of some strongly convex bounded domain in $\mathbb{R}^{n+1}$.

Proof of Corollary 1.1 For $0 \le t \le 1$ and $0 < \epsilon < 1$, set

$$F(t,X) = [(1-t)(C_n^k)^{-\frac{1}{k}}|X|^{1+\epsilon} + tF^{-\frac{1}{k}}(X)]^{-k}.$$

Consider

$$S_k(\kappa_1,\kappa_2,\cdots,\kappa_n)(X^t) = F(t,X^t). \tag{2.24}$$

We use degree theory. Following the same lines of the proof in [5], C^4 norms of convex solutions of (2.24) are under control independent of t and ϵ. (Since we treat convex hypersurfaces, the gradient estimate is automatic once there is a C^0 estimate, so condition (1.4) is not needed in this case.) By Theorem 1.2 and continuity of degree argument, the strict convexity is preserved for each $t \ge 0$. It is straightforward to calculate that the degree at $t = 0$ is not vanishing. From this, we may conclude that there is a strictly convex solution of (2.24) at $t = 1$. If in addition $F(X)$ satisfies the condition (1.4), as in [5] any two solutions are endpoints of a one-parameter family of homothetic dilations, all of which are solutions.

Theorem 1.2 implies the following stronger result, which even allows the eigenvalues of Hessian of $F^{-\frac{1}{k}}(X)$ to be negative.

Theorem 2.1 *For any constant $1 > \beta > 0$, there is a positive constant $\gamma > 0$ such that if $F(X)$ is a smooth positive function in the region $r_1 < |X| < r_2$ satisfying the barriers condition* (1.3), $\frac{\inf F}{\sup F} \geq \beta$, $\frac{\sup F}{\|F\|_{C^2}} \geq \beta$, *and*

$$(F^{-\frac{1}{k}})_{ij}(X) \geq -\gamma\delta_{ij} \quad \text{on } r_1 < |X| < r_2, \tag{2.25}$$

then there is a C^∞ convex hypersurface M satisfying (1.5).

Proof We argue by contradiction. If the result is not true, for some $0 < \beta < 1$, there is a sequence of positive functions $F_l \in C^{1,1}$ such that $\sup F_l = 1$, $\inf F_l \geq \beta$, $\|F_l\|_{C^{1,1}} \leq \frac{1}{\beta}$, $(F^{-\frac{1}{k}})_{ij}(X) \geq -\frac{1}{l}\delta_{ij}$, F_l satisfies (1.3) and equation (1.5) has no convex solution. As in the proof of Corollary 1.1, we consider equation (2.24). For each l, there is $0 < t_l < 1$, such that the equation

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(X^t) = \widetilde{F}_(X)$$

with $\widetilde{F}_l(X) = [(1-t_l)(C_n^k)^{-\frac{1}{k}}|X|^{1+\epsilon} + t_l F_l^{-\frac{1}{k}}(X)]^{-k}$ has a convex solution u_l with

$$h_{ij}(X) = 0 \text{ at some point } x_l, \text{ and } \|u_l\|_{C^{3,\alpha}} \leq C, \text{ independent of } l.$$

Therefore, there exist subsequences, which we still denote by F_l and u_l,

$$F_l \to F \quad \text{in } C^{1,\alpha}, \quad u_l \to u \quad \text{in } C^{3,\alpha},$$

for some positive $F \in C^{1,1}$ with $(F_{ij}^{-\frac{1}{k}})(X) \geq 0$, and u satisfies equation (1.5), $h_{ij}(X) \geq 0$ and vanishes at some point x_0. This is a contradiction to Theorem 1.2.

3 The Homogeneous Weingarten Curvature Problem

We consider the homogeneous Weingarten curvature problem in this section. Since equation (1.2) is invariant under dilations, there is no C^0 bound in general. To solve the equation, we need to establish the Harnack inequality for solutions of (1.2). This is the main part of the proof in this section. We will follow the ideas in [10] to consider the following auxiliary equation first

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(X) = f\Big(\frac{X}{|X|}\Big)|X|^{-p}, \quad \forall\, X \in M,\ 1 \leq k \leq n-1, \tag{3.1}$$

where f is a prescribed positive function on $\mathbb{S}^n$ and M is a starlike hypersurface in $\mathbb{R}^{n+1}$. Since M is starshaped, let ρ be the radial function as in (1.2). The following is the equation for ρ.

$$S_k(\kappa_1, \cdots, \kappa_n)(x) = f(x)\rho^{-p} \quad \text{on } \mathbb{S}^n. \tag{3.2}$$

We first derive an upper bound of $|\nabla^2\rho|$ estimates for the k-admissible solution ρ of equation (3.2) for any $p \in [k, k+1]$ assuming C^1 boundedness.

Lemma 3.1 *If M is a starlike hypersurface in $\mathbb{R}^{n+1}$ with respect to the origin, f is a C^2 positive function on $\mathbb{S}^n$, $k > 1, p \in [k, k+1]$, and if M is a C^4 k-admissible solution of equation* (3.1)*, then we have the mean curvature $H \le C$ for some constant C depending only on $k, n, \frac{|\nabla f|}{f}, \frac{|\nabla^2 f|}{f}, \|\rho\|_{C^1}$ and $\|\frac{1}{\rho}\|_{C^0}$ (independent of p). In turn, $\max\limits_{x\in\mathbb{S}^n}|\nabla^2\rho(x)| \le C$.*

Proof Let $F(X) = f\big(\frac{X}{|X|}\big)$ and $\varphi(X) = [|X|^{-p}F(X)]^{\frac{1}{k}}$. The equation in Lemma 3.1 becomes

$$G(\lambda\{h_{ij}\})(X) = [S_k(\lambda\{h_{ij}\})]^{\frac{1}{k}}(X) = \varphi(X) \quad \text{on } M. \tag{3.3}$$

Assume that the function $P = \log H - \log\langle X, e_{n+1}\rangle$ attains its maximum at $X_o \in M$. Then at X_o we have

$$P_i = \frac{H_i}{H} - \frac{\langle X, e_{n+1}\rangle_i}{\langle X, e_{n+1}\rangle} = 0, \quad P_{ii} = \frac{H_{ii}}{H} - \frac{\langle X, e_{n+1}\rangle_{ii}}{\langle X, e_{n+1}\rangle}.$$

Let $G^{ij} = \frac{\partial G(\lambda\{h_{ij}\})}{\partial h_{ij}}$, and choose suitable $\{e_1, e_2, \cdots, e_n\}$ on the neighborhood of $X_o \in M$ such that at X_o the matrix $\{h_{ij}\}$ is diagonal. Then at X_o, the matrix $\{G^{ij}\}$ is also diagonal and positive definitive. At X_o,

$$\sum_{ij=1}^n G^{ij}P_{ij} = \frac{\sum\limits_{i=1}^n G^{ii}H_{ii}}{H} - \frac{\sum\limits_{i=1}^n G^{ii}\langle X, e_{n+1}\rangle_{ii}}{\langle X, e_{n+1}\rangle} \le 0. \tag{3.4}$$

From this inequality we shall obtain an upper bound of H.

We set $|A|^2 = \sum\limits_{i=1}^n h_{ii}^2$. From (2.6), we have

$$\begin{aligned}\sum_{i=1}^n G^{ii}H_{ii} &= \sum_{i=1}^n G^{ii}\Big(\sum_{l=1}^n h_{llii}\Big) = \sum_{i=1}^n G^{ii}\sum_{l=1}^n (h_{iill} + h_{ii}h_{ll}^2 - h_{ll}h_{ii}^2)\\ &= \sum_{il=1}^n G^{ii}h_{iill} + |A|^2\sum_{i=1}^n G^{ii}h_{ii} - H\sum_{i=1}^n G^{ii}h_{ii}^2 \ge \sum_{l=1}^n \varphi_{ll} + |A|^2\varphi - H\sum_{i=1}^n G^{ii}h_{ii}^2.\end{aligned}$$

And from (2.2) and (2.7),

$$\begin{aligned}\sum_{i=1}^n G^{ii}\langle X, e_{n+1}\rangle_{ii} &= \sum_{i=1}^n G^{ii}\Big[\sum_{l=1}^n h_{iil}\langle X, e_l\rangle + h_{ii} - h_{ii}^2\langle X, e_{n+1}\rangle\Big]\\ &= \sum_{l=1}^n\Big(\sum_{i=1}^n G^{ii}h_{iil}\Big)\langle X, e_l\rangle + \sum_{i=1}^n G^{ii}h_{ii} - \langle X, e_{n+1}\rangle\sum_{i=1}^n G^{ii}h_{ii}^2\\ &= \sum_{l=1}^n \varphi_l\langle X, e_l\rangle + \varphi - \langle X, e_{n+1}\rangle\sum_{i=1}^n G^{ii}h_{ii}^2.\end{aligned}$$

So from (3.4), at X_o we have the following inequality

$$|A|^2 + \sum_{l=1}^n \frac{\varphi_{ll}}{\varphi} - \sum_{l=1}^n \frac{H\varphi_l}{\varphi\langle X, e_{n+1}\rangle}\langle X, e_l\rangle - \frac{H}{\langle X, e_{n+1}\rangle} \le 0. \tag{3.5}$$

Let F_A, F_{AB} be the ordinary Euclidian differential in $\mathbb{R}^{n+1}$. Since

$$\frac{\varphi_l}{\varphi} = \frac{1}{k}\Big[- p|X|^{-2}\langle X, e_l\rangle + \sum_{A=1}^{n+1} \frac{F_A}{F} X_l^A \Big],$$

we have

$$\begin{aligned}\sum_{l=1}^{n} \frac{\varphi_{ll}}{\varphi} &= H\Big[\frac{p}{k}|X|^{-2}\langle X, e_{n+1}\rangle - \frac{1}{k}\sum_{A=1}^{n+1}\frac{F_A}{F}e_{n+1}^A\Big] + \frac{1}{k}\sum_{l=1}^{n}\sum_{A,B=1}^{n+1}\frac{F_{AB}}{F}X_l^A X_l^B \\ &\quad + \sum_{l=1}^{n}\sum_{AB=1}^{n+1}\frac{F_A F_B}{F^2}X_l^A X_l^B + \frac{p}{k}\Big[1+\frac{p}{k}\Big]|X|^{-2} - \frac{p}{k}\Big[2+\frac{p}{k}\Big]|X|^{-4}\langle X, e_{n+1}\rangle^2 \\ &\quad - \frac{2p}{k^2}|X|^{-2}\sum_{l=1}^{n}\sum_{A=1}^{n+1}\frac{F_A}{F}X_l^A\langle X, e_l\rangle.\end{aligned}$$

As $|A|^2 \geq \frac{1}{n}H^2$, by (3.5) there exists a positive constant C depending only on the k, n, $\frac{|\nabla f|}{f}$, $\frac{|\nabla^2 f|}{f}$ such that $H(X_o) \leq C$. Again from C^1 bound, we have $\max H \leq C$. The proof of the lemma is complete.

One may also derive C^1-estimates if C^0 bound is assumed. Instead, we will derive the Harnack inequality directly, that will imply C^0 and C^1 bounds. It is convenient to introduce a new function $v = -\log \rho$. Then the first and second fundamental forms become

$$\begin{aligned} g_{ij} &= e^{-2v}[\delta_{ij} + v_i v_j], \\ h_{ij} &= e^{-v}(1+|\nabla v|^2)^{-\frac{1}{2}}[\delta_{ij} + v_i v_j + v_{ij}]. \end{aligned}$$

And

$$[g^{ij}]^{\frac{1}{2}} = e^v\Big[\delta_{ij} - \frac{v_i v_j}{\sqrt{1+|\nabla v|^2}\,(1+\sqrt{1+|\nabla v|^2}\,)}\Big].$$

So if we let

$$\begin{aligned} \bar{g}^{ij} &= \Big[\delta_{ij} - \frac{v_i v_j}{\sqrt{1+|\nabla v|^2}\,(1+\sqrt{1+|\nabla v|^2}\,)}\Big], \\ \bar{h}_{lm} &= \delta_{lm} + v_l v_m + v_{lm}, \\ a_{ij} &= \bar{g}^{il}\bar{h}_{lm}\bar{g}^{mj}, \end{aligned} \tag{3.6}$$

then the matrix in (2.1) becomes

$$A_{ij} = e^v(1+|\nabla v|^2)^{-\frac{1}{2}} a_{ij}, \tag{3.7}$$

and equation (3.2) turns into

$$S_k(\lambda\{a_{ij}\}) = e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} f(x) \quad \text{on } \mathbb{S}^n. \tag{3.8}$$

First we have the easy case $p > k$.

Proposition 3.1 *Suppose $p > k$. For any $f(x) \in C^2(\mathbb{S}^n)$, $n \geq 2$, $f > 0$, there exists a unique k-admissible starlike hypersurface M which satisfies* (3.1)*. If in addition f satisfies*

$$|X|^{\frac{p}{k}} f\Big(\frac{X}{|X|}\Big)^{-\frac{1}{k}} \quad \textit{is a convex function in } \mathbb{R}^{n+1} \setminus \{0\}, \tag{3.9}$$

then M is a strictly convex hypersurface.

Proof of Proposition 3.1 For any positive function $f \in C^2(\mathbb{S}^n)$, for $0 \leq t \leq 1$, set $f_t = [1 - t + t f^{-\frac{1}{k}}]^{-k}$. We consider the equation

$$S_k(\kappa_1, \cdots, \kappa_n)(x) = f_t(x)\rho^{-p} \quad \text{on } \mathbb{S}^n, \tag{3.10}$$

Set $I = \{t \mid (3.10) \text{ solvable}\}$.

We first consider C^0-estimates. Let

$$l = \min_{\mathbb{S}^n} \rho \quad \text{and} \quad L = \max_{\mathbb{S}^n} \rho.$$

If $x_o \in \mathbb{S}^n$ such that $\rho(x_o) = L$, then at x_o, we have

$$\nabla\rho = 0 \quad \text{and} \quad \{\rho_{ij}\} \leq 0.$$

It follows that at x_o,

$$\kappa_i(x_o) \geq L^{-1}, \quad \forall\, 1 \leq i \leq n.$$

Evaluating (3.10) at x_o and using the above, we have $L \leq \left[\frac{\max_{\mathbb{S}^n} f_t}{C_n^k}\right]^{\frac{1}{p-k}}$. The similar argument also yields $l \geq \left[\frac{\min_{\mathbb{S}^n} f_t}{C_n^k}\right]^{\frac{1}{p-k}}$.

With the C^0-estimates, the arguments in [5] immediately yield the C^1-estimates. Together with Lemma 3.1, we have

$$\|\rho\|_{C^2(\mathbb{S}^n)} \leq C \quad \text{and} \quad \Big\|\frac{1}{\rho}\Big\|_{C^2(\mathbb{S}^n)} \leq C, \tag{3.11}$$

where C depends only on $p, k, n, \|f\|_{C^2(\mathbb{S}^n)}$ and $\min_{\mathbb{S}^n} f$. (In the case $k = 1$, (3.11) follows from the standard quasilinear theory. The regularity assumption on f can also be reduced.)

Now the Evens-Krylov theorem and the Schauder theorem imply that I is closed. The openness is from the implicit function theorem since the linearized operator of (3.8) is invertible when $p > k$. The method of continuity yields the existence. The uniqueness follows easily from the Strong Maximum Principle and the dilation property of equation (3.1) for $p > k$.

Since f_t satisfies the convexity condition (3.9) in Theorem 1.1 for $0 \leq t \leq 1$, the strict convexity comes from Theorem 1.2.

We now deal with equation (3.1) for the case $p = k$ in the rest of this section. The equation is in the following form

$$S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)(x) = f(x)\rho^{-k}, \quad \forall\, x \in \mathbb{S}^n. \tag{3.12}$$

In order to bound $\frac{\max\rho}{\min\rho}$, we turn to estimate $|\nabla\log\rho| = |\nabla v|$. We follow an argument in [10] to make use of the result in Proposition 3.1 with some refined estimates for ρ_r with $p_r = k + \frac{1}{r}$. We hope to get the convergence of ρ_r as r tends to infinity. It turns out that the limit of ρ_r will satisfy equation (3.12) but with f replaced by γf for some positive γ. We will show that the constant γ is unique.

Lemma 3.2 *For $1 \le k \le n$ suppose f is a positive C^1 function on $\mathbb{S}^n$. Suppose ρ is a C^3 k-admissible solution of equation* (3.2) *with $p \in [k, k+1]$. If $k < n$, we further assume that f satisfies*

$$\delta_f =: \min_{x\in\mathbb{S}^n, d_1\le s\le d_2} \Big\{k\Big(\Big(\frac{(n-k)s}{nf(x)}\Big)^{\frac{1}{k}} + \Big(\frac{nf(x)}{(n-k)s}\Big)^{\frac{1}{k}}\Big) - \frac{|\nabla f(x)|}{f(x)}\Big\} > 0, \tag{3.13}$$

where $d_1 = \min f$, $d_2 = \max f$. Then $\max\limits_{\mathbb{S}^n} |\nabla\log\rho(x)| \le C$, for some constant C depending only on k, n, δ_f, $\max\frac{|\nabla f|}{f}$ (and independent of p). In particular,

$$1 \le \frac{\max\rho}{\min\rho} \le C.$$

Remark 3.1 If $k = p$, from the proof below, the gradient estimate Lemma 3.2 can be established under simpler and weaker condition

$$\min_{x\in\mathbb{S}^n} \Big\{k\Big(\frac{(C_{n-1}^k)^{\frac{1}{k}}}{f^{\frac{1}{k}}} + \frac{f^{\frac{1}{k}}}{(C_{n-1}^k)^{\frac{1}{k}}}\Big) - \frac{|\nabla f|}{f}\Big\} > 0.$$

From the counter-example of Treibergs, it can be shown that this condition is sharp for the gradient estimate of equation (3.8) when $1 \le k \le n-1$.

Proof We work on equation (3.8) to get gradient estimates for v. Let $P = |\nabla v|^2$ attain its maximum at $x_o \in \mathbb{S}^n$. Then

$$P_i = \sum_{k=1}^n v_k v_{ki} = 0 \quad \text{at } x_o. \tag{3.14}$$

Let $\{e_1, e_2, \cdots, e_n\}$ be the standard orthonormal frame at the neighborhood of x_o. Take e_1 such that

$$v_1 = |\nabla v|, \quad v_i = 0, \quad i \ge 2, \tag{3.15}$$

and $e_2, \cdots, e_n$ such that $\{v_{ij}\}(x_o)$ is diagonal. It follows that at x_o,

$$v_{11} = 0, \quad v_{ij} = 0, \quad i \ne j,$$

so the matrices $\{\bar g^{ij}\}, \{\bar h^{ij}\}$ and $\{a_{ij}\}$ are diagonal at the point, and $\bar g^{11} = \frac{1}{\sqrt{1+|\nabla v|^2}}$, $\bar h_{11} = 1 + |\nabla v|^2$, $a_{11} = 1$; and for all $i > 1$, $\bar g^{ii} = 1$, $\bar h_{ii} = a_{ii} = 1 + v_{ii}$.

Let $F^{ij} = \frac{\partial S_k}{\partial a_{ij}}$, so $\{F^{ij}\}$ is diagonal at x_o. Differentiating equation (3.8) gives

$$F^{ij} a_{ijs} = e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}}[(p-k)v_s f + f_s]. \tag{3.16}$$

From (3.6),

$$a_{ijs} = (\bar{g}^{il}\bar{h}_{lm}\bar{g}^{mj})_s, \quad v_s\bar{g}^{mj}_s = 0 = v_s\bar{g}^{il}_s,$$

we have

$$v_s a_{ijs} = \bar{g}^{il} v_s v_{lms} \bar{g}^{mj}. \tag{3.17}$$

Couple (3.16) and (3.17), and we have

$$v_s F^{ij} a_{ijs} = F^{ij}\bar{g}^{il} v_s v_{lms}\bar{g}^{mj} = e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} v_s[(p-k)v_s f + f_s]. \tag{3.18}$$

On the other hand,

$$\begin{aligned} v_s F^{ij} a_{ijs} &= F^{ij}\bar{g}^{il} v_s v_{lms}\bar{g}^{mj} = F^{lm}\bar{g}^{il} v_s v_{ijs}\bar{g}^{mj} \\ &= F^{lm}\bar{g}^{il}\bar{g}^{mj} v_s[v_{sij} - v_s\delta_{ij} + v_j\delta_{si}] \\ &= F^{lm}\bar{g}^{il}\bar{g}^{mj} v_s v_{sij} - |\nabla v|^2 \sum_{ilm} F^{lm}\bar{g}^{il}\bar{g}^{mi} + \sum_{ijlm} F^{lm}\bar{g}^{il}\bar{g}^{mj} v_i v_j. \end{aligned}$$

Let $\overline{F}^{ij} = \sum_{lm} F^{lm}\bar{g}^{il}\bar{g}^{mj}$, so at x_o, $\overline{F}^{ij}$ is diagonal with $\overline{F}^{11} = \frac{F^{11}}{1+v_1^2}$ and $\overline{F}^{ii} = F^{ii}$ for $i > 1$. Then we have

$$\sum_{ijs} v_s F^{ij} a_{ijs} = \sum_{ijs} \overline{F}^{ij} v_s v_{sij} - |\nabla v|^2 \sum_i \overline{F}^{ii} + \sum_{ij} \overline{F}^{ij} v_i v_j.$$

From (3.18), (3.19) and (3.15), we have

$$\sum_{ijs} \overline{F}^{ij} v_s v_{sij} = e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} v_s[(p-k)v_s f + f_s] + |\nabla v|^2 \sum_{i=2}^{n} \overline{F}^{ii}. \tag{3.19}$$

Since $\overline{F}^{ij}$ is positive definite and

$$P_{ij} = \sum_s v_{si}v_{sj} + \sum_s v_s v_{sij},$$

we have at x_o,

$$\overline{F}^{ij} P_{ij} = \sum_{ijs} \overline{F}^{ij} v_{si} v_{sj} + \sum_{ijs} \overline{F}^{ij} v_s v_{sij} \leq 0. \tag{3.20}$$

From (3.19) and (3.20) it follows that at x_o,

$$\sum_{i=2}^{n} \overline{F}^{ii}(v_1^2 + v_{ii}^2) + e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}}[(p-k)v_1^2 f + v_1 f_1] \leq 0, \tag{3.21}$$

i.e., we obtain the following inequality

$$\sum_{i=2}^{n} \overline{F}^{ii}(v_1^2 + v_{ii}^2) + e^{(p-k)v}(1+|\nabla v|^2)^{\frac{k}{2}} v_1 f_1 \leq 0. \tag{3.22}$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the eigenvalues of the matrix $\{a_{ij}\}$, at the point,

$$\lambda_1 = 1, \quad \lambda_2 = 1 + v_{22}, \quad \cdots, \quad \lambda_n = 1 + v_{nn}; \tag{3.23}$$

and for $i \geq 2$,

$$\overline{F}^{ii} = S_{k-1}(\lambda|i), \quad v_{ii}^2 = \lambda_i^2 - 2\lambda_i + 1. \tag{3.24}$$

Then equation (3.8) becomes

$$S_k(\lambda) = e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}} f(x) \quad \text{on } \mathbb{S}^n. \tag{3.25}$$

From (3.22) and (3.24) we have

$$\begin{aligned}(1 + v_1^2)\sum_{i=2}^{n} S_{k-1}(\lambda \mid i) &+ \sum_{i=2}^{n} \lambda_i^2 S_{k-1}(\lambda \mid i) \\ - 2\sum_{i=2}^{n} \lambda_i S_{k-1}(\lambda \mid i) &+ e^{(p-k)v}(1 + |\nabla v|^2)^{\frac{k}{2}} v_1 f_1 \leq 0,\end{aligned} \tag{3.26}$$

since

$$\sum_{i=2}^{n} S_{k-1}(\lambda \mid i) = (n-k)S_{k-1}(\lambda) + S_{k-2}(\lambda \mid 1), \tag{3.27}$$

$$\begin{aligned}&\sum_{i=2}^{n} \lambda_i^2 S_{k-1}(\lambda \mid i) - 2\sum_{i=2}^{n} \lambda_i S_{k-1}(\lambda \mid i) \\ &= \sum_{i=1}^{n} \lambda_i^2 S_{k-1}(\lambda \mid i) - 2\sum_{i=1}^{n} \lambda_i S_{k-1}(\lambda \mid i) + S_{k-1}(\lambda \mid 1) \\ &= S_1(\lambda)S_k(\lambda) - (k+1)S_{k+1}(\lambda) - 2kS_k(\lambda) + S_{k-1}(\lambda \mid 1).\end{aligned} \tag{3.28}$$

Put (3.27) and (3.28) into (3.26). It follows that

$$\begin{aligned}&(1 + v_1^2)(n-k)S_{k-1}(\lambda) + S_1(\lambda)S_k(\lambda) - (k+1)S_{k+1}(\lambda) \\ &+ e^{(p-k)v} v_1 f_1 (1 + v_1^2)^{\frac{k}{2}} - 2kS_k(\lambda) + (1 + v_1^2)S_{k-2}(\lambda \mid 1) + S_{k-1}(\lambda \mid 1) \leq 0.\end{aligned} \tag{3.29}$$

We also note that if x_1 and x_2 are minimum and maximum points of v respectively, from equation (3.25), we have

$$e^{(p-k)v(x_1)} \geq \frac{C_n^k}{f(x_1)} \geq \frac{C_n^k}{\max f}, \quad e^{(p-k)v(x_2)} \leq \frac{C_n^k}{f(x_2)} \leq \frac{C_n^k}{\min f}. \tag{3.30}$$

So $\forall\, x$,

$$C_n^k \frac{\max f}{\min f} \geq e^{(p-k)v} f \geq C_n^k \frac{\min f}{\max f}. \tag{3.31}$$

This fact will be used later on.

We divide the proof into two cases.

Case 1 $k = n$.

As $S_{n+1}(\lambda) = 0$, and both $S_{n-2}(\lambda \mid 1)$ and $S_{n-1}(\lambda \mid 1)$ are positive, the above inequality takes a simpler form

$$S_1(\lambda)S_n(\lambda) + e^{(p-n)v} v_1 f_1 (1 + v_1^2)^{\frac{n}{2}} \leq 2nS_n(\lambda).$$

Since $\lambda_1 = 1$, $S_n(\lambda) = S_{n-1}(\lambda \mid 1)$. By the Newton-MacLaurin inequality,

$$S_1(\lambda) > S_1(\lambda \mid 1) \geq (n-1)S_{n-1}(\lambda \mid 1)^{\frac{1}{n-1}} = (n-1)S_n(\lambda)^{\frac{1}{n-1}}.$$

In turn, we get

$$(n-1)S_n(\lambda)^{\frac{n}{n-1}} - e^{(p-n)v}v_1|f_1|(1+v_1^2)^{\frac{n}{2}} \leq 2nS_n(\lambda). \tag{3.32}$$

(3.25), (3.31) and (3.32) yield that at the point,

$$(n-1)(1+v_1^2)^{\frac{n}{2(n-1)}}\Big(\frac{\min f}{\max f}\Big)^{\frac{1}{n-1}} - (1+v_1^2)^{\frac{1}{2}}\frac{|\nabla f|}{f} \leq 2n.$$

Since $\frac{n}{2(n-1)} > \frac{1}{2}$ and $\frac{\min f}{\max f}$ is bounded from below by a positive constant $\big($depending only on the upper bound of $\frac{|\nabla f|}{f}\big)$, we obtain an upper bound for $|\nabla v|$.

Case 2 $k < n$.

Claim

$$(k+1)S_{k+1}(\lambda) \leq (k+1)S_k(\lambda) + (n-k-1)(C_{n-1}^k)^{-\frac{1}{k}}S_k(\lambda)^{\frac{1}{k}+1}. \tag{3.33}$$

Proof of Claim If $S_{k+1}(\lambda) \leq 0$, it is automatic. We may assume $S_{k+1}(\lambda) > 0$. As $\lambda \in \Gamma_k$, we get $\lambda \in \Gamma_{k+1}$. In turn $(\lambda \mid 1) \in \Gamma_k$. We have

$$S_{k+1}(\lambda) = S_{k+1}(\lambda \mid 1) + S_k(\lambda \mid 1) \leq S_{k+1}(\lambda \mid 1) + S_k(\lambda). \tag{3.34}$$

If $S_{k+1}(\lambda \mid 1) \leq 0$, we are done. Thus we may assume $S_{k+1}(\lambda \mid 1) > 0$. Again as $(\lambda \mid 1) \in \Gamma_k$, this gives $(\lambda \mid 1) \in \Gamma_{k+1}$.

By the Newton-MacLaurin inequality,

$$\begin{aligned} S_{k+1}(\lambda \mid 1) &\leq C_{n-1}^{k+1}(C_{n-1}^k)^{-\frac{k+1}{k}}(S_k(\lambda \mid 1))^{\frac{k+1}{k}} \leq C_{n-1}^{k+1}(C_{n-1}^k)^{-\frac{k+1}{k}}S_k(\lambda)^{\frac{k+1}{k}} \\ &= \frac{n-k-1}{k+1}(C_{n-1}^k)^{-\frac{1}{k}}S_k^{\frac{1}{k}+1}(\lambda). \end{aligned} \tag{3.35}$$

The Claim now follows from (3.34)-(3.35).

Now back to the proof of the lemma. If $S_k(\lambda \mid 1) \leq 0$, we will have $S_{k-1}(\lambda) \geq S_k(\lambda)$. From (3.33), (3.29) and the Newton-MacLaurin inequality, we get

$$(1+v_1^2)(n-k)S_k(\lambda) - \frac{|\nabla f|}{f}(1+v_1^2)^{\frac{1}{2}}S_k(\lambda) - (3k+1)S_k(\lambda) \leq 0.$$

From this we obtain an upper bound of $|\nabla v|$.

We may now assume $S_k(\lambda \mid 1) > 0$, i.e., $(\lambda \mid 1) \in \Gamma_k$ in the rest of the proof. From the Newton-MacLaurin inequality,

$$S_1(\lambda) > S_1(\lambda \mid 1) \geq (n-1)(C_{n-1}^k)^{-\frac{1}{k}}S_k^{\frac{1}{k}}(\lambda \mid 1).$$

Similarly,

$$S_1(\lambda \mid 1) \geq (n-1)(C_{n-1}^{k-1})^{-\frac{1}{k-1}}S_{k-1}^{\frac{1}{k-1}}(\lambda \mid 1).$$

From this, we get

$$\Big(S_1(\lambda)+\frac{n-1}{n-k}\Big)^k\geq S_1^k(\lambda)+\frac{k(n-1)}{n-k}S_1^{k-1}(\lambda)$$
$$\geq\frac{(n-1)^k}{C_{n-1}^k}(S_k(\lambda\mid 1)+S_{k-1}(\lambda\mid 1))=\frac{(n-1)^k}{C_{n-1}^k}S_k(\lambda).$$

That is

$$S_1(\lambda)>(n-1)(C_{n-1}^k)^{-\frac{1}{k}}S_k^{\frac{1}{k}}(\lambda)-\frac{n-1}{n-k}. \tag{3.36}$$

Since

$$(n-k)S_{k-1}(\lambda)+S_{k-2}(\lambda\mid 1)=(n-k)S_{k-1}(\lambda\mid 1)+(n-k+1)S_{k-2}(\lambda\mid 1),$$

and $S_k(\lambda)=S_k(\lambda\mid 1)+S_{k-1}(\lambda\mid 1)$, we get

$$[(n-k)S_{k-1}(\lambda)+S_{k-2}(\lambda\mid 1)]^k=\sum_{0\leq j\leq k}C_k^j(n-k)^{k-j}(n-k+1)^jS_{k-1}^{k-j}(\lambda\mid 1)S_{k-2}^j(\lambda\mid 1),$$
$$k^kC_{n-1}^kS_k^{k-1}(\lambda)=\sum_{0\leq j\leq k-1}k^kC_{n-1}^kC_{k-1}^jS_k^{k-1-j}(\lambda\mid 1)S_{k-1}^j(\lambda\mid 1).$$

Again using the Newton-MacLaurin inequality on $S_l(\lambda\mid 1)$, it is elementary to check that for $0\leq j\leq k-1$,

$$C_k^j(n-k)^{k-j}(n-k+1)^jS_{k-1}^{k-j}(\lambda\mid 1)S_{k-2}^j(\lambda\mid 1)\geq k^kC_{n-1}^kC_{k-1}^jS_k^{k-1-j}(\lambda\mid 1)S_{k-1}^j(\lambda\mid 1),$$

that is,

$$(n-k)S_{k-1}(\lambda)+S_{k-2}(\lambda\mid 1)\geq k(C_{n-1}^k)^{\frac{1}{k}}S_k^{\frac{k-1}{k}}(\lambda). \tag{3.37}$$

Combining (3.33), (3.37), (3.36), (3.25) and (3.29), we obtain

$$(1+v_1^2)^{\frac{1}{2}}\Big(k(A+A^{-1})-\frac{|\nabla f|}{f}\Big)\leq C, \tag{3.38}$$

where $A=e^{\frac{(k-p)v}{k}}\Big(\frac{C_{n-1}^k}{f}\Big)^{\frac{1}{k}}$ and C is a constant under control.

In view of condition (3.13), and by (3.30), we get $(1+v_1^2)^{\frac{1}{2}}\delta_f\leq C$. The proof is complete.

Since (1.7) implies (3.13), Theorem 1.3 is a consequence of the following proposition.

Proposition 3.2 *Suppose $n\geq 2,\ 1\leq k\leq n$, and suppose f is a positive smooth function on $\mathbb{S}^n$. If $k<n$, we assume f satisfies condition* (3.13). *Then there exists a unique constant $\gamma>0$ satisfying* (1.8) *and a smooth k-admissible hypersurface M satisfying equation* (1.9). *The solution is unique up to homothetic dilations. Furthermore, for $1\leq k<n$, if in addition $|X|f\big(\frac{X}{|X|}\big)^{-\frac{1}{k}}$ is convex in $\mathbb{R}^{n+1}\setminus\{0\}$, then M is strictly convex.*

Proof of Proposition 3.2 First we deal with the existence of solution and γ. For all $r\in\mathbf{Z}^+$, from Proposition 3.1, we let $\rho_r=|X_r|$ be the unique solution of equation (3.2) with $p=k+\frac{1}{r}$. We rescale ρ, letting $\tilde\rho_r=\frac{\rho_r}{l_r}$ with $l_r=\min\rho_r$. Now $\tilde\rho_r$ satisfies

$$S_k(\tilde k_1,\tilde k_2,\cdots,\tilde k_n)(x)=\tilde\rho^{-k-\frac{1}{r}}\tilde f_r(x)\quad\text{on }\mathbb{S}^n,$$

where $\tilde{f} = {l_r}^{-\frac{1}{r}} f$. From (3.30), $\frac{C_n^k \min_{\mathbb{S}^n} f}{\max_{\mathbb{S}^n} f} \le \tilde{f} \le \frac{C_n^k \max_{\mathbb{S}^n} f}{\min_{\mathbb{S}^n} f}$.

If f satisfies the conditions in the proposition, by Lemmas 3.1 and 3.2, there exists a positive constant C independent of r, such that $1 \le \tilde{\rho}_r$ and $\|\tilde{\rho}_r\|_{C^2} \le C$. The Evans-Krylov theorem gives $\|\tilde{\rho}_r\|_{C^{l,\alpha}} \le C_{l,\alpha}$ with $C_{l,\alpha}$ $(l \ge 2,\ 0 < \alpha < 1)$ independent of r. So, there is a subsequence $r_j \to \infty$, such that $\tilde{\rho}_{r_j} \to \rho$ in $C^{l,\alpha}(\mathbb{S}^n)$, and ${l_{r_j}}^{-\frac{1}{r_j}} \to \gamma$ for some positive constant γ. (3.30) implies (1.8) and the radial graph of ρ satisfies (1.9). The higher regularity of ρ follows from the standard elliptic theory.

We now turn to the uniqueness. Let $M(\rho) = S_k(\kappa_1, \kappa_2, \cdots, \kappa_n)\rho^k$ and suppose $\exists\, \gamma_0,\ \gamma_1$, $\rho_0 > 0$ and $\rho_1 > 0$ satisfying (1.9) respectively. We may assume $\gamma_0 \ge \gamma_1$, so we have

$$M(\rho_0) - M(\rho_1) = (\gamma_0 - \gamma_1) f \ge 0.$$

Since M is invariant under scaling, we may assume $\rho_0 \le \rho_1$, and $\rho_0(x_o) = \rho_1(x_o)$ at some point $x_o \in \mathbb{S}^n$. Let $\rho_t = t\rho_1 + (1-t)\rho_0$. Since $\rho_t = \rho_0$ and $\nabla \rho_t = \nabla \rho_0$ at x_o, the first fundamental forms of ρ_t are same at x_o for all $0 \le t \le 1$. Therefore ρ_t is k-admissible for all $0 \le t \le 1$ at x_o. By the continuity of the second derivatives, there is a neighborhood of x_o such that ρ_t is k-admissible for all $0 \le t \le 1$. We have, in the neighborhood of x_o,

$$\begin{aligned} M(\rho_1) - M(\rho_0) &= \int_0^1 \frac{\partial}{\partial t} M_t dt \\ &= \sum_{i,j=1}^n b^{ij}(\rho_1, \rho_0)(\rho_1 - \rho_0)_{ij} + \sum_{i=1}^n c^i(\rho_1, \rho_0)(\rho_1 - \rho_0)_i + d(\rho_1, \rho_0)(\rho_1 - \rho_0). \end{aligned}$$

By the Strong Maximum Principle, $\rho_1 = \rho_0$ everywhere and $\gamma_1 = \gamma_0$.

Finally we discuss the convexity. It is easy to check that the convexity of $|X| f^{-\frac{1}{k}}\big(\frac{X}{|X|}\big)$ implies the convexity of $|X|^{\frac{p}{k}} f^{-\frac{1}{k}}\big(\frac{X}{|X|}\big)$ for any $p \ge k$. When $1 \le k \le n-1$, from Proposition 3.1, we know that the solution $M = \{\rho(x)x : \mathbb{S}^n \to \mathbb{R}^{n+1}\}$ is convex if f satisfies the convex condition in Theorem 1.3. The strict convexity follows from Theorem 1.2.

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