The Christofel-Minkowski Problem III: Existence and Convexity of Admissible Solutions

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1 Introduction

This paper is a sequel to [15] on geometric fully nonlinear partial differential equations associated to the Christoffel-Minkowski problem. In [15], we considered the existence of *convex* solutions of the following equation:

$$S_k(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where S_k is the k^{th} elementary symmetric function and u_{ij} the second-order covariant derivatives of u with respect to orthonormal frames on $\mathbb{S}^n$, and where a function $u \in C^2(\mathbb{S}^n)$ is called *convex* if

$$(u_{ij} + u\delta_{ij}) > 0 \quad \text{on } \mathbb{S}^n. \tag{1.2}$$

It is known that (e.g., see [11, 24]) $\forall v \in C^2(\mathbb{S}^n)$,

$$\int_{\mathbb{S}^n} x_m S_k(v_{ij}(x) + v(x)\delta_{ij})dx = 0 \quad \forall m = 1, \dots, n+1.$$

A necessary condition for equation (1.1) to have a solution is

$$\int_{\mathbb{S}^n} x_i \varphi(x)dx = 0 \quad \forall i = 1, \dots, n+1. \tag{1.3}$$

Condition (1.3) is also sufficient for the Minkowski problem, which corresponds to $k = n$ in equation (1.1). In this case, equation (1.1) is the Monge-Ampère equation corresponding to the Minkowski problem

$$\det(u_{ij} + u\delta_{ij}) = \varphi \quad \text{on } \mathbb{S}^n. \tag{1.4}$$

The Minkowski problem has been settled completely by Nirenberg [21] and Pogorelov [22] for dimension 2 and by Cheng and Yau [6] and Pogorelov [24] for general dimensions. From their work, for any positive function $\varphi \in C^2(\mathbb{S}^n)$ satisfying

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the necessary condition (1.3), the Monge-Ampère equation (1.4) always has a *convex* solution.

At the other end $k = 1$, equation (1.1) corresponds to the Christoffel problem and has the following simple form:

$$\Delta u + nu = \varphi \quad \text{on } \mathbb{S}^n, \tag{1.5}$$

where Δ is the Beltrami-Laplace operator of the round unit sphere. The operator $L = \Delta + n$ is linear and self-adjoint. From linear elliptic theory, equation (1.1) is solvable if and only if φ is orthogonal to the kernel of the operator $L = \Delta + n$. Since n is the second eigenvalue of the operator $-\Delta$, the kernel of L is exactly $\text{span}\{x_1, \dots, x_{n+1}\}$. Therefore, condition (1.3) is necessary and sufficient for the solvability of equation (1.5). In general, a solution to equation (1.5) is not necessarily convex (this is the point Christoffel overlooked when he made the premature claim in [8]). Alexandrov [1] constructed some positive analytic function φ satisfying (1.3) such that equation (1.1) has no convex solution. The convexity of solution u to equation (1.1) is equivalent to a positive lower bound of the eigenvalues of the spherical Hessian $(u_{ij} + u\delta_{ij})$, which in turn are exactly the principal radii of the convex hypersurface with u as its support function. Alexandrov's examples indicate that when $k < n$, there exists no such bound. Equation (1.5) is linear on $\mathbb{S}^n$, a necessary and sufficient condition for the existence of convex solutions of (1.5) was found by reading off from the explicit construction of the Green function by Firey [9].

For the intermediate cases $1 < k < n$, the situation is much more delicate. Let's first define the admissible solutions for equation (1.1). Let $\mathcal{S}$ be the space consisting all $n \times n$ symmetric matrices. For any symmetric matrix $A \in \mathcal{S}$, $S_k(A)$ is defined to be $S_k(\lambda)$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ are the eigenvalues of A. Γ_k defined in [10] can be written equivalently as the connected cone in $\mathcal{S}$ containing the identity matrix determined by

$$\Gamma_k = \{A \in \mathcal{S} : S_1(A) > 0, \dots, S_k(A) > 0\}. \tag{1.6}$$

By the works of [4, 17, 19], *k-convex* functions are the natural class of functions where equation (1.1) is elliptic.

DEFINITION 1.1 For $1 \le k \le n$, let Γ_k be as in (1.6). If $u \in C^2(\mathbb{S}^n)$, we say u is *k-convex* if $W(x) = \{u_{ij}(x) + u(x)\delta_{ij}\}$ is in Γ_k for each $x \in \mathbb{S}^n$. We observe that u is *convex* on $\mathbb{S}^n$ if u is n-convex. Furthermore, u is called an *admissible* solution of (1.1) if u is k-convex and satisfies (1.1).

When $k \ne n$, the class of admissible solutions of equation (1.1) is much larger (e.g., [4]). We treated the intermediate Christoffel-Minkowski problem in [15] as a convexity problem for fully nonlinear equations and a sufficient condition was found there. The convexity is a fundamental problem in the theory of nonlinear elliptic partial differential equations. Equation (1.1) is a special form of some general, fully nonlinear equations related to Weingarten curvature functions. One particular class of equations is the following:

$$\frac{S_k(u_{ij}+\delta_{ij}u)}{S_l(u_{ij}+\delta_{ij}u)} = \varphi \quad \text{on } \mathbb{S}^n, \tag{1.7}$$

where $0 \le l < k \le n$. It is known that admissible solutions of equation (1.7) are exactly k-convex functions. In the special case $k = n$, the equation is related to the problem of prescribing j^{th} Weingarten curvature $W_j(\kappa)$ of a convex hypersurface $M \subset \mathbb{R}^{n+1}$ proposed by Alexandrov [2] and Chern [7], where $W_j(\kappa) = S_j(\kappa_1, \dots, \kappa_n)$ and $\kappa = (\kappa_1, \dots, \kappa_n)$ are the principal curvatures of M. When $k = n$, admissible solutions of (1.7) are exactly convex functions; the problem was addressed in [11]. For general $0 \le l < k \le n$, equation (1.7) corresponds to the problem of prescribing the quotient of Weingarten curvatures on outer normals of a convex hypersurface in $\mathbb{R}^{n+1}$. In this case, admissible solutions of (1.7) are not necessarily convex. As a first result of this paper, we establish a convexity criterion for equation (1.7).

THEOREM 1.2 (Full Rank Theorem) *Suppose u is an admissible solution of* (1.7) *such that $W = (u_{ij} + \delta_{ij}u)$ is semidefinite on $\mathbb{S}^n$. If $\{(\varphi^{-1/(k-l)})_{ij} + \varphi^{-1/(k-l)}\delta_{ij}\}$ is semipositive definite everywhere on $\mathbb{S}^n$, then W is positive definite on $\mathbb{S}^n$.*

Another objective of this paper is regarding the existence of admissible solutions of equation (1.1). We note that when $k = 1$, equation (1.1) is exactly (1.5). (1.3) is the necessary and sufficient condition for (1.1) to be solvable. When $k = n$, admissible solutions of (1.1) are exactly convex functions. The existence of admissible solutions follows from the works of Nirenberg, Cheng and Yau, and Pogorelov. Though a sufficient condition for the existence of a convex solution of equation (1.1) was given in [15], the general existence of an admissible solution to equation (1.1) was left open. Here we prove that condition (1.3) is also the necessary and sufficient condition for the existence of admissible solutions of equation (1.1).

THEOREM 1.3 (Existence) *Let $\varphi(x) \in C^{1,1}(\mathbb{S}^n)$ be a positive function, and suppose φ satisfies* (1.3). *Then equation* (1.1) *has a solution. More precisely, there exist a constant C depending only on n, α,* $\min\varphi$, *and* $\|\varphi\|_{C^{1,1}}(\mathbb{S}^n)$ *and a* $C^{3,\alpha}$ $(\forall\alpha : 0 < \alpha < 1)$ *k-convex solution u of* (1.1) *such that*

$$\|u\|_{C^{3,\alpha}}(\mathbb{S}^n) \le C. \tag{1.8}$$

Furthermore, if $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$ ($l \ge 2, \gamma > 0$), then u is $C^{2+l,\gamma}$. If φ is analytic, u is analytic.

Alexandrov [2] and Pogorelov [23] studied some general forms of fully nonlinear geometric equations on $\mathbb{S}^n$ under various structural conditions. They obtained some regularity estimates under the assumption that the solution is convex. We will extend their regularity estimates for admissible solutions in Proposition 2.7. We will also prove a uniqueness result for admissible solutions in Proposition 3.1.

The uniqueness result, together with the regularity estimates, enable us to establish existence of admissible solutions under general structural conditions in Section 3 via a degree argument. One consequence of our existence results in Section 3 together with Theorem 1.2 is the following:

THEOREM 1.4 *Suppose there is an automorphic group $\mathcal{G}$ of $\mathbb{S}^n$ that has no fixed points. Suppose φ is a smooth positive $\mathcal{G}$-invariant function on $\mathbb{S}^n$ and the spherical Hessian $\{(\varphi^{\frac{-1}{k-l}})_{ij} + \varphi^{\frac{-1}{k-l}}\delta_{ij}\}$ is semi-positive definite everywhere, then equation* (1.7) *has a $\mathcal{G}$-invariant convex, smooth solution u. In particular, for such φ, there is a strictly convex, smooth hypersurface $M \subset \mathbb{R}^{n+1}$ such that the quotient of Weingarten curvatures $W_{n-l}(\kappa)/W_{n-k}(\kappa)$ on the outer normals of M is exactly φ.*

We remark that the reason to impose a group-invariant condition in Theorem 1.4 is the same as in [11], since for $l \neq 0$, equation (1.7) does not have a variational structure. For this reason, it is found in [11] that condition (1.3) is neither sufficient nor necessary for the existence of admissible solutions of (1.7).

The organization of the paper is as follows: In the next section, we will establish a priori estimates for general, fully nonlinear equations on $\mathbb{S}^n$ under some structure conditions. In Section 3 we prove a general existence result containing Theorem 1.3 as a special case. Theorem 1.4 will also be proved there. Finally, we prove Theorem 1.2 in Section 4.

2 Structural Conditions and Regularity Estimates

We establish the a priori estimates for admissible solutions of equation (1.1) in this section. We note that for any solution $u(x)$ of (1.1), $u(x) + l(x)$ is also a solution of the equation for any linear function $l(x) = \sum_{i=1}^{n+1} a_i x_i$. We will confine ourselves to solutions satisfying the following orthogonal condition:

$$\int_{\mathbb{S}^n} x_i u \, dx = 0 \quad \forall i = 1, \dots, n+1. \tag{2.1}$$

When u is convex, it is a support function of some convex body Ω. Condition (2.1) implies that the Steiner point of Ω coincides with the origin.

If $k = 1$, equation (1.1) is a linear, a priori estimates for solution u satisfies (2.1) follows from standard linear elliptic theory. When $k = n$, equation (1.1) is the Monge-Ampère equation, the admissible solutions are exactly the convex functions; the a priori estimates were obtained in [6, 21, 24]. For the intermediate case $1 < k < n$, the a priori estimates for convex solutions of equation (1.1) were proved in [15]. Here we establish a priori estimates for admissible solutions. We note that equation (1.1) will be uniformly elliptic once C^2 estimates are established for u (see [4]). By the Evans-Krylov theorem and the Schauder theory, one may obtain higher-derivative estimates for u. Therefore, we only need to get C^2 estimates for u.

In fact, the a priori estimates we will prove are valid for a general class of fully nonlinear elliptic equations on $\mathbb{S}^n$. We consider the following equation:

$$Q(u_{ij} + u\delta_{ij}) = \tilde{\varphi} \quad \text{on } \mathbb{S}^n. \tag{2.2}$$

Following [4], we specify some structure conditions so that (2.2) is elliptic. Let Γ be an open, symmetric subset in $\mathbb{R}^n$; that is, for $\lambda \in \Gamma$ and any permutation σ, $\sigma \cdot \lambda = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) \in \Gamma$. We assume

$$\Gamma \text{ is a convex cone and } \Gamma \subseteq \Gamma_1, \tag{2.3}$$

where $\Gamma_1 = \{\lambda \mid \sum_{j=1}^n \lambda_j > 0\}$. It is clear that $(1, \dots, 1) \in \Gamma$. We assume that Q is a $C^{2,\gamma}$ function defined in $\Gamma \subseteq \Gamma_1$ for some $0 < \gamma < 1$ and satisfies the following conditions in Γ:

$$\frac{\partial Q}{\partial \lambda_i}(\lambda) > 0 \quad \text{for } i = 1, \dots, n \text{ and } \lambda \in \Gamma, \tag{2.4}$$

$$Q \text{ is concave in } \Gamma, \tag{2.5}$$

and for $M > 0$, there is $\delta_M > 0$ such that for $\lambda \in \Gamma$ with $Q(\lambda) \le M$,

$$\sum_{i=1}^n \frac{\partial Q}{\partial \lambda_i}(\lambda) \ge \delta_M. \tag{2.6}$$

Set

$$\tilde{\Gamma} = \{W \mid W \text{ is a symmetric matrix whose eigenvalues } \lambda = (\lambda_1, \dots, \lambda_n) \in \Gamma\}.$$

We note that since $\Gamma \subset \Gamma_1$, for $W \in \tilde{\Gamma}$, the eigenvalues λ_i of W satisfy $|\lambda_i| \le (n-1)\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of W. From a result in Section 3 in [4], the fact that Q is concave in Γ implies Q is concave in $\tilde{\Gamma}$ and condition (2.4) implies $(\partial Q/\partial W_{ij})$ is positive definite for all $W = (W_{ij}) \in \tilde{\Gamma}$. We will simply write Γ for $\tilde{\Gamma}$ in the rest of the paper.

Remark 2.1. We note that $S_k^{1/k}$ and the general quotient operator $(S_k/S_l)^{1/(k-l)}$, $0 \le l < k \le n$, satisfy the structure conditions (2.3)–(2.6) with $\Gamma = \Gamma_k$, and one may take $\delta_M = 1$ for all $M > 0$.

DEFINITION 2.2 We say a function $u \in C^2(\mathbb{S}^n)$ is Γ-*admissible* if $W(x) = (u_{ij}(x) + \delta_{ij}u(x)) \in \Gamma$ for all $x \in \mathbb{S}^n$. If u is Γ-admissible and satisfies equation (2.2), we call u an *admissible solution* of (2.2).

Condition (2.4) is a monotonicity condition that is natural for the ellipticity of equation (2.2) as we will see that the concavity condition (2.5) is also crucial for C^2 and $C^{2,\alpha}$ estimates. Condition (2.6) appears artificial, but it follows from some natural conditions on Q. For example, in order for equation (2.2) to have an admissible solution for some $\tilde{\varphi}$ with $\sup \tilde{\varphi} = M$, there must exist $W \in \Gamma$ such that $Q(W) = M$. By conditions (2.3)–(2.5), we have

$$Q(t_0 I) \ge M \quad \text{for some } t_0 > 0, \tag{2.7}$$

where I is the identity matrix.

LEMMA 2.3 *Suppose that Q satisfies* (2.3)–(2.5)*. Set $Q^{ij}(W) = \partial Q(W)/\partial W_{ij}$ for $W = (W_{ij}) \in \Gamma$.*

(i) *If Q satisfies* (2.7) *and*

$$\overline{\lim}_{t\to+\infty} Q(tW) > -\infty \quad \text{for all } W \in \Gamma, \tag{2.8}$$

then there is a $\delta_M > 0$ depending on Q and t_0 in (2.7) *such that* (2.6) *is true.*

(ii) *If Q satisfies*

$$\overline{\lim}_{t\to+\infty} Q(tW_1 + W_2) > -\infty \quad \text{for all } W_1, W_2 \in \Gamma, \tag{2.9}$$

then $\sum_{i,j} Q^{ij}(W)W_{ij} > 0$ for all $W \in \Gamma$.

We also refer the reader to [14] for a related treatment of (2.3)–(2.5) and (2.7).

PROOF: By the concavity condition (2.5),

$$Q(tI) \le Q(W) + \sum_{i,j} Q^{ij}(W)(t\delta_{ij} - W_{ij}). \tag{2.10}$$

The concavity condition (2.5) and (2.8) implies that $\frac{d}{dt}Q(tW) \ge 0$ for all $W \in \Gamma$. That is, $\sum_{i,j} Q^{ij}(W)W_{ij} \ge 0$ for all $W \in \Gamma$. By the monotonicity condition (2.4), there exists $\epsilon > 0$ such that $Q(2t_0 I) \ge M + \epsilon$. Since $Q(W) \le M$, (2.6) follows from (2.10) by letting $t = 2t_0$.

We now prove the second statement in the lemma. Since Γ is open, for each $W \in \Gamma$ there is $\delta > 0$ such that $\tilde{W} = W - \delta I \in \Gamma$. In turn, $t\tilde{W} + \delta I \in \Gamma$ for all $t > 0$. Set $g(t) = Q(t\tilde{W} + \delta I)$. By the concavity of Q and condition (2.9), we have $g'(1) \ge 0$; that is, $\sum_{i,j} Q^{ij}(W)\tilde{W}_{ij} \ge 0$. In turn, by condition (2.4) we get $\sum_{i,j} Q^{ij}(W)W_{ij} \ge \delta \sum_i Q^{ii}(W) > 0$. □

We now turn our attention to a priori estimates of solutions to equation (2.2). In [5], Caffarelli, Nirenberg, and Spruck treated similar equations related to the prescribing Weingarten curvature functions of hypersurfaces in $\mathbb{R}^n$. The main difference here is there is no barrier assumption for equation (2.2); we need to work out the C^0 estimate. We follow the arguments in [11] to obtain an upper bound on the largest eigenvalue of the matrix $(u_{ij} + \delta_{ij}u)$ first. We then come back to deal with the C^0 bound.

PROPOSITION 2.4 *Suppose Q satisfies the structural conditions* (2.3)–(2.6) *and $u \in C^4(\mathbb{S}^n)$ is an admissible solution of equation* (2.2)*; then there is $C > 0$ depending only on $Q(I)$ in* (2.7)*, δ in* (2.6)*, and $\|\varphi\|_{C^2}$ such that*

$$0 < \lambda_{\max} \le C, \tag{2.11}$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $(u_{ij} + \delta_{ij}u)$. In particular, for any eigenvalue $\lambda_i(x)$ of $(u_{ij}(x) + \delta_{ij}u(x))$,

$$|\lambda_i(x)| \le (n-1)C \quad \forall x \in \mathbb{S}^n. \tag{2.12}$$

PROOF: When $Q = S_k^{1/k}$ and u is convex, this is the Pogorelov-type estimate (e.g., [24]). Here we will deal with general admissible solutions of Q under the structural conditions. It seems that the moving-frames method is more appropriate for equation (2.2) on $\mathbb{S}^n$. We set $W = \{u_{ij} + \delta_{ij} u\}$.

(2.12) follows from (2.11) and the fact that $\Gamma \subset \Gamma_1$. Also, the positivity of $\lambda_{\max}$ follows from the assumption that $\Gamma \subset \Gamma_1$. We need to estimate the upper bound of $\lambda_{\max}$. Assume the maximum value of $\lambda_{\max}$ is attained at a point $x_0 \in \mathbb{S}^n$ and in the direction e_1, so we can take $\lambda_{\max} = W_{11}$ at x_0. We choose an orthonormal local frame $e_1, \dots, e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal, so W is also diagonal at x_0.

For the standard metric on $\mathbb{S}^n$, we have the following commutator identity:

$$W_{11ii} = W_{ii11} - W_{ii} + W_{11}.$$

By the assumption, (Q^{ij}) is positive definite. Since $W_{11ii} \le 0$ at x_0, it follows that at this point

$$0 \ge Q^{ii} W_{11ii} = Q^{ii} W_{ii11} - Q^{ii} W_{ii} + W_{11} Q^{ii}. \tag{2.13}$$

By concavity condition (2.5),

$$\begin{aligned} \sum_i Q^{ii}(W) W_{ii} &\le \sum_i Q^{ii}(W) + Q(W) - Q(I) \\ &= \sum_i Q^{ii}(W) + \tilde{\varphi} - Q(I). \end{aligned} \tag{2.14}$$

We differentiate equation (2.2) twice in the e_1-direction and obtain

$$\begin{aligned} Q^{ij} W_{ijk1} &= \nabla_1 \tilde{\varphi}, \\ Q^{ij,rs} W_{ij1} W_{rs1} + Q^{ij} W_{ij11} &= \tilde{\varphi}_{11}. \end{aligned}$$

By the concavity of Q, at x_0 we have

$$Q^{ii} W_{ii11} \ge \tilde{\varphi}_{11}. \tag{2.15}$$

Combining (2.14), (2.15), and (2.13), we see that

$$0 \ge \tilde{\varphi}_{11} - \sum_i Q^{ii} - \tilde{\varphi} + W_{11} \sum_{i=1}^n Q^{ii} + Q(I).$$

By assumption, $\tilde{\varphi} \le M$ for some $M > 0$. From condition (2.6), $\sum_{i=1}^n Q^{ii} \ge \delta_M > 0$. It follows that $W_{11} \le C$. □

COROLLARY 2.5 *If* $u \in C^4(\mathbb{S}^n)$ *is an admissible solution of equation* (1.1) (*so* $W(x) = (u_{ij}(x) + u(x)\delta_{ij}) \in \Gamma_k \ \forall x \in \mathbb{S}^n$), *then* $0 < \max_{x \in \mathbb{S}^n} \lambda_{\max}(x) \le C$.

In order to obtain a C^2 bound, we need a C^0 bound for u. In the case of the Minkowski problem ($k = n$), such a crucial C^0 bound was established by Cheng and Yau [6] and for general k with the convexity assumption in [15]. The arguments rely on the convexity assumption. Here we use the a priori bounds in Proposition 2.4 to get a C^0 bound for general admissible solutions of equation (2.2). A similar argument was also used in [11].

LEMMA 2.6 *For any* Γ*-admissible function* u*, there is a constant* C *depending only on* n*,* $\max_{x\in\mathbb{S}^n}\lambda_{\max}(x)$*, and* $\max_{\mathbb{S}^n}|u|$ *such that*

$$\|u\|_{C^2}\le C. \tag{2.16}$$

PROOF: The bound on the second derivatives follows directly from the fact that $W(x)=(u_{ij}(x)+\delta_{ij}u(x))\in\Gamma\subset\Gamma_1$. The bound on the first derivatives follows from interpolation. □

Now we establish the C^0 estimate. The proof is based on a rescaling argument.

PROPOSITION 2.7 *Suppose* Q *satisfies structure conditions* (2.3)–(2.6)*. If* u *is an admissible solution of equation* (2.2) *and* u *satisfies* (2.1)*, then there exists a positive constant* C *depending only on* n*,* k*,* $\|\tilde\varphi\|_{C^2}$*, and* Q *such that*

$$\|u\|_{C^2}\le C. \tag{2.17}$$

PROOF: We need only get a bound on $\|u\|_{C^0}$. Suppose there is no such bound; then there exist u^l, $l=1,2,\dots$, satisfying (2.1), a constant $\tilde C$ independent of l, and $Q(W^l)=\tilde\varphi^l$ (where $W^l=(u^l_{ij}+\delta_{ij}u^l)$), with $\tilde\varphi^l$ satisfying

$$\|\tilde\varphi^l\|_{C^2}\le\tilde C,\quad \sup\tilde\varphi\le 1,\ \ \|u^l\|_{L^\infty}\ge l.$$

Let $v^l=u^l/\|u^l\|_{L^\infty}$; then

$$\|v^l\|_{L^\infty}=1. \tag{2.18}$$

By Proposition 2.4, we have for any eigenvalue $\lambda_i(W^l(x))$ of $W^l(x)$,

$$|\lambda_i(W^l(x))|\le(n-1)\lambda_{\max}(W^l)\le C, \tag{2.19}$$

where $\lambda_{\max}(W^l)$ is the maximum of the largest eigenvalues of W^l on $\mathbb{S}^n$ and the constant C is independent of l. Let $\tilde W^l=(v^l_{ij}+\delta_{ij}v^l)$; from (2.19) v^l satisfies the following estimates:

$$|\lambda_i(\tilde W^l(x))|\le(n-1)\lambda_{\max}(\tilde W^l)\le\frac{C}{\|u^l\|_{L^\infty}}\longrightarrow 0. \tag{2.20}$$

In particular, $\Delta v^l+nv^l\to 0$.

On the other hand, by Lemma 2.6, (2.18), and (2.20), we have

$$\|v^l\|_{C^2}\le C.$$

Hence there exists a subsequence $\{v^{l_i}\}$ and a function $v\in C^{1,\alpha}(\mathbb{S}^n)$ satisfying (2.1) such that

$$v^{l_i}\longrightarrow v\quad\text{in } C^{1,\alpha}(\mathbb{S}^n)\text{ with }\|v\|_{L^\infty}=1. \tag{2.21}$$

In the distribution sense we have

$$\Delta v+nv=0\quad\text{on }\mathbb{S}^n.$$

By linear elliptic theory, v is in fact smooth. Since v satisfies (2.1), we conclude that $v\equiv 0$ on $\mathbb{S}^n$. This is a contradiction to (2.21). □

The higher regularity would follow from the Evans-Krylov theorem and the Schauder theory if we can ensure uniform ellipticity for equation (2.2). That can be guaranteed by the following condition:

$$\overline{\lim}_{W\to\partial\Gamma} Q(W) = 0. \tag{2.22}$$

THEOREM 2.8 *Suppose Q satisfies the structure conditions* (2.3)–(2.6) *and condition* (2.22)*, and $\tilde{\varphi} > 0$ on $\mathbb{S}^n$; then for each $0 < \alpha < 1$, there exists a constant C depending only on n, α, $\min\tilde{\varphi}$, $\|\tilde{\varphi}\|_{C^{1,1}}(\mathbb{S}^n)$, and Q such that*

$$\|u\|_{C^{3,\alpha}}(\mathbb{S}^n) \le C \tag{2.23}$$

for all admissible solutions u of (2.2) *satisfying* (2.1)*. If in addition $Q \in C^l$ for some $l \ge 2$, then there exists a constant C depending only on n, l, α, $\min\tilde{\varphi}$, $\|\tilde{\varphi}\|_{C^{l,1}}(\mathbb{S}^n)$, and Q such that*

$$\|u\|_{C^{l+1,\alpha}}(\mathbb{S}^n) \le C. \tag{2.24}$$

In particular, estimate (2.24) *is true for any admissible solution of* (1.1) *and* (2.1) *with $\tilde{\varphi} = \varphi^{1/k}$.*

PROOF: We verify that equation (2.2) is uniformly elliptic. By Proposition 2.7 and condition (2.22), the set $\{W(x) \in \Gamma \mid Q(W(x)) = \tilde{\varphi}(x)\ \forall x \in \mathbb{S}^n\}$ is compact in Γ. Since $Q \in C^1$, equation (2.2) is uniformly elliptic by condition (2.4). $\square$

3 Existence via Degree Theory

The main object of this section is to establish an existence result for equation (1.1). With the a priori estimates established in the previous section, one may wish to apply the continuity method to get the existence. This leads to the study of the linearized operator L of the Hessian operator in (1.1). L is self-adjoint (e.g., [6, 24]). In the cases $k = 1, n$, the kernel of L is exactly the span of the linear coordinate functions $x_1, \dots, x_{n+1}$. By the standard implicit function theorem, L is surjective to some appropriate function space modulus $\operatorname{span}\{x_1, \dots, x_{n+1}\}$. The continuity method yields the existence. For the case $1 < k < n$, we are not able to verify that the kernel of L is $\operatorname{span}\{x_1, \dots, x_{n+1}\}$, though it contains $\operatorname{span}\{x_1, \dots, x_{n+1}\}$.

We will use a degree theory argument for the existence. In fact, the argument applies to equation (2.2). In order to compute the degree, we need a uniqueness result. The following uniqueness result is known when u is a support function of some convex body, e.g., by Alexandrov's moving-planes method. But we need to treat the uniqueness problem for general admissible solutions. If equation (2.2) carries a variational structure, such a uniqueness result can be proved by integral formulas as in [7]. Here we use a simple argument involving a priori estimates to obtain a general uniqueness result in this direction.

PROPOSITION 3.1 *Suppose that Q satisfies condition* (2.4) *and* (2.5). *If u is an admissible solution of the equation*

$$Q(u_{ij}+\delta_{ij}u)=Q(I)\quad \text{on } \mathbb{S}^n, \tag{3.1}$$

then $u=1+\sum_{j=1}^{n+1}a_jx_j$ for some constants $a_1,\dots,a_{n+1}$.

PROOF: By concavity, for $W=(W_{ij})\in\Gamma$,

$$\begin{aligned} Q(I) &\le Q(W)+\sum_{i,j}Q^{ij}(W)(\delta_{ij}-W_{ij})\\ &= Q(W)+\sum_{i}^{n}Q^{ii}(W)-\sum_{i,j}^{n}Q^{ij}(W)W_{ij}. \end{aligned}\tag{3.2}$$

Also, by the symmetry,

$$Q^{11}(I)=\cdots=Q^{nn}(I)=\frac{\sum_{i=1}^n Q^{ii}(I)}{n}.$$

If u is an admissible solution of (3.1), we know $u\in C^2$ by definition. Since $Q\in C^{2,\gamma}$, by the Evans-Krylov theorem and the Schauder theory, $u\in C^{4,\gamma}$. Let $W(x)=(u_{ij}(x)+\delta_{ij}u(x))$ and $H(x)=\operatorname{tr}W(x)=\Delta u(x)+nu(x)$. Since

$$Q^{jj}(I)=\frac{\sum_{i=1}^n Q^{ii}(I)}{n}\quad\forall j,$$

by concavity, for all $x\in\mathbb{S}^n$,

$$\begin{aligned} Q(W(x)) &\le Q(I)+\sum_{i,j}Q^{ij}(I)(W_{ij}(x)-\delta_{ij})\\ &= Q(I)+\frac{\sum_{i=1}^n Q^{ii}(I)}{n}H(x)-\sum_{i=1}^{n}Q^{ii}(I). \end{aligned}$$

Because $Q(W(x))=Q(I)$ and $\sum_{i=1}^n Q^{ii}(I)>0$, we get

$$H(x)\ge n\quad\forall x\in\mathbb{S}^n. \tag{3.3}$$

We want to show $H(x)\le n$ for all $x\in\mathbb{S}^n$. Suppose that the maximum value of $H(x)$ is attained at a point $x_0\in\mathbb{S}^n$. We choose an orthonormal local frame $e_1,\dots,e_n$ near x_0 such that $u_{ij}(x_0)$ is diagonal, so $W=\{u_{ij}+\delta_{ij}u\}$ is also diagonal at x_0. For the standard metric on $\mathbb{S}^n$, we have the following commutator identity:

$$H_{ii}=\Delta W_{ii}-nW_{ii}+H.$$

Since $Q(W(x))=Q(I)$, it follows from (3.2) that

$$\sum_{i=1}^n Q^{ii}(W)\ge\sum_{i=1}^n Q^{ii}(W)W_{ii}.$$

Since $H_{ii} \le 0$ at x_0,

$$0 \ge \sum_{i=1}^n Q^{ii}(W) H_{ii} = \sum_{i=1}^n Q^{ii}(W)\Delta W_{ii} - n\sum_{i=1}^n Q^{ii}(W) W_{ii} + H\sum_{i=1}^n Q^{ii}(W)$$

$$\ge \sum_{i=1}^n Q^{ii}(W)\Delta W_{ii} - n\sum_{i=1}^n Q^{ii}(W) + H\sum_{i=1}^n Q^{ii}(W). \tag{3.4}$$

Applying Δ to $Q(W) = Q(I)$ and by the concavity of Q, we obtain at x_0

$$Q^{ii}(W)\Delta W_{ii} \ge \Delta Q(I) = 0. \tag{3.5}$$

It follows from (3.5) and (3.4) that

$$n\sum_{i=1}^n Q^{ii}(W) \ge H\sum_{i=1}^n Q^{ii}(W).$$

Since $\sum_{i=1}^n Q^{ii}(W) > 0$, we get $n \ge H(x_0)$. Combining (3.3), we conclude that $H(x) = n$ $\forall x \in \mathbb{S}^n$. Therefore, $u - 1 \in \operatorname{span}\{x_1, \dots, x_{n+1}\}$. □

For $\alpha > 0$, $l \ge 0$ an integer, we set

$$\mathcal{A}^{l,\alpha} = \{f \in C^{l,\alpha}(\mathbb{S}^n) : f \text{ satisfying (2.1)}\}. \tag{3.6}$$

For $R > 0$ fixed, let

$$\mathcal{O}_{\mathcal{R}} = \{w \in \mathcal{A}^{l,\alpha} : w \text{ is } \Gamma\text{-admissible and } \|w\|_{C^{l,\alpha}(\mathbb{S}^n)} < R\}. \tag{3.7}$$

In addition to the structural conditions on Q in the previous section, we need some further conditions on Q in (2.2) to ensure a general existence result. We assume that there is a smooth, strictly monotonic positive function F defined in $R_+ = (0,\infty)$ such that $\forall u \in C^2(\mathbb{S}^n)$ with $W = (u_{ij} + u\delta_{ij}) \in \Gamma_k$, Q satisfies the orthogonal condition

$$\int_{\mathbb{S}^n} F(Q(W(x)))x_m = 0 \quad \forall m = 1, \dots, n+1. \tag{3.8}$$

PROPOSITION 3.2 *Suppose Q satisfies the structural conditions* (2.3)–(2.6)*,* (2.22)*, and the orthogonal condition* (3.8)*. Then for any positive $\tilde\varphi \in C^{1,1}(\mathbb{S}^n)$ with $\varphi(x) = F(\tilde\varphi(x))$ satisfying* (2.1)*, equation* (2.2) *has an admissible solution $u \in \mathcal{A}^{3,\alpha}$ $\forall 0 < \alpha < 1$ satisfying*

$$\|u\|_{C^{3,\alpha}(\mathbb{S}^n)} \le C,$$

where C is a constant depending only on Q, α, $\min\varphi$*, and $\|\varphi\|_{C^{1,1}(\mathbb{S}^n)}$. Furthermore, if $Q \in C^{l,\gamma}$ and $\varphi(x) \in C^{l,\gamma}(\mathbb{S}^n)$, $l \ge 2$, $\gamma > 0$, then u is $C^{2+l,\gamma}$.*

PROOF: For each fixed $0 < \tilde\varphi \in C^\infty(\mathbb{S}^n)$ with $\varphi = F(\tilde\varphi)$ satisfying (2.1) and for $0 \le t \le 1$, we define

$$T_t(u) = F(Q(\{u_{ij} + u\delta_{ij}\})) - t\varphi - (1-t)Q(I). \tag{3.9}$$

T_t is a nonlinear differential operator that maps $\mathcal{A}^{l+2,\alpha}$ into $\mathcal{A}^{l,\alpha}$. If R is sufficiently large, $T_t(u) = 0$ has no solution on $\partial\mathcal{O}_{\mathcal{R}}$ by the a priori estimates in Theorem 2.8. Therefore, the degree of T_t is well-defined (e.g., [20]). Since degree is a homotopic invariant,

$$\deg(T_0, \mathcal{O}_{\mathcal{R}}, 0) = \deg(T_1, \mathcal{O}_{\mathcal{R}}, 0).$$

At $t = 0$, by Proposition 3.1, $u = 1$ is the unique solution of (2.2) in $\mathcal{O}_{\mathcal{R}}$. We may compute the degree using the formula

$$\deg(T_0, \mathcal{O}_{\mathcal{R}}, 0) = \sum_{\mu_j > 0} (-1)^{\beta_j},$$

where μ_j are the eigenvalues of the linearized operator of T_0 and β_j is its multiplicity. Since Q is symmetric, it is easy to show that the linearized operator of T_0 at $u = 1$ is

$$L = \nu(\Delta + n)$$

for some constant $\nu > 0$. Because the eigenvalues of the Beltrami-Laplace operator Δ on $\mathbb{S}^n$ are strictly less than $-n$ except for the first two eigenvalues 0 and $-n$, there is only one positive eigenvalue of L with multiplicity 1, namely $\mu = n\nu$. Therefore,

$$\deg(T_1, \mathcal{O}_{\mathcal{R}}, 0) = \deg(T_0, \mathcal{O}_{\mathcal{R}}, 0) = -1.$$

That is, there is an admissible solution of equation (2.2). The regularity and estimates of the solution follow directly from Theorem 2.8. □

We now prove Theorem 1.3.

PROOF: Since $Q(W) = S_k^{1/k}(W)$ satisfies conditions (2.3)–(2.6) and (2.22), Theorem 1.3 follows from the above proposition. The orthogonal condition (3.8) follows from (1.3). □

Remark 3.3. Since the C^2 a priori bound in Proposition 2.7 is independent of the lower bound of $\tilde{\varphi}$ (we note it is used only for the $C^{2,\alpha}$ estimate), Proposition 3.2 can be used to prove the existence of $C^{1,1}$ solutions to equation (2.2) in the degenerate case. To be more precise, if Q satisfies the structural conditions (2.3)–(2.6), (2.22), and the orthogonal condition (3.8), then for any nonnegative $\tilde{\varphi} \in C^{1,1}(\mathbb{S}^n)$ with $\varphi(x) = F(\tilde{\varphi}(x))$ satisfying (2.1), equation (2.2) has a solution $u \in C^{1,1}(\mathbb{S}^n)$.

For equation (1.1), we can do a little better. One can prove that if $\varphi \geq 0$ satisfies (1.3) and $\varphi^{1/(k-1)} \in C^{1,1}$, then equation (1.1) has a $C^{1,1}$ solution (see [12, 13] for similar results for the degenerate Monge-Ampère equation). For this, we need only rework Proposition 2.4. Instead, we estimate $H = \Delta u + nu$. Following along the same line as in the proof of Proposition 2.4, the desired estimate can be obtained using two facts: (1) for $f = \varphi^{1/(k-1)}$, we have $|\nabla f(x)|^2 \leq Cf(x)$ for all $x \in \mathbb{S}^n$, where C depends only on the $C^{1,1}$ norm of f, and (2) for $k > 1$ and $Q = S_k^{1/k}$,

$$\sum_{i=1}^{n} Q^{ii}(W) \geq \frac{1}{k} S_k^{-\frac{1}{k(k-1)}}(W) S_1^{\frac{1}{k-1}}(W)$$

(for a proof, see fact 3.5 on page 1429 in [16]).

The structural conditions (2.3)–(2.6) and (2.22) are satisfied for the quotient operator

$$Q(W) = (\frac{S_k(W)}{S_l(W)})^{\frac{1}{k-l}}$$

with $\Gamma = \Gamma_k$ for any $0 \le l < k$. Also, the unique solution of $Q(W) = 1$ is constant in $\mathcal{A}^{2,\alpha}$ by Proposition 3.1. Unfortunately, the orthogonal condition (3.8) is not valid in general by some simple examples in [11]. Nevertheless, as in [11], we have the following existence result:

PROPOSITION 3.4 *Suppose Q satisfies the structural conditions* (2.3)–(2.6) *and* (2.22). *Assume $\tilde{\varphi} \in C^{l,1}(\mathbb{S}^n)$, $l \ge 1$, is a positive function. Suppose there is an automorphic group $\mathcal{G}$ of $\mathbb{S}^n$ that has no fixed points. If $\tilde{\varphi}$ is invariant under $\mathcal{G}$, i.e., $\tilde{\varphi}(g(x)) = \tilde{\varphi}(x)$ for all $g \in \mathcal{G}$ and $x \in \mathbb{S}^n$, then there exists a $\mathcal{G}$-invariant admissible function $u \in C^{l+2,\alpha}$, $\forall\, 0 < \alpha < 1$, such that u satisfies equation* (2.2). *Moreover, there is a constant C depending only on α,* $\min \tilde{\varphi}$, *and* $\|\tilde{\varphi}\|_{C^{l,1}}(\mathbb{S}^n)$ *such that*

$$\|u\|_{C^{l+1,\alpha}}(\mathbb{S}^n) \le C.$$

In particular, for any positive $\mathcal{G}$-invariant positive $\varphi \in C^{1,1}(\mathbb{S}^n)$, equation (1.7) *has a k-convex $\mathcal{G}$-invariant solution.*

PROOF: We only sketch the main arguments of the proof since any $\mathcal{G}$-invariant function is orthogonal to $\mathrm{span}\{x_1, \dots, x_{n+1}\}$ by [11]. Therefore, $u = 1$ is the unique $\mathcal{G}$-invariant solution of (2.2) by Proposition 3.1. We again use degree theory. This time, we consider $\mathcal{G}$-invariant function spaces

$$\tilde{\mathcal{A}}^{l,\alpha} = \{f \in C^{l,\alpha}(\mathbb{S}^n) : f \text{ is } \mathcal{G}\text{-invariant}\}$$

and

$$\tilde{\mathcal{O}_{\mathcal{R}}} = \{w \text{ is } k\text{-convex}, w \in \tilde{\mathcal{A}}^{l,\alpha} : \|w\|_{C^{l,\alpha}(\mathbb{S}^n)} < R\}.$$

One may compute that the degree of Q is not vanishing as in the proof of Theorem 3.2 (see also [11]). $\square$

We will prove Theorem 1.2 in the next section. Here we will use it together with Proposition 3.4 to prove Theorem 1.4.

PROOF: For $0 \le t \le 1$, we define

$$\varphi_t = (1 - t + t\varphi^{\frac{-1}{k-l}})^{-k+l}.$$

Certainly φ_t is $\mathcal{G}$-invariant and $\{({\varphi_t}^{-1/(k-l)})_{ij} + {\varphi_t}^{-1/(k-l)}\delta_{ij}\}$ is semipositive definite everywhere on $\mathbb{S}^n$. We consider equation

$$\frac{S_k}{S_l}(u^t_{ij} + u^t\delta_{ij}) = \varphi_t. \tag{3.10}$$

Applying degree theory as in the proof of Proposition 3.4, there exists an admissible solution u^t of equation (3.10) for each $0 \le t \le 1$. When $t = 0$, $u^0 = 1$ is the unique solution by Proposition 3.1 and it is convex. By the continuity of the degree argument and Theorem 1.2, u^t is convex for all $0 \leq t \leq 1$. $\square$

4 A Convexity Criterion for Spherical Quotient Equations

Now we turn to the convexity of the solutions of equation (1.7). In order to prove the full rank theorem (Theorem 1.2), as in [15], we need to establish the following deformation lemma for the Hessian quotient equation (1.7). The proof below follows along the lines of the proof in [15] by exploring some special algebraic structural properties of the quotient operator. The proof involves some direct but lengthy computations. In a forthcoming article, we will deal with this type of convexity problem for general elliptic, concave, fully nonlinear equations.

For $W = \{u_{ij} + \delta_{ij} u\}$, we rewrite (1.7) in the form

$$F(W) = \frac{S_k(W)}{S_l(W)} = \varphi \quad \text{on } \mathbb{S}^n \tag{4.1}$$

and let

$$F^{\alpha\beta} := \frac{\partial F}{\partial W_{\alpha\beta}}, \quad F^{\alpha\beta,rs} := \frac{\partial^2 F}{\partial W_{\alpha\beta} \partial W_{rs}}. \tag{4.2}$$

We note that $F^{\alpha\beta}$ is positive definite for $W \in \Gamma_k$.

LEMMA 4.1 (Deformation Lemma) *Let $O \subset \mathbb{S}^n$ be an open subset, and suppose $u \in C^4(O)$ is a solution of* (1.7) *in O and that the matrix $W = \{W_{ij}\}$ is semipositive definite. Suppose further that there is a positive constant $C_0 > 0$ such that for a fixed integer $(n-1) \geq m \geq k$, $S_m(W(x)) \geq C_0$ for all $x \in O$. Let $\phi(x) = S_{m+1}(W(x))$, and let $\tau(x)$ be the largest eigenvalue of $\{-(\varphi^{-1/(k-l)})_{ij}(x) - \delta_{ij}\varphi^{-1/(k-l)}(x)\}$. Then there are constants C_1 and C_2 depending only on $\|u\|_{C^3}$, $\|\varphi\|_{C^{1,1}}$, n, and C_0 such that the following differential inequality holds in O*:

$$\begin{aligned} \sum_{\alpha,\beta}^{n} F^{\alpha\beta}(x)\phi_{\alpha\beta}(x) \leq (k-l)(n-m)\varphi^{\frac{k-l+1}{k-l}}(x) S_m(W(x))\tau(x) \\ + C_1|\nabla\phi(x)| + C_2\phi(x), \end{aligned} \tag{4.3}$$

where the $F^{\alpha\beta}$ are defined by (4.2).

PROOF: The proof will follow mainly the arguments in [15], which in turn were motivated by Caffarelli and Friedman [3] and Korevaar and Lewis [18].

For two functions defined in an open set $O \subset \mathbb{S}^n$, $y \in O$, we say that $h(y) \lesssim k(y)$ provided there exist positive constants c_1 and c_2 such that

$$(h-k)(y) \leq (c_1|\nabla\phi| + c_2\phi)(y). \tag{4.4}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constants c_1 and c_2 depending only on $\|u\|_{C^3}$, $\|\varphi\|_{C^2}$, n, and C_0 (independent of y and O).

Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$. We shall show that

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim (k-l)(n-m)\varphi^{\frac{k-l+1}{k-l}} S_m(W)\tau. \tag{4.5}$$

For any $z \in O$, let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of W at z. Since $S_m(W) \geq C_0 > 0$ and $u \in C^3$, for any $z \in \mathbb{S}^n$, there is a positive constant $C > 0$ depending only on $\|u\|_{C^3}$, $\|\varphi\|_{C^2}$, n, and C_0 such that $\lambda_1 \geq \cdots \geq \lambda_m \geq C$.

Let $G = \{1, \dots, m\}$ and $B = \{m+1, \dots, n\}$ be the "good" and "bad" sets of indices. Define $S_k(W \mid i) = S_k((W \mid i))$ where $(W \mid i)$ means matrix W excluding the i^{th} column and i^{th} row, and $(W \mid ij)$ means matrix W excluding columns i and j and rows i and j. Let $\Lambda_G = (\lambda_1, \dots, \lambda_m)$ be the "good" eigenvalues of W at z; for simplicity of notation, we also write $G = \Lambda_G$ if there is no confusion. In the following, all calculations are taken at the point z using the relation $\lesssim$ with the understanding that the constants in (4.4) are under control.

For each $z \in O$ fixed, we choose a local orthonormal frame $e_1, \dots, e_n$ so that W is diagonal at z, and $W_{ii} = \lambda_i$ $\forall i = 1, \dots, n$. Let

$$S^{ij} = \frac{\partial S_{m+1}(W)}{\partial W_{ij}}, \quad S^{ij,rs} = \frac{\partial^2 S_{m+1}(W)}{\partial W_{ij}\partial W_{rs}}.$$

We note that S^{ij} is diagonal at the point since W is diagonal. Notice that $\phi_\alpha = \sum_{i,j} S^{ij} W_{ij\alpha}$, and we find that (as W is diagonal at z),

$$0 \sim \phi(z) \sim \left(\sum_{i\in B} W_{ii}\right) S_m(G) \sim \sum_{i \in B} W_{ii} \quad (\text{so } W_{ii} \sim 0,\ i \in B). \tag{4.6}$$

This relation yields that, $\forall t$, $1 \leq t \leq m$,

$$\begin{gathered} S_t(W) \sim S_t(G), \quad S_t(W \mid j) \sim \begin{cases} S_t(G \mid j) & \text{if } j \in G, \\ S_t(G) & \text{if } j \in B, \end{cases} \\ S_t(W|ij) \sim \begin{cases} S_t(G \mid ij) & \text{if } i, j \in G, \\ S_t(G \mid j) & \text{if } i \in B,\, j \in G, \\ S_t(G) & \text{if } i, j \in B, i \neq j. \end{cases} \end{gathered} \tag{4.7}$$

Also,

$$0 \sim \phi_\alpha \sim S_m(G) \sum_{i\in B} W_{ii\alpha} \sim \sum_{i \in B} W_{ii\alpha}. \tag{4.8}$$

According to [15],

$$S^{ij} \sim \begin{cases} S_m(G) & \text{if } i = j \in B, \\ 0 & \text{otherwise,} \end{cases} \tag{4.9}$$

$$S^{ij,rs} = \begin{cases} S_{m-1}(W|ir) & \text{if } i=j,\ r=s,\ i\neq r, \\ -S_{m-1}(W|ij) & \text{if } i\neq j,\ r=j,\ s=i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Since $\phi_{\alpha\alpha} = \sum_{i,j}[S^{ij,rs}W_{rs\alpha}W_{ij\alpha} + S^{ij}W_{ij\alpha\alpha}]$, by combining (4.6), (4.8), and (4.10), it follows that for any $\alpha \in \{1,\ldots,n\}$,

$$\begin{aligned} \phi_{\alpha\alpha} &= \sum_{i\neq j} S_{m-1}(W \mid ij)W_{ii\alpha}W_{jj\alpha} - \sum_{i\neq j} S_{m-1}(W \mid ij)W_{ij\alpha}^2 + \sum_i S^{ii}W_{ii\alpha\alpha} \\ &= \Bigg(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}\Bigg) S_{m-1}(W \mid ij)W_{ii\alpha}W_{jj\alpha} \\ &\quad - \Bigg(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in B\\ i\neq j}} + \sum_{\substack{i,j\in G\\ i\neq j}}\Bigg) S_{m-1}(W \mid ij)W_{ij\alpha}^2 + \sum_i S^{ii}W_{ii\alpha\alpha}. \end{aligned} \tag{4.11}$$

From (4.8) and (4.7), we have

$$\sum_{\substack{i\in B\\ j\in G}} S_{m-1}(W \mid ij)W_{ii\alpha}W_{jj\alpha} \sim \Bigg[\sum_{j\in G} S_{m-1}(G \mid j)W_{jj\alpha}\Bigg]\sum_{i\in B} W_{ii\alpha} \sim 0. \tag{4.12}$$

Similarly,

$$\sum_{\substack{i\in G\\ j\in B}} S_{m-1}(W \mid ij)W_{ii\alpha}W_{jj\alpha} \sim 0. \tag{4.13}$$

By (4.8), $\forall i \in B$ fixed and $\forall\alpha$,

$$-W_{ii\alpha} \sim \sum_{\substack{j\in B\\ j\neq i}} W_{jj\alpha}.$$

Then, (4.7) yields

$$\sum_{\substack{i,j\in B\\ i\neq j}} S_{m-1}(W \mid ij)W_{ii\alpha}W_{jj\alpha} \sim -S_{m-1}(G)\sum_{i\in B} W_{ii\alpha}^2, \tag{4.14}$$

and

$$\sum_{j\in G, i\in B} S_{m-1}(W \mid ij)W_{ij\alpha}^2 \sim \sum_{i\in B, j\in G} S_{m-1}(G \mid j)W_{ij\alpha}^2. \tag{4.15}$$

Inserting (4.7) and (4.12)–(4.15) into (4.11), we obtain, as in [15],

$$\phi_{\alpha\alpha} \sim \sum_i S^{ii}W_{ii\alpha\alpha} - 2\sum_{\substack{i\in B\\ j\in G}} S_{m-1}(G \mid j)W_{ij\alpha}^2 - S_{m-1}(G)\sum_{i,j\in B} W_{ij\alpha}^2. \tag{4.16}$$

So we have

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} = \sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \sim S_m(G)\sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}W_{ii\alpha\alpha} - 2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}} S_{m-1}(G \mid j)F^{\alpha\alpha}W_{ij\alpha}^2 - S_{m-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2. \tag{4.17}$$

Since F is homogeneous of order $k-l$, $\sum_\alpha F^{\alpha\alpha}W_{\alpha\alpha} = (k-l)\varphi$. Commuting the covariant derivatives, it follows that

$$\begin{aligned}\sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}W_{ii\alpha\alpha} &= \sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}(W_{\alpha\alpha ii} + W_{ii} - W_{\alpha\alpha})\\ &\sim \sum_{\alpha=1}^{n}\sum_{i\in B} F^{\alpha\alpha}W_{\alpha\alpha ii} - (n-m)(k-l)\varphi. \end{aligned}\tag{4.18}$$

Now we compute $\sum_{\alpha=1}^{n} F^{\alpha\alpha}W_{\alpha\alpha ii}$ for $i \in B$. Differentiating equation (4.1), we have

$$\varphi_i = \sum_{\alpha,\beta} F^{\alpha\beta}W_{\alpha\beta i}, \quad \varphi_{ii} = \sum_{\alpha,\beta,r,s} F^{\alpha\beta,rs}W_{\alpha\beta i}W_{rsi} + \sum_{\alpha,\beta} F^{\alpha\beta}W_{\alpha\beta ii}.$$

So for any $i \in B$, we get

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}W_{\alpha\alpha ii} &= \varphi_{ii} - \sum_{\alpha\neq\beta}\bigg[\frac{S_{k-2}(W \mid \alpha\beta)}{S_l} - 2\frac{S_{k-1}(W \mid \alpha)S_{l-1}(W \mid \beta)}{S_l^2}\\ &\qquad - \frac{S_k S_{l-2}(W \mid \alpha\beta)}{S_l^2} + 2\frac{S_k S_{l-1}(W \mid \alpha)S_{l-1}(W \mid \beta)}{S_l^3}\bigg]W_{\alpha\alpha i}W_{\beta\beta i}\\ &\quad + 2\sum_{\alpha=1}^{n}\left[\frac{S_{k-1}(W \mid \alpha)S_{l-1}(W \mid \alpha)}{S_l^2} - \frac{S_k S_{l-1}^2(W \mid \alpha)}{S_l^3}\right]W_{\alpha\alpha i}^2\\ &\quad + \sum_{\alpha\neq\beta}\left[\frac{S_{k-2}(W \mid \alpha\beta)}{S_l} - \frac{S_k S_{l-2}(W \mid \alpha\beta)}{S_l^2}\right]W_{\alpha\beta i}^2.\end{aligned}\tag{4.19}$$

By (4.6)–(4.10), we regroup it as

$$\begin{aligned}\sum_{\alpha=1}^{n} F^{\alpha\alpha}W_{\alpha\alpha ii} &\sim \varphi_{ii} + \sum_{\alpha\neq\beta}\left[\frac{S_{k-2}(W \mid \alpha\beta)}{S_l(G)} - \frac{S_k S_{l-2}(W \mid \alpha\beta)}{S_l^2(G)}\right]W_{\alpha\beta i}^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\alpha\in B}\left[\frac{S_{k-2}(G)}{S_l(G)}-2\,\frac{S_{k-1}(G)S_{l-1}(G)}{S_l^2(G)}-\frac{S_k(G)S_{l-2}(G)}{S_l^2(G)}\right.\\
&\qquad\left.+2\,\frac{S_k(G)S_{l-1}^2(G)}{S_l^3(G)}\right]W_{\alpha\alpha i}^2\\
&-\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\left[\frac{S_{k-2}(G\mid\alpha\beta)}{S_l(G)}-2\,\frac{S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^2(G)}\right.\\
&-\left.\frac{S_k(G)S_{l-2}(G\mid\alpha\beta)}{S_l^2(G)}+2\,\frac{S_k(G)S_{l-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^3(G)}\right]W_{\alpha\alpha i}W_{\beta\beta i}\\
&+2\sum_{\alpha=1}^{n}\left[\frac{S_{k-1}(W\mid\alpha)S_{l-1}(W\mid\alpha)}{S_l^2(G)}-\frac{S_k(G)S_{l-1}^2(W\mid\alpha)}{S_l^3(G)}\right]W_{\alpha\alpha i}^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{\alpha=1}^{n}F^{\alpha\alpha}\phi_{\alpha\alpha}\\
&\quad\sim S_m(G)\sum_{i\in B}\varphi_{ii}-(n-m)(k-l)S_m(G)\varphi\\
&\qquad+S_m(G)\sum_{i\in B}\left[\sum_{\alpha\in B}\left\{\frac{S_{k-2}(G)}{S_l(G)}-\frac{S_k(G)S_{l-2}(G)}{S_l^2(G)}\right\}W_{\alpha\alpha i}^2\right.\\
&\qquad-\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\left\{\frac{S_{k-2}(G\mid\alpha\beta)}{S_l(G)}-2\frac{S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^2(G)}\right.\\
&\qquad-\left.\frac{S_k(G)S_{l-2}(G\mid\alpha\beta)}{S_l^2(G)}+2\frac{S_k(G)S_{l-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^3(G)}\right\}W_{\alpha\alpha i}W_{\beta\beta i}\\
&\qquad+2\sum_{\alpha\in G}\left\{\frac{S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\alpha)}{S_l^2(G)}-\frac{S_k(G)S_{l-1}^2(G\mid\alpha)}{S_l^3(G)}\right\}W_{\alpha\alpha i}^2\\
&\qquad+\left.\sum_{\alpha\neq\beta}\left\{\frac{S_{k-2}(W\mid\alpha\beta)}{S_l(G)}-\frac{S_k(G)S_{l-2}(W\mid\alpha\beta)}{S_l^2(G)}\right\}W_{\alpha\beta i}^2\right]\\
&\qquad-2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}}S_{m-1}(G\mid j)F^{\alpha\alpha}W_{ij\alpha}^2-S_{m-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B}F^{\alpha\alpha}W_{ij\alpha}^2.
\end{aligned}
\tag{4.20}
$$

We first treat the following three terms in the above formula:

Set

$$
\begin{aligned}
A = {} & S_m(G)\sum_{i\in B}\sum_{\alpha\neq\beta}\frac{S_{k-2}(W\mid\alpha\beta)}{S_l(G)}W^2_{\alpha\beta i}\\
& -2\sum_{\alpha=1}^{n}\sum_{\substack{i\in B\\ j\in G}}S_{m-1}(G\mid j)F^{\alpha\alpha}W^2_{ij\alpha}\\
& - S_m(G)\sum_{i\in B}\sum_{\alpha\neq\beta}\frac{S_k(G)S_{l-2}(W\mid\alpha\beta)}{S_l^2(G)}W^2_{\alpha\beta i}.
\end{aligned}
\tag{4.21}
$$

We want to show that

$$
\begin{aligned}
A \lesssim {} & S_m(G)\sum_{i\in B}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in B}}\left[\frac{S_{k-2}(G)}{S_l(G)}-\frac{S_k(G)S_{l-2}(G)}{S_l^2(G)}\right]W^2_{\alpha\beta i}\\
& -2\sum_{\substack{i\in B\\ \alpha\in G}}\left[\frac{S_{m-1}(G\mid\alpha)S_{k-1}(G\mid\alpha)}{S_l(G)}\right.\\
& \left.\qquad - \frac{S_k(G)S_{m-1}(G\mid\alpha)S_{l-1}(G\mid\alpha)}{S_l^2(G)}\right]W^2_{\alpha\alpha i}.
\end{aligned}
\tag{4.22}
$$

Indeed, since

$$
F^{\alpha\alpha}=\frac{S_{k-1}(W\mid\alpha)}{S_l(G)}-\frac{S_k(W)S_{l-1}(W\mid\alpha)}{S_l^2(G)},
\tag{4.23}
$$

by the definition of A, we have

$$
\begin{aligned}
S_l^2(G)A = {} & S_m(G)\sum_{i\in B}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}+\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in B}}+2\sum_{\substack{\alpha\in B\\ \beta\in G}}\right)\Big[S_l(G)S_{k-2}(W\mid\alpha\beta)\\
& \qquad - S_k(G)S_{l-2}(W\mid\alpha\beta)\Big]W^2_{\alpha\beta i}\\
& -2\sum_{i\in B}\left(\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}+\sum_{\alpha=\beta\in G}+\sum_{\substack{\alpha\in B\\ \beta\in G}}\right)\Big[S_l(G)S_{m-1}(G\mid\beta)S_{k-1}(W\mid\alpha)\\
& \qquad - S_k(G)S_{m-1}(G\mid\beta)S_{l-1}(W\mid\alpha)\Big]W^2_{\alpha\beta i}.
\end{aligned}
$$

Now we should prove that the two terms

$$
\sum_{i\in B}\sum_{\substack{\alpha\in B\\ \beta\in G}} \quad\text{and}\quad \sum_{i\in B}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}
$$

on the right-hand side of the previous equality are nonpositive. More precisely, we prove that

$$\begin{aligned}\sum_{i\in B}\sum_{\substack{\alpha\in B\\ \beta\in G}}\big[&S_m(G)S_l(G)S_{k-2}(G\mid\beta)\\ &-S_m(G)S_k(G)S_{l-2}(G\mid\beta)-S_l(G)S_{m-1}(G\mid\beta)S_{k-1}(G)\\ &+S_k(G)S_{m-1}(G\mid\beta)S_{l-1}(G)\big]W^2_{\alpha\beta i}\lesssim 0.\end{aligned}\tag{4.24}$$

As usual, we only need to prove that for each $i\in B$, the term is nonpositive. For $\beta\in G$, $S_t(G)=S_t(G\mid\beta)+S_{t-1}(G\mid\beta)W_{\beta\beta}$ where $t\in\{l,l-1,k,k-1\}$. By the Newton-MacLaurin inequality, we have

$$\begin{aligned}&W_{\beta\beta}S_l(G)S_{k-2}(G\mid\beta)-W_{\beta\beta}S_k(G)S_{l-2}(G\mid\beta)\\ &\quad -S_l(G)S_{k-1}(G)+S_k(G)S_{l-1}(G)\\ &\qquad =W_{\beta\beta}[S_l(G\mid\beta)+W_{\beta\beta}S_{l-1}(G\mid\beta)]S_{k-2}(G\mid\beta)\\ &\qquad\quad -W_{\beta\beta}[S_k(G\mid\beta)+W_{\beta\beta}S_{k-1}(G\mid\beta)]S_{l-2}(G\mid\beta)\\ &\qquad\quad -[S_l(G\mid\beta)+W_{\beta\beta}S_{l-1}(G\mid\beta)][S_{k-1}(G\mid\beta)+W_{\beta\beta}S_{k-2}(G\mid\beta)]\\ &\qquad\quad +[S_k(G\mid\beta)+W_{\beta\beta}S_{k-1}(G\mid\beta)][S_{l-1}(G\mid\beta)+W_{\beta\beta}S_{l-2}(G\mid\beta)]\\ &\qquad =S_k(G\mid\beta)S_{l-1}(G\mid\beta)-S_l(G\mid\beta)S_{k-1}(G\mid\beta)\\ &\qquad\lesssim 0.\end{aligned}\tag{4.25}$$

We now treat the term

$$\sum_{i\in B}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\cdot$$

We shall prove that it is also nonpositive. In fact, for any $i\in B$, we have

$$\begin{aligned}&\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\big[S_l(G)S_m(G)S_{k-2}(G\mid\alpha\beta)-S_m(G)S_k(G)S_{l-2}(G\mid\alpha\beta)\\ &\qquad -2S_l(G)S_{m-1}(G\mid\beta)S_{k-1}(G\mid\alpha)+2S_k(G)S_{m-1}(G\mid\beta)S_{l-1}(G\mid\alpha)\big]\\ &\quad=\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}S_{m-1}(G\mid\beta)\big[2\{S_k(G)S_{l-1}(G\mid\alpha\beta)-S_l(G)S_{k-1}(G\mid\alpha\beta)\}\\ &\qquad\qquad +W_{\beta\beta}\{S_k(G)S_{l-2}(G\mid\alpha\beta)-S_l(G)S_{k-2}(G\mid\alpha\beta)\}\big]\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}} \Big[2S_{m-1}(G \mid \beta)\{S_k(G \mid \alpha\beta)S_{l-1}(G \mid \alpha\beta) - S_l(G \mid \alpha\beta)S_{k-1}(G \mid \alpha\beta)\} \\
&\qquad + S_m(G)\{S_k(G \mid \alpha\beta)S_{l-2}(G \mid \alpha\beta) - S_l(G \mid \alpha\beta)S_{k-2}(G \mid \alpha\beta)\} \\
&\qquad + S_m(G)(W_{\alpha\alpha} - W_{\beta\beta})\big\{S_{k-2}(G \mid \alpha\beta)S_{l-1}(G \mid \alpha\beta) \\
&\qquad\qquad - S_{l-2}(G \mid \alpha\beta)S_{k-1}(G \mid \alpha\beta)\big\}\Big] \\
&= 2\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}} S_{m-1}(G \mid \beta)\big[S_k(G \mid \alpha\beta)S_{l-1}(G \mid \alpha\beta) - S_l(G \mid \alpha\beta)S_{k-1}(G \mid \alpha\beta)\big] \\
&\quad + S_m(G)\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}} \big[S_k(G \mid \alpha\beta)S_{l-2}(G \mid \alpha\beta) - S_l(G \mid \alpha\beta)S_{k-2}(G \mid \alpha\beta)\big] \\
&\lesssim 0.
\end{aligned}
$$

Here we have again used the Newton-MacLaurin inequality. So (4.22) follows.

Combining (4.20) and (4.22), we have

$$
\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \lesssim S_m(G)\sum_{i\in B}\left[\varphi_{ii} - \frac{k-l+1}{k-l}\frac{\varphi_i^2}{\varphi} - (k-l)\varphi\right] + I_1 + I_2, \tag{4.26}
$$

where

$$
\begin{aligned}
I_1 &= S_m(G)\sum_{i\in B}\left[\sum_{\alpha\in B}\frac{S_{k-2}(G)}{S_l(G)}W_{\alpha\alpha i}^2 - \sum_{\alpha\in B}\frac{S_k(G)S_{l-2}(G)}{S_l^2(G)}W_{\alpha\alpha i}^2\right] \\
&\quad - S_{m-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 \\
&\quad + S_m(G)\sum_{i\in B}\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in B}}\left[\frac{S_{k-2}(G)}{S_l(G)} - \frac{S_k(G)S_{l-2}(G)}{S_l^2(G)}\right]W_{\alpha\beta i}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = \sum_{i\in B}\Bigg\{&\left(1+\frac{1}{k-l}\right)S_m(G)\frac{\varphi_i^2}{\varphi}\\
&- S_m(G)\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\left[\frac{S_{k-2}(G\mid\alpha\beta)}{S_l(G)} - 2\frac{S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^2(G)}\right.\\
&\left. - \frac{S_k(G)S_{l-2}(G\mid\alpha\beta)}{S_l^2(G)} + 2\frac{S_k(G)S_{l-1}(G\mid\alpha)S_{l-1}(G\mid\beta)}{S_l^3(G)}\right]W_{\alpha\alpha i}W_{\beta\beta i}\\
&+ 2S_m(G)\sum_{\alpha\in G}\left[\frac{S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\alpha)}{S_l^2(G)} - \frac{S_k(G)S_{l-1}^2(G\mid\alpha)}{S_l^3(G)}\right]W_{\alpha\alpha i}^2\\
&- 2\sum_{\alpha\in G}\left[\frac{S_{m-1}(G\mid\alpha)S_{k-1}(G\mid\alpha)}{S_l(G)} - \frac{S_k(G)S_{m-1}(G\mid\alpha)S_{l-1}(G\mid\alpha)}{S_l^2(G)}\right]W_{\alpha\alpha i}^2\Bigg\}.
\end{aligned}
$$

CLAIM: $I_1 \lesssim 0$ and $I_2 \lesssim 0$.

If the claim is true, it follows from (4.26) that

$$
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \lesssim S_m(G)\sum_{i\in B}\left[\varphi_{ii} - \frac{k-l+1}{k-l}\frac{\varphi_i^2}{\varphi} - (k-l)\varphi\right]. \tag{4.27}
$$

Thus (4.5) follows from (4.27).

PROOF OF CLAIM: First by induction and the Newton-MacLaurin inequality we have the following inequality:

$$
\begin{aligned}
&S_m(G)S_l(G)S_{k-2}(G) - S_{m-1}(G)S_{k-1}(G)S_l(G)\\
&\qquad - S_m(G)S_k(G)S_{l-2}(G) + S_k(G)S_{l-1}(G)S_{m-1}(G) \le 0.
\end{aligned} \tag{4.28}
$$

On the other hand, it is clear by (4.23) that

$$
\sum_{\alpha=1}^{n}\sum_{i,j\in B} F^{\alpha\alpha}W_{ij\alpha}^2 \ge \sum_{i\in B}\sum_{\alpha,\beta\in B}\left[\frac{S_{k-1}(G)}{S_l(G)} - \frac{S_k(G)S_{l-1}(G)}{S_l(G)^2}\right]W_{\alpha\beta i}^2. \tag{4.29}
$$

If we put (4.29) into I_1 and use (4.28), we obtain

$$
\begin{aligned}
S_l^2(G)I_1 \lesssim \big[&S_l(G)S_l(G)S_{k-2}(G) - S_{m-1}(G)S_{k-1}(G)S_l(G)\\
&- S_m(G)S_k(G)S_{l-2}(G) + S_k(G)S_{m-1}(G)S_{l-1}(G)\big]\sum_{i\in B}\sum_{\alpha,\beta\in B}W_{\alpha\beta i}^2\\
\le 0&.
\end{aligned}
$$

To treat I_2, it follows from (4.7) and (4.8) that

$$
\varphi_i \sim \sum_{\alpha\in G} F^{\alpha\alpha}W_{\alpha\alpha i} \quad \text{for } i\in B. \tag{4.30}
$$

Using (4.23), we need only verify the following inequality for each $i \in B$:

$$\begin{aligned}
\sum_{\alpha\in G}\Bigg\{&\frac{2}{W_{\alpha\alpha}}\big[S_l^2(G)S_k(G)S_{k-1}(G\mid\alpha)-S_k^2(G)S_l(G)S_{l-1}(G\mid\alpha)\big]W_{\alpha\alpha i}^2\\
&+\frac{2}{k-l}S_l(G)S_k(G)S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\alpha)W_{\alpha\alpha i}^2\\
&+\bigg[\left(1-\frac{1}{k-l}\right)S_k^2(G)S_{l-1}^2(G\mid\alpha)\\
&\qquad-\left(1+\frac{1}{k-l}\right)S_l^2(G)S_{k-1}^2(G\mid\alpha)\bigg]W_{\alpha\alpha i}^2\Bigg\}\\
+\sum_{\substack{\alpha\neq\beta\\ \alpha,\beta\in G}}\bigg[&S_l^2(G)S_k(G)S_{k-2}(G\mid\alpha\beta)-S_l(G)S_k^2(G)S_{l-2}(G\mid\alpha\beta)\\
&+\left(1-\frac{1}{k-l}\right)S_k^2(G)S_{l-1}(G\mid\alpha)S_{l-1}(G\mid\beta)\\
&+\frac{2}{k-l}S_l(G)S_k(G)S_{k-1}(G\mid\alpha)S_{l-1}(G\mid\beta)\\
&-\left(1+\frac{1}{k-l}\right)S_l^2(G)S_{k-1}(G\mid\alpha)S_{k-1}(G\mid\beta)\bigg]W_{\alpha\alpha i}W_{\beta\beta i}\\
\geq 0.&
\end{aligned}$$

This follows from the fact that the matrix

$$\left(f^{\alpha\beta}+2\frac{f^{\alpha}}{\lambda_\alpha}\delta_{\alpha\beta}\right)$$

is semipositive definite (e.g., [25]) for

$$f(\lambda)=-\left(\frac{S_k}{S_l}\right)^{-\frac{1}{k-l}}(\lambda)$$

when $\lambda=(\lambda_1,\ldots,\lambda_m)$ where each λ_i, $1\leq i\leq m$, is a positive number, and the claim is proved. □

With the claim proven, so is Lemma 4.1. □

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