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The Existence of Convex Body with Prescribed Curvature Measures

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1 Introduction

Curvature measure and surface area measure are the basic notions in the classical differential geometry. They play fundamental roles in the theory of convex bodies. They are closely related to the differential geometry and integral geometry of convex hypersurfaces. The Minkowski problem is the problem of prescribing nth surface area measure on $\mathbb{S}^n$. The Christoffel problem concerns the prescribing the first surface area measure (e.g. see [1, 3, 6, 7, 14, 17, 19]). The general problem of prescribing surface area measures is called the Christoffel–Minkowski problem, we refer [12] for an updated account. The problem of prescribing zeroth curvature measure is called the Alexandrov problem, which is a counterpart to Minkowski problem. The problem is equivalent to solve a Monge–Ampère-type equation on $\mathbb{S}^n$. The existence and uniqueness were obtained by Alexandrov [2]. The regularity of the Alexandrov problem in elliptic case was proved by Pogorelov [18] for $n = 2$ and by Oliker [16] for higher-dimension case. The general regularity results (degenerate case) of the problem were obtained in [9]. The general problem of prescribing $(n-k)$th curvature measure for case $k \le n$ is an interesting counterpart of the Christoffel–Minkowski problem. It has been discussed in literature (e.g. [20]). Nevertheless, very little is known except for the Alexandrov problem.

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In this article, we are concerned with the existence of convex bodies with the prescribed $(n-k)$th curvature measure for $1 \le k < n$.

We start with the definitions of curvature measures and surface area measures for convex bodies with smooth boundary. Let Ω be a bounded convex body in $\mathbb{R}^{n+1}$ with C^2 boundary M, the corresponding curvature measures and surface area measures of Ω can be defined according to some geometric quantities of M. Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the principal curvatures of M at point x, let $W_k(x) = S_k(\kappa(x))$ be the kth Weingarten curvature of M at x (where S_k is the kth elementary symmetric function). In particular, W_1, W_2, and W_n are the mean curvature, the scalar curvature, and the Gauss–Kronecker curvature, respectively. The kth curvature measure of Ω is defined as

$$\mathcal{C}_k(\Omega, \beta) := \int_{\beta \cap M} W_{n-k} dF_n$$

for every Borel measurable set β in $\mathbb{R}^{n+1}$, where dF_n is the volume element of the induced metric of $\mathbb{R}^{n+1}$ on M. Since M is convex, M is star-shaped about some point. We may assume that the origin is inside of Ω. Since M and $\mathbb{S}^n$ is diffeomorphic through radial correspondence R_M. Then the kth curvature measure can also be defined as a measure on each Borel set β in $\mathbb{S}^n$ (e.g. see [20]):

$$\mathcal{C}_k(M, \beta) = \int_{R_M(\beta)} W_{n-k} dF_n.$$

We note that $\mathcal{C}_k(M, \mathbb{S}^n)$ is the kth quermassintegral of Ω. Similarly, if M is strictly convex, let $r_1, \ldots, r_n$ be the principal radii of curvature of M, $P_k = S_k(r_1, \ldots, r_n)$. The kth surface area measure of Ω then can be defined as

$$\mathcal{S}_k(\Omega, \beta) := \int_{\beta} P_k d\sigma_n$$

for every Borel set β in $\mathbb{S}^n$, where $d\sigma_n$ is standard volume element on $\mathbb{S}^n$.

We are interested in the problem of prescribing $(n-k)$th curvature measure in a differential geometrical setting. Suppose $1 \le k < n$ is a given integer, we consider.

1.1 Curvature measure problem

For each positive function $f \in C^2(\mathbb{S}^n)$, find a convex hypersurface M as a graph over $\mathbb{S}^n$, such that $\mathcal{C}_{n-k}(M, \beta) = \int_\beta f d\sigma$ for each Borel set β in $\mathbb{S}^n$, where $d\sigma$ is the standard volume element on $\mathbb{S}^n$.

We would like to deduce the problem to a fully nonlinear partial differential equation on $\mathbb{S}^n$. If M is of class C^2, then

$$\mathcal{C}_{n-k}(M,\beta) = \int_{R_M(\beta)} S_k dF = \int_{\beta} S_k g d\sigma_n. \tag{1.1}$$

where g is the density of dF with respect to standard volume element $d\sigma_n$ on $\mathbb{S}^n$. If we view M as a graph over $\mathbb{S}^n$, we may write $X(x) = \rho(x)x, x \in \mathbb{S}^n, \forall X \in M$. The density function g in (1.1) can be computed (see (2.4) in the next section) as

$$g = \rho^{n-1}(\rho^2 + |\nabla\rho|^2)^{\frac{1}{2}}.$$

Therefore by (1.1), the problem of prescribing $(n-k)$th curvature measure can be reduced to the following curvature equation:

$$S_k(\kappa_1, \kappa_2, \ldots, \kappa_n) = f\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbb{S}^n, \tag{1.2}$$

where $f > 0$ is the given function on $\mathbb{S}^n$.

This type of equations can be put in a more general form:

$$S_k(\kappa_1, \kappa_2, \ldots, \kappa_n) = g(x, \rho, \nabla\rho), \quad 1 \le k \le n \quad \text{on} \quad \mathbb{S}^n. \tag{1.3}$$

A solution of (1.2) is called admissible if at each point $X \in M$, its principal curvatures $(\kappa_1, \kappa_2, \ldots, \kappa_n)$ is in the Garding's cone:

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \quad S_i(\lambda) > 0, \quad \forall i \le k.\}$$

For $k < n$, an admissible solution is not necessarily a convex solution. The issue of convexity of admissible solution when $k < n$ arising naturally as in the Christoffel–Minkowski problem [12]. Lemma 2.4 in the next section states that admissible solution to Equation (1.2) is unique if it exists. Therefore, some condition on f is necessary to ensure the existence of convex solutions when $k < n$. The other challenging problem for Equation (1.2) is the lack of some appropriate C^2 a priori estimates for admissible solutions. Equation (1.2) is similar to the equation of prescribing Weingarten curvature equation in [5, 11]. But there is a difference: $g^{1/k}(x, \rho, \nabla\rho)$ may not be necessary convex in $\nabla\rho$. This makes the matter delicate. The problem of C^2 a priori estimates for admissible solutions of Equation (1.2) is still open. We will discuss this in the last section of the article.

Equation (1.2) was studied in an unpublished notes [10] by Yan Yan Li and the first author. The uniqueness and C^1 estimates were established for admissible solutions in [10]. But the issue of convexity and C^2 estimates for Equation (1.2) were left open (except for $k = 1$ and $k = n$, the first case follows from the theory of quasi-linear equations and the latter case was dealt with in [9, 16]).

We now state our main results.

Theorem 1.1. Suppose $f(x) \in C^2(\mathbb{S}^n)$, $f > 0$, $n \geq 2$, $1 \leq k \leq n-1$. If f satisfies the condition

$$|X|^{\frac{n+1}{k}} f\left(\frac{X}{|X|}\right)^{-\frac{1}{k}} \quad \text{is a strictly convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{1.4}$$

then there exists a unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies (1.2). □

When $k = 1$ or 2, the strict convex condition (1.4) can be weakened.

Theorem 1.2. Suppose $k = 1$, or 2 and $k < n$, and suppose $f(x) \in C^2(\mathbb{S}^n)$ is a positive function. If f satisfies

$$|X|^{\frac{n+1}{k}} f\left(\frac{X}{|X|}\right)^{-\frac{1}{k}} \quad \text{is a convex function in} \quad \mathbb{R}^{n+1} \setminus \{0\}, \tag{1.5}$$

then there exists unique strictly convex hypersurface $M \in C^{3,\alpha}, \alpha \in (0,1)$ such that it satisfies Equation (1.2). □

Since the Alexandrov problem (Gauss curvature measure problem) has already been solved [2, 9, 16, 18], Theorem 1.2 yields solutions to two other important measures, the mean curvature measure and scalar curvature measure under convex condition (1.5). The class of functions on $\mathbb{S}^n$ satisfying condition (1.5) is quite wide. For example, if $v > 0$ is a function on $\mathbb{S}^n$ with $(v_{ij} + \delta_{ij} v) > 0$, then $f = v^{-(n+1)}$ is in this class. This is because the homogeneous degree one extension $h = |X| v^{-\frac{1}{n+1}}(\frac{X}{|X|})$ of $v^{-\frac{1}{n+1}}$ is convex in $\mathbb{R}^{n+1}$ and it is strictly convex except in the radial direction. It is easy to see that $h^{\frac{n+1}{k}}$ satisfies (1.5) since $\frac{n+1}{k} > 1$. We note that $(v_{ij} + \delta_{ij} v) > 0$ is equivalent to say that v is a support function of a strictly convex body.

The plan of the article is as follows. In Section 2, we derive uniqueness and C^1 bound of the solutions of (1.3). Theorem 1.1 will be proved in Section 3. The novel feature is the C^2 estimates. Instead of obtaining an upper bound of the principal curvatures, we look for a lower bound of the principal curvatures by transforming (1.2) to a new equation for the support function on $\mathbb{S}^n$ through Gauss map. Section 4 is devoted to the proof of Theorem 1.2. The key part is the C^2 estimates for the case $k = 2$, which we use a special structure of S_2. Then we establish a deformation lemma 4.3 as in [11, 12] to ensure the convexity of solutions in the process of applying the method of continuity. In the last section, we discuss C^2 estimates for admissible solutions of curvature type Equations (1.3).

2 Uniqueness and C^1 Boundness

We first recall some relevant geometric quantities of a smooth closed hypersurface $M \subset \mathbb{R}^{n+1}$. We assume the origin is inside the body enclosed by M.

$A, B, \ldots$ will be from 1 to $n+1$ and Latin from 1 to n, the repeated indices denote summation over the indices. Covariant differentiation will simply be indicated by indices.

Let M^n be a n-dimension closed hypersurface immersed in $\mathbb{R}^{n+1}$. We choose an orthonormal frame in $\mathbb{R}^{n+1}$ such that $\{e_1, e_2, \ldots, e_n\}$ are tangent to M and e_{n+1} is the outer normal. Let $\{\omega_A\}$ and $\{\omega_{A,B}\}$ be the corresponding coframe and the connection forms, respectively. We will use the same notions for the pull-back of them through the immersion. Therefore, on M,

$$\omega_{n+1} = 0.$$

The second fundamental form is defined by the symmetric matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{2.1}$$

We recall the following fundamental formulas of a hypersurface in $\mathbb{R}^{n+1}$:

$$\begin{aligned}
X_{ij} &= -h_{ij}e_{n+1} \quad \text{(Gauss formula)}\\
(e_{n+1})_i &= h_{ij}e_j \quad \text{(Weingarten equation)}\\
h_{ijk} &= h_{ikj} \quad \text{(Codazzi formula)}\\
R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \quad \text{(Gauss equation)},
\end{aligned} \tag{2.2}$$

where R_{ijkl} is the curvature tensor. And we have the following formulas:

$$\begin{aligned} h_{ijkl} &= h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk}, \\ h_{ijkl} &= h_{klij} + (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} + (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}, \\ (e_{n+1})_{ii} &= \sum_{j=1}^{n} h_{iij}e_j - \sum_{j=1}^{n} h_{ij}^2 e_{n+1}. \end{aligned} \tag{2.3}$$

Since M is star-shaped with respect to origin, the position vector X of M can be written as $X(x) = \rho(x)x, \quad x \in \mathbb{S}^n$, where ρ is a smooth function on $\mathbb{S}^n$. Let $\{e_1, \ldots, e_n\}$ be smooth local orthonormal frame on $\mathbb{S}^n$, let ∇ be the gradient on $\mathbb{S}^n$ and covariant differentiation will simply be indicated by indices. Then in terms of ρ, the metric of M is given by

$$g_{ij} = \rho^2\delta_{ij} + \rho_i\rho_j.$$

So the area factor

$$g = (\det g_{ij})^{\frac{1}{2}} = \rho^{n-1}(\rho^2 + |\nabla\rho|^2)^{\frac{1}{2}}. \tag{2.4}$$

The second fundamental form of M is

$$h_{ij} = (\rho^2 + |\nabla\rho|^2)^{-\frac{1}{2}}(\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}) \tag{2.5}$$

and the unit outer normal of the hypersurface M in $\mathbb{R}^{n+1}$ is

$$\mathbf{N} = \frac{\rho x - \nabla\rho}{\sqrt{\rho^2 + |\nabla\rho|^2}}. \tag{2.6}$$

The principal curvature $(\kappa_1, \kappa_2, \ldots, \kappa_n)$ of M are the eigenvalue of the second fundamental form with respect to the metric satisfying the following equation:

$$\det(h_{ij} - kg_{ij}) = 0.$$

Equation (1.3) can be expressed as differential equations on the radial function ρ and position vector X, respectively. From (2.6) we have

$$\begin{aligned} \langle X, N\rangle &= \rho^2(\rho^2 + |\nabla\rho|^2)^{-1/2}, \\ S_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) &= |X|^{-(n+1)} f\left(\frac{X}{|X|}\right)\langle X, N\rangle, \quad \forall X \in M. \end{aligned} \tag{2.7}$$

Equation (1.2) is equivalent to Equation (2.7).

Definition 2.1. For $1 \leq k \leq n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, \dots, \quad S_k(\lambda) > 0\}.$$

A C^2 surface M is called k-admissible if at every point $X \in M$, $(\kappa_1, \kappa_2, \dots, \kappa_n) \in \Gamma_k$. □

The following four lemmas had been proved in [10]; for the completeness, we provide the proofs here. First we get the C^0 estimates.

Lemma 2.2. If M satisfies (2.7), then

$$\left(\frac{\min_{\mathbb{S}^n} f}{C_n^k}\right)^{1/(n-k)} \leq \min_{\mathbb{S}^n} |X| \leq \max_{\mathbb{S}^n} |X| \leq \left(\frac{\max_{\mathbb{S}^n} f}{C_n^k}\right)^{1/(n-k)}. \quad \square$$

Proof. Since M is compact, $|X|$ attains a maximum R at some point X_1. Let B_R be a ball of radius R centered at the origin. We note that $M \subset B_R$, and M and ∂B_R have the same outer normal $\frac{X_1}{|X_1|}$ at X_1. We have $\langle X_1, N\rangle = R$ and

$$\kappa_i(X_1) \geq R, \quad \forall i = 1, \cdots, n.$$

Hence,

$$f\left(\frac{X_1}{|X_1|}\right) = S_k(\kappa(x_1), \cdots, \kappa(X_1)) \geq C_n^k R^{n-k}.$$

In turn,

$$\max_{\mathbb{S}^n} |X| \leq \left(\frac{\max_{\mathbb{S}^n} f}{C_n^k}\right)^{1/(n-k)}.$$

The inequality $(\frac{\min_{\mathbb{S}^n} f}{C_n^k})^{1/(n-k)} \leq \min_{\mathbb{S}^n} |X|$ can be shown in a similar way. ■

The next is the gradient estimate for general admissible solution ρ.

Lemma 2.3. If M satisfies (2.7), then there exist a constant C depending only on $n, k, \min_{\mathbb{S}^n} f, |f|_{C^1}$ such that

$$\max_{S^n} |\nabla \rho| \leq C. \quad \square$$

Proof. Since we have the lower and upper bound for the solution of (2.7), thus we only need prove

$$\langle X, N\rangle \geq C.$$

For any local orthonormal frame $(e_1, \cdots, e_n)$ on M, we have the following formulas:

$$(|X|^2)_i = 2\langle X, e_i\rangle, \tag{2.8}$$

$$(|X|^2)_{ij} = 2\delta_{ij} - 2h_{ij}\langle X, N\rangle, \tag{2.9}$$

$$\langle X, N\rangle_i = h_{ik}\langle X, e_k\rangle, \tag{2.10}$$

$$\langle X, N\rangle_{ij} = h_{ijk}\langle X, e_k\rangle + h_{ij} - h_{ik}h_{kj}\langle X, N\rangle. \tag{2.11}$$

Set

$$\phi(X) = |X|^{-(n+1)} f\left(\frac{X}{|X|}\right).$$

Equation (2.7) becomes

$$S_k(h_{ij}) = \phi(X)\langle X, N\rangle. \tag{2.12}$$

Let

$$P(X) = \gamma(t) - \log\langle X, N\rangle, \quad t = |X|^2,$$

of which the function $\gamma(t)$ will be determined later in the article.

Assume $P(X)$ attains its maximum at a point $X_o \in M$. If at X_o, $\langle X, e\rangle = 0$ for all $e \in T_{X_o}M$, we have $\langle X, N\rangle = |X|^2$, there is nothing to proof. So we may assume $\langle X, N\rangle^2 < |X|^2$ at X_o and we may choose the smooth local orthonormal frame $\{e_1, \ldots, e_n\}$ on M such that at X_o,

$$\langle X, e_i\rangle = 0, \quad i \geq 2.$$

From now on, all the calculations will be done at X_o. Since

$$P_i(X) = -\frac{\langle X, N\rangle_i}{\langle X, N\rangle} + \gamma'(t)(|X|^2)_i,$$

we have

$$\frac{\langle X, N\rangle_i}{\langle X, N\rangle} = 2\gamma'(t)\langle X, e_i\rangle. \tag{2.13}$$

For $i = 1$, we have

$$\frac{h_{11}\langle X, e_1\rangle}{\langle X, N\rangle} = 2\gamma'(t)\langle X, e_1\rangle.$$

By the assumption, $\langle X, e_1\rangle \neq 0$. We have

$$h_{11} = 2\gamma'(t)\langle X, N\rangle.$$

Again from (2.13), we have for

$$h_{1i} = 0, i \geq 2.$$

Now we may choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ on M such that (h_{ij}) is diagonal at X_o. Using (2.13), we have

$$F^{ii}h_{ii1} = F^{ij}h_{ij1} = (\phi_1 + 2\phi\gamma'(t)\langle X, e_1\rangle)\langle X, N\rangle, \tag{2.14}$$

where $F^{ij} = \frac{S_k(h_{ij})}{\partial h_{ij}}$. Since for any $1 \leq i \leq n$,

$$\begin{aligned} P_{ii} &= -\frac{\langle X, N\rangle_{ii}}{\langle X, N\rangle} + \frac{{\langle X, N\rangle_i}^2}{\langle X, N\rangle^2} + \gamma''(t)((|X|^2)_i)^2 + \gamma'(t)(|X|^2)_{ii} \\ &= -\frac{1}{\langle X, N\rangle}\big[h_{ii1}\langle X, e_1\rangle + h_{ii} - h_{ii}^2\langle X, N\rangle\big] \\ &\quad + [(\gamma'(t))^2 + \gamma''(t)]((|X|^2)_i)^2 + \gamma'(t)(|X|^2)_{ii}. \end{aligned}$$

From (2.14), we get

$$
\begin{aligned}
F^{ii}P_{ii} &= -\frac{\langle X, e_1\rangle}{\langle X, N\rangle}F^{ii}h_{ii1} - \frac{F^{ii}h_{ii}}{\langle X, N\rangle} + F^{ii}h_{ii}^2 \\
&\quad + 4[(\gamma'(t))^2 + \gamma''(t)]\langle X, e_1\rangle^2 F^{11} + \gamma'(t)F^{ii}[2 - 2h_{ii}\langle X, N\rangle] \\
&= -\phi_1\langle X, e_1\rangle - 2\phi\gamma'(t)\langle X, e_1\rangle^2 - k\phi + F^{ii}h_{ii}^2 \\
&\quad + 4[(\gamma'(t))^2 + \gamma''(t)]\langle X, e_1\rangle^2 F^{11} + 2\gamma'(t)F^{ii} - 2k\langle X, N\rangle^2\gamma'(t)\phi \\
&\le 0.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\langle X, N\rangle^2[2\gamma'(t)\phi(k-1) + 4((\gamma'(t))^2 + \gamma''(t))F^{11}] \ge 4[(\gamma'(t))^2 + \gamma''(t)]|X|^2F^{11} - \phi_1\langle X, e_1\rangle \\
&\quad - k\phi - 2\phi\gamma'(t)|X|^2 + 2\gamma'(t)F^{ii} + F^{ii}h_{ii}^2.
\end{aligned} \tag{2.15}
$$

Now let

$$
\gamma(t) = \frac{\alpha}{t},
$$

where $\alpha > 0$ will be determined later. We have

$$
2\gamma'(t)\phi(k-1) + 4((\gamma'(t))^2 + \gamma''(t))F^{11} \le \left(\frac{4\alpha^2}{t^4} + \frac{8\alpha}{t^3}\right)F^{11}. \tag{2.16}
$$

Now we treat the right-hand side in (2.15), where we shall use some properties for the elementary symmetry functions (see for example [13]).

Let's take α large enough such that

$$
-\phi_1\langle X, e_1\rangle - k\phi - 2\phi\gamma'(t)|X|^2 = \frac{2\alpha}{t^2}|X|^2\phi - \phi_1\langle X, e_1\rangle - k\phi \ge 0. \tag{2.17}
$$

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$, for any $1 \le i \le n$, then $(\lambda|i) = (\lambda_1, \ldots, \hat{\lambda_i}, \ldots, \lambda_n) \in \Gamma_{k-1}$ and $S_{k-2}(\lambda|i) > 0$, where the hat means this element has been deleted.

We have

$$
\begin{aligned}
S_{k-1}(\lambda) &= S_{k-2}(\lambda|1)\lambda_1 + S_{k-1}(\lambda|1), \\
\sum_i F^{ii} &= (n-k+1)S_{k-1}(h_{ij}).
\end{aligned}
$$

Since at X_o,

$$h_{11} = -\frac{2\alpha}{t^2}\langle X, N\rangle < 0,$$

we have

$$S_{k-1}(h_{ij}) < S_{k-1}(\lambda|1) = F^{11}.$$

So for α large enough, there is a positive constant C_o such that we have

$$\begin{aligned} &4[(\gamma'(t))^2 + \gamma''(t)]|X|^2F^{11} + 2\gamma'(t)F^{ii} + F^{ii}h_{ii}^2 \\ &\quad\geq 4\left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3}\right)|X|^2F^{11} - \frac{2(n-k+1)\alpha}{t^2}S_{k-1}(h_{ij}) \\ &\quad\geq C_oF^{11}. \end{aligned} \tag{2.18}$$

By (2.15)–(2.18), there exists a positive constant that depends only on $n, k, \min_{S^n} f, |f|_{C^1}$, such that $\max_{S^n}|\nabla\rho| \leq C$. ∎

The proof of the gradient estimate in Lemma 2.3 follows similar arguments in [5]. One difference is the choice of the test function. Also, a barrier condition $\frac{\partial(\rho f)}{\partial\rho} \leq 0$ was imposed in [5]. Equation (2.7) considered here differs from Equation (1) in [5] by a factor $\langle X, N\rangle$. This factor arises naturally from the geometric situation. It is also the reason we are able to prove Lemma 2.3 without any barrier condition. It is of interest to know when can one obtain a gradient estimate for general curvature equation (1.3).

Let's denote $\lambda = \lambda(\rho) = (\lambda_1(\rho), \cdots, \lambda_n(\rho))$ to be the eigenvalues of the second fundamental form (h_{ij}) with respect to the first fundamental form (g_{ij}) of the spherical graph defined by ρ. For the rest of this section, we set $F(\lambda) = S_k^{1/k}(\lambda)$. Equation (1.2) can be written as

$$F(\lambda) \equiv F(\lambda_1, \ldots, \lambda_n) = f^{1/k}\rho^{(1-n)/k}(\rho^2 + |\nabla\rho|^2)^{-1/(2k)} \equiv K(x, \rho, \nabla\rho).$$

The following is the uniqueness result of the problem.

Lemma 2.4. Suppose $1 \leq k < n$, $\lambda(\rho_i) \in \Gamma_k$, $i = 1, 2$. Suppose ρ_1, ρ_2 are solutions of (1.2). Then $\rho_1 \equiv \rho_2$. □

Proof. We prove it by contradiction. Suppose $\rho_2 > \rho_1$ somewhere on $\mathbb{S}^n$. Take $t \geq 1$ such that

$$t\rho_1 \geq \rho_2 \quad \text{on } \mathbb{S}^n, \quad t\rho_1 = \rho_2 \quad \text{at some point } P \in \mathbb{S}^n.$$

Obviously, $\lambda(t\rho_1) = t^{-1}\lambda(\rho_1)$, and therefore $F(\lambda(t\rho_1)) = t^{-1}F(\lambda(\rho_1))$. It is clear that

$$\begin{aligned} &K(x, t\rho_1, \nabla(t\rho_1)) = t^{-n/k}K(x, \rho_1, \nabla\rho_1) \\ &= t^{-n/k}F(\lambda(\rho_1)) \leq t^{-1}F(\lambda(\rho_1)) = F(\lambda(t\rho_1)). \end{aligned}$$

It follows that

$$F(\lambda(t\rho_1)) - K(x, t\rho_1, \nabla(t\rho_1)) \geq 0, \quad F(\lambda(\rho_2)) - K(x, \rho_2, \nabla\rho_2) = 0.$$

Hence

$$\tilde{L}(t\rho_1 - \rho_2) \geq 0,$$

where $\tilde{L}$ is a linear elliptic operator. The strong maximum principle yields $t\rho_1 - \rho_2 \equiv 0$ on $\mathbb{S}^n$. Since $n > k$, from Equation (1.2), we conclude that $t = 1$. ■

The following lemma will also be used in this article.

Lemma 2.5. Let L denote the linearized operator of $F(\lambda) - K(x, \rho, \nabla\rho)$ at a solution ρ of (1.2). If w satisfies $Lw = 0$ on $\mathbb{S}^n$, then $w \equiv 0$ on $\mathbb{S}^n$. □

Proof. We write $F(x, \rho, \nabla\rho, \nabla^2\rho) \equiv F(\lambda)$. The linearized operator L at ρ is defined as

$$L(u) = \sum_{i,j} \frac{\partial F}{\partial \rho_{ij}} u_{ij} + \sum_k \left(\frac{\partial F}{\partial \rho_k} - \frac{\partial K}{\partial \rho_k} \right) u_k + \left(\frac{\partial F}{\partial \rho} - \frac{\partial K}{\partial \rho} \right) u.$$

We have

$$F(x, t\rho, \nabla(t\rho), \nabla^2(t\rho)) = F(\lambda(t\rho)) = F(\lambda(\rho)/t).$$

Applying $\frac{d}{dt}\big|_{t=1}$ to above identity,

$$\sum_{i,j}\frac{\partial F}{\partial \rho_{ij}}\rho_{ij}+\sum_k \frac{\partial F}{\partial \rho_k}\rho_k+\frac{\partial F}{\partial \rho}\rho=-\sum_i \lambda_i F_{\lambda_i}=-F.$$

It is easy to see that

$$K(x,t\rho,\nabla(t\rho))=t^{-n/k}K(x,\rho,\nabla\rho).$$

Applying $\frac{d}{dt}\big|_{t=1}$ to this equation,

$$\sum_k \frac{\partial K}{\partial \rho_k}\rho_k+\frac{\partial K}{\partial \rho}\rho=-n/kK(x,\rho,\nabla\rho).$$

It follows that

$$L\rho=-F(\lambda)+n/kK(x,\rho,\nabla\rho)=(n/k-1)K(x,\rho,\nabla\rho)>0.$$

Set $w=z\rho$, we get

$$0=Lw=L(z\rho)\equiv L'z+zL\rho,$$

where $L'z=\rho\sum_{i,j}\frac{\partial F}{\partial \rho_{ij}}z_{ij}+$ first-order derivatives of z. By (2.5), we have $(\frac{\partial F}{\partial \rho_{ij}})<0$. Therefore at minimum point of z, $L'z\le 0$. Since $L\rho>0$, the minimum value of z must be nonnegative. Similarly, the maximum value of z must be nonpositive. That is $w\equiv 0$. ■

3 Proof of Theorem 1.1

In this section we prove C^2 estimates for convex of solution of Equation (1.3). For the mean curvature measure case ($k=1$), a gradient bound is enough for a C^2 a priori bound by the standard theory of quasi-linear elliptic equations. In the rest of this section, we will assume $k>1$.

For the C^2 estimates for admissible solutions of (1.3), it is equivalent to estimate the upper bounds of principal curvatures. If the hypersurface is strictly convex, it is simple to observe that a positive lower bound on the principal curvatures implies an upper bound of the principal curvatures. This follows from Equation (1.3) and the

Newton–Maclaurin inequality,

$$S_n^{\frac{1}{n}}(\lambda) \leq \left[\frac{S_k}{C_n^k}\right]^{\frac{1}{k}}(\lambda).$$

This is the starting point of our approach in this section. To achieve such a lower bound, we shall use the inverse Gauss map and consider the equation for the support function of the hypersurface. The role of the Gauss map here should be compared with the role of the Legendre transformation on the graph of convex surface in a domain in $\mathbb{R}^n$. Since M is curved and compact, the Gauss map fits into the picture neatly. This way, we can make use of some special features of the support function. We note that a lower bound on the principal curvature is an upper bound on the principal radii. And the principal radii are exactly the eigenvalues of the spherical Hessian of the support function. Therefore, we are led to get a C^2 bound on the support function of M.

Let $X : M \to \mathbb{R}^{n+1}$ be a closed strictly convex smooth hypersurface in $\mathbb{R}^{n+1}$. We may assume the X is parameterized by the inverse Gauss map

$$X : \quad \mathbb{S}^n \quad \to \quad \mathbb{R}^{n+1}.$$

The support function of X is defined by

$$u(x) = \langle x, X(x) \rangle, \quad \text{at} \quad x \in \mathbb{S}^n.$$

Let $e_1, e_2, \ldots, e_n$ be a smooth local orthonormal frame on $\mathbb{S}^n$, we know that the second fundamental form of X is

$$h_{ij} = u_{ij} + u\delta_{ij},$$

and the metric of X is

$$g_{ij} = \sum_{l=1}^{n} h_{il} h_{jl}.$$

The principal radii of curvature are the eigenvalues of matrix (with respect to the standard metric on $\mathbb{S}^n$)

$$W_{ij} = u_{ij} + u\delta_{ij}.$$

Equation (1.2) can be rewritten as an equation on support function u:

$$F(W_{ij}) = \left[\frac{\det W_{ij}}{S_{n-k}(W_{ij})}\right]^{\frac{1}{k}}(x) = G(X)u^{-\frac{1}{k}} \quad \text{on} \quad \mathbb{S}^n, \tag{3.1}$$

where X is position vector of hypersurface, and

$$G(X) = |X|^{\frac{n+1}{k}} f^{-\frac{1}{k}}\left(\frac{X}{|X|}\right).$$

Equation (3.1) is similar to the equation in [8], where a problem of prescribing Weingarten curvature was considered. The position function and the support function have the following explicit form:

$$X(x) = \sum_{i=1}^{n} u_i e_i + ux, \quad \text{on} \quad x \in \mathbb{S}^n.$$

Straightforward computations yield

$$X_l = u_{il}e_i + u_i(e_i)_l + u_l x + ux_l = u_{il}e_i - xu_i\delta_{il} + u_l x + ue_l = W_{il}e_i, \tag{3.2}$$

$$\begin{aligned}\sum_{l=1}^{n} X_{ll} &= \sum_{i,l=1}^{n}[W_{ill}e_i + W_{il}(e_i)_l] \\ &= \sum_{i=1}^{n}\left[\sum_{l=1}^{n} W_{ll}\right]_i e_i + \sum_{i,l=1}^{n} W_{il}(-x\delta_{il}) \\ &= \sum_{i=1}^{n}\left[\sum_{l=1}^{n} W_{ll}\right]_i e_i - x\sum_{l=1}^{n} W_{ll}.\end{aligned} \tag{3.3}$$

The following is a key lemma.

Lemma 3.1. If $G(X)$ is strictly convex function in $\mathbb{R}^{n+1} \setminus \{o\}$, and $u(x)$ satisfies (3.1), then

$$\max(\Delta u + nu) \le C, \tag{3.4}$$

where the constant C depends only on $n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f, |\nabla f|_{C^0}$, and $|\nabla^2 f|_{C^0}$. In turn,

$$|\nabla^2 \rho| \le C. \tag{3.5}$$

□

Proof. Since we have already obtained C^1 bound in Lemma 2.3, to get (3.5), we only need to prove (3.4). Let

$$H = \sum_{l=1}^{n} = \Delta u + nu$$

and assume the maximum of H attains at some point $x_o \in \mathbb{S}^n$. We choose an orthonormal frame $e_1, e_2, \ldots, e_n$ near x_o such that $u_{ij}(x_o)$ is diagonal (so is $W_{ij} = u_{ij} + u\delta_{ij}$ at x_o). The following formula for commuting covariant derivatives are elementary:

$$(\Delta u)_{ii} = \Delta(u_{ii}) + 2\Delta u - 2nu_{ii}.$$

So we have

$$H_{ii} = (\Delta u)_{ii} + nu_{ii} = \Delta(W_{ii}) - nW_{ii} + H. \tag{3.6}$$

Let $F^{ij} = \frac{\partial F(W)}{\partial W_{ij}}$. At x_o the matrix F^{ij} is positive definite, diagonal. Setting the eigenvalues of W_{ij} at x_o as $\lambda(W_{ij}) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$),

$$F^{ii} = \frac{1}{k}\left(\frac{S_n}{S_{n-k}}\right)^{\frac{1}{k}}\left[\frac{S_{n-1}(\lambda|i)}{S_{n-k}} - \frac{S_n S_{n-k-1}(\lambda|i)}{{S_{n-k}}^2}\right].$$

The following facts are known (e.g. see [8]):

$$\sum_{i=1}^{n} F^{ii} W_{ii} = F, \quad \sum_{i=1}^{n} F^{ii} \geq \left(C_n^{n-k}\right)^{-\frac{1}{k}}.$$

Now at x_o, we have

$$H_i = 0, \quad H_{ij} \leq 0 \tag{3.7}$$

At x_o, we have

$$\begin{aligned} 0 \ &\geq \sum_{i,j=1}^{n} F^{ij} H_{ij} = \sum_{i=1}^{n} F^{ii} H_{ii} = \sum_{i=1}^{n} F^{ii} \Delta(W_{ii}) - n \sum_{i=1}^{n} F^{ii} W_{ii} + H \sum_{i=1}^{n} F^{ii} \\ &\geq \sum_{i=1}^{n} F^{ii} \Delta(W_{ii}) - nF + \left(C_n^{n-k}\right)^{-\frac{1}{k}} H. \end{aligned} \tag{3.8}$$

From Equation (3.1),

$$F^{ij}W_{ijl} = \left[G(X)u^{-\frac{1}{k}}\right]_l, \quad F^{ij}W_{ijll} + F^{ij,st}W_{ijl}W_{stl} = \left[G(X)u^{-\frac{1}{k}}\right]_{ll}.$$

By the concavity of F, we get

$$\sum_{i=1}^{n} F^{ii}\Delta(W_{ii}) \geq \sum_{l=1}^{n}\left[G(X)u^{-\frac{1}{k}}\right]_{ll}.$$

Combining this with (3.8), we have the following inequality at x_o:

$$\sum_{l=1}^{n}\left[G(X)u^{-\frac{1}{k}}\right]_{ll} - nF + \left(C_n^{n-k}\right)^{-\frac{1}{k}}H \leq 0. \tag{3.9}$$

Now we treat the term $[G(X)u^{-\frac{1}{k}}]_{ll}$. In the following, the repeated indices on α, β denote summation over the indices from $1, 2, \ldots, n+1$. Denote $G_\alpha = \frac{\partial G}{\partial X^\alpha}$, $G_{\alpha\beta} = \frac{\partial^2 G}{\partial X^\alpha \partial X^\beta}$.

$$\begin{aligned}
\left[G(X)u^{-\frac{1}{k}}\right]_l &= G_\alpha X_l^\alpha u^{-\frac{1}{k}} + G(X)\left(-\frac{1}{k}\right)u^{-\frac{1}{k}-1}u_l,\\
\sum_{l=1}^{n}\left[G(X)u^{-\frac{1}{k}}\right]_{ll} &= G_{\alpha\beta}X_l^\alpha X_l^\beta u^{-\frac{1}{k}} + G_\alpha X_{ll}^\alpha u^{-\frac{1}{k}}\\
&\quad - \frac{2}{k}G_\alpha X_l^\alpha u^{-\frac{1}{k}-1}u_l + \frac{1}{k}\left(\frac{1}{k}+1\right)G(X)u^{-\frac{1}{k}-2}|Du|^2 - \frac{1}{k}G(X)u^{-\frac{1}{k}-1}u_{ll}.
\end{aligned}$$

Using (3.2) and (3.3), it follows that at x_o,

$$\begin{aligned}
\sum_{l=1}^{n}\left[G(X)u^{-\frac{1}{k}}\right]_{ll} &= G_{\alpha\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - \left[G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}\right]H - \frac{2}{k}\left(G_\alpha e_l^\alpha u_l W_{ll}\right)u^{-\frac{1}{k}-1}\\
&\quad + \frac{1}{k}\left(\frac{1}{k}+1\right)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}}.
\end{aligned} \tag{3.10}$$

By (3.10), at x_o (3.9) becomes

$$\begin{aligned}
&G_{\alpha\beta}e_l^\alpha e_l^\beta W_{ll}^2 u^{-\frac{1}{k}} - \left[G_\alpha x^\alpha u^{-\frac{1}{k}} + \frac{1}{k}G(X)u^{-\frac{1}{k}-1}\right]H - nF + \left(C_n^{n-k}\right)^{-\frac{1}{k}}H\\
&\quad - \frac{2}{k}\left(G_\alpha e_l^\alpha u_l W_{ll}\right)u^{-\frac{1}{k}-1} + \frac{1}{k}\left(\frac{1}{k}+1\right)G(X)u^{-\frac{1}{k}-2}|Du|^2 + +\frac{n}{k}G(X)u^{-\frac{1}{k}} \leq 0.
\end{aligned} \tag{3.11}$$

If $G(X)$ is strictly convex in $\mathbb{R}^{n+1} \setminus \{o\}$, then there exist a uniform constant $c_o > 0$ such that

$$\sum_{\alpha\beta=1}^{n} G_{\alpha,\beta} e_l^{\alpha} e_l^{\beta} \geq c_o, \quad l = 1, 2, \ldots, n.$$

Since $\sum_{l=1}^{n} W_{ll}^2 \geq \frac{H^2}{n}$, we obtain $H(x_o) \leq C$. ∎

Proof of Theorem 1.1. For any positive function $f \in C^2(\mathbb{S}^n)$, for $0 \leq t \leq 1$ and $1 \leq k \leq n-1$, set $f_t(x) = [1 - t + t f^{-\frac{1}{k}}(x)]^{-k}$. We consider the following family of equations for $0 \leq t \leq 1$:

$$S_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(x) = f_t(x)\rho^{1-n}(\rho^2 + |\nabla\rho|^2)^{-1/2}, \quad \text{on} \quad \mathbb{S}^n, \tag{3.12}$$

where $n \geq 2$. We want to find solutions in the class of strictly convex hypersurfaces. Let $I = \{t \in [0,1] : \text{such that (3.12) is solvable}\}$. Since $\rho = [C_n^k]^{-\frac{1}{n-2}}$ is a solution for $t = 0$, I is not empty. By Lemmas 2.3 and 3.5, $o0 < \rho \in C^{1,1}(\mathbb{S}^n)$ and the principal curvatures of the solution hypersurface are bounded from below and above. The Evans–Krylov theorem yields $\rho \in C^{2,\alpha}(\mathbb{S}^n)$ and

$$||\rho||_{C^{2,\alpha}(\mathbb{S}^n)} \leq C, \tag{3.13}$$

where C depends only on $n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f$, $|\nabla f|_{C^0}$ and $|\nabla^2 f|_{C^0}$, and α. The a priori estimates guarantee I is closed. The openness is from Lemma 2.5 and the implicit function theorem. So we have the existence. The uniqueness of the solution for $t \in [0,1]$ is from Lemma 2.4. This completes the proof of Theorem 1.1. ∎

Remark 3.2. We suspect the strict convexity condition (1.4) can be weakened. For the cases $k = 1, 2$, this is verified in Theorem 1.2. □

4 Proof of Theorem 1.2

In this section, we will first prove the C^2 estimate for the scalar curvature measure case under the convexity assumption of the solution. The proof of Theorem 1.2 is different from the proof of Theorem 1.1 in this section. Due to the weakened condition, we are not able to obtained a positive lower bound for the principal curvatures directly. Instead, we will use special structure of the elementary symmetric function S_2 to get an upper bound of principal curvatures for convex solutions of (1.3). Since the convexity of solutions is not guaranteed for Equation (1.3) when $k < n$, we will use condition (1.5) and a deformation lemma to prove the existence of a strictly convex solution of (1.3) in the next section, as in [12].

We consider the following prescribed scalar curvature measure equation:

$$S_2(\lambda\{h_{ij}\})(X) = |X|^{-(n+1)} f\left(\frac{X}{|X|}\right)\langle X, N\rangle, \quad \forall X \in M. \tag{4.1}$$

Lemma 4.1. Let f be a C^2 positive function on $\mathbb{S}^n$ and let M be a star-shaped hypersurface in $\mathbb{R}^{n+1}$ with respect to the origin, if M is a convex solution surface of Equation (4.1) and for the function $\rho = |X|$ on $\mathbb{S}^n$, the following estimates hold:

$$\|\rho\|_{C^2} \le C, \tag{4.2}$$

where the constant C depends only on $n, k, \min_{\mathbb{S}^n} f$ and $\|f\|_{C^2}$. □

Proof. C^1 estimates were already obtained in Lemma 2.3 in the Section 2. We only need to get an upper bound of the mean curvature H.

Let

$$F(X) = f\left(\frac{X}{|X|}\right), \quad \phi(X) = |X|^{-(n+1)}F(X), \tag{4.3}$$

Equation (4.1) becomes

$$S_2(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) = \phi(X)\langle X, e_{n+1}\rangle, \quad \text{on} \quad M. \tag{4.4}$$

Assume the function $P = H + \frac{a}{2}|X|^2$ attains its maximum at $X_o \in M$, where a is a constant, which will be determined later. At X_o, we have

$$P_i = H_i + a\langle X, e_i\rangle = 0, \tag{4.5}$$

$$P_{ii} = H_{ii} + a[1 - h_{ii}\langle X, e_{n+1}\rangle]. \tag{4.6}$$

Let $F^{ij} = \frac{\partial S_2(\lambda\{h_{ij}\})}{\partial h_{ij}}$, we choose a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ in a neighborhood of $X_o \in M$ such that at X_o, the matrix $\{h_{ij}\}$ is diagonal. Then at X_o, the matrix $\{F^{ij}\}$

is also diagonal and positive definitive. At X_o,

$$\sum_{ij=1}^{n} F^{ij} P_{ij} = \sum_{i=1}^{n} F^{ii} H_{ii} + a \sum_{i=1}^{n} F^{ii} - a\langle X, e_{n+1}\rangle \sum_{i=1}^{n} F^{ii} h_{ii} \leq 0. \tag{4.7}$$

In what follows, all the calculations will be done at $X_o \in M$. First we deal with the term $\sum_{i=1}^{n} F^{ii} H_{ii}$. From (4.5) and (2.3), we have

$$\begin{aligned}
\sum_{i=1}^{n} F^{ii} H_{ii} &= \sum_{i=1}^{n} F^{ii} \left(\sum_{j=1}^{n} h_{jjii} \right) = \sum_{i=1}^{n} F^{ii} \sum_{j=1}^{n} \left(h_{iijj} + h_{ii} h_{jj}^2 - h_{jj} h_{ii}^2 \right) \\
&= \sum_{ij=1}^{n} F^{ii} h_{iijj} + |A|^2 \sum_{i=1}^{n} F^{ii} h_{ii} - H \sum_{i=1}^{n} F^{ii} h_{ii}^2,
\end{aligned}$$

where $|A|^2 = \sum_{i=1}^{n} h_{ii}^2$.

We treat the term $\sum_{ij=1}^{n} F^{ii} h_{iijj}$. Differentiate Equation (4.4) twice, by (2.2),

$$\begin{aligned}
\sum_{ij=1}^{n} F^{ii} h_{iijj} &= \sum_{j=1}^{n} [\phi(X)\langle X, e_{n+1}\rangle]_{jj} + \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k\neq l} h_{jkk} h_{jll} \\
&= \Delta\phi \langle X, e_{n+1}\rangle + 2 \sum_{j=1}^{n} \phi_j h_{jj} \langle X, e_j\rangle + \phi \sum_{j=1}^{n} \langle X, e_{n+1}\rangle_{jj} \\
&\quad + \sum_{j,k\neq l} h_{jkl}^2 - \sum_{j,k,l} h_{jkk} h_{jll} + \sum_{j,k} h_{jkk}^2.
\end{aligned}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
\sum_{i=1}^{n} \langle X, e_{n+1}\rangle_{ii} &= \sum_{i,l=1}^{n} [h_{il} \langle X, e_l\rangle]_i \\
&= \sum_{i=1}^{n} \left[\sum_{l=1}^{n} h_{iil} \langle X, e_l\rangle + h_{ii} - h_{ii}^2 \langle X, e_{n+1}\rangle \right] \\
&= \sum_{l=1}^{n} H_l \langle X, e_l\rangle + H - |A|^2 \langle X, e_{n+1}\rangle \\
&= -a \sum_{i=1}^{n} \langle x, e_i\rangle^2 + H - |A|^2 \langle X, e_{n+1}\rangle.
\end{aligned}$$

In turn, (4.4)–(4.5) yield the following estimate:

$$\sum_{ij=1}^{n} F^{ii} h_{iijj} \geq -|A|^2 S_2(h_{ij}) + \phi H + \Delta\phi\langle X, e_{n+1}\rangle + 2\sum_{j=1}^{n} \phi_j h_{jj}\langle X, e_j\rangle$$
$$-a\phi\sum_{i=1}^{n}\langle x, e_i\rangle^2 - a^2\sum_{i=1}^{n}\langle x, e_i\rangle^2. \tag{4.8}$$

It is easy to compute that

$$\begin{aligned}\sum_{i=1}^{n} F^{ii} &= (n-1)H,\\ \sum_{i=1}^{n} F^{ii} h_{ii} &= 2S_2(h_{ij}),\\ \sum_{i=1}^{n} F^{ii} h_{ii}^2 &= HS_2(h_{ij}) - 3S_3(h_{ij}),\\ |A|^2 &= H^2 - 2S_2(h_{ij}).\end{aligned} \tag{4.9}$$

Combining the (4.7)–(4.9), we get

$$a(n-1)H + \phi H + 2\sum_{i=1}^{n}\phi_i h_{ii}\langle X, e_i\rangle + \Delta\phi\langle X, e_{n+1}\rangle + 3HS_3(h_{ij})$$
$$\leq 2S_2(h_{ij})^2 + 2a\langle X, e_{n+1}\rangle S_2(h_{ij}) + [a\phi + a^2]\sum_{i=1}^{n}\langle X, e_i\rangle^2. \tag{4.10}$$

Let F_A, F_{AB} be the ordinary Euclidean differentiations of function F in $\mathbb{R}^{n+1}$. Using (2.2), we compute

$$\begin{aligned}\phi_i &= -(n+1)|X|^{-(n+3)}\langle X, e_i\rangle F(X) + |X|^{-(n+1)}\sum_{A=1}^{n+1} F_A X_i^A,\\ \Delta\phi &= \sum_{i=1}^{n}\phi_{ii} = H\left[(n+1)|X|^{-(n+3)}\langle X, e_{n+1}\rangle F - |X|^{-(n+1)}\sum_{A=1}^{n+1} F_A e_{n+1}^A\right]\\ &\quad - 2(n+1)|X|^{-(n+3)}\sum_{i=1}^{n}\sum_{A=1}^{n+1}\langle X, e_i\rangle F_A X_i^A - n(n+1)|X|^{-(n+3)}F\\ &\quad + |X|^{-(n+1)}\sum_{A,B=1}^{n+1}\sum_{i=1}^{n} F_{AB} X_i^A X_i^B + (n+1)(n+3)|X|^{-(n+5)}F\sum_{i=1}^{n}\langle X, e_i\rangle^2.\end{aligned}$$

As the solution is convex,

$$S_3(h_{ij}) \geq 0, \quad 0 \leq h_{ii} \leq H.$$

If a is suitably large, we get the following mean curvature estimate:

$$\max H \leq C(n, \max_{\mathbb{S}^n} f, \min_{\mathbb{S}^n} f, |\nabla f|_{C^0}, |\nabla^2 f|_{C^0}). \tag{4.11}$$

This finishes the proof of the lemma. ■

Since the C^2 estimates in Lemma 4.1 are only valid for convex solutions, in order to carry on the method of continuity, we need to show the convexity is preserved during the process.

Theorem 4.2. Suppose M is a convex hypersurface and satisfies Equation (2.7) for $k < n$ with the second fundamental form $W = \{h_{ij}\}$ and $|X|^{\frac{n+1}{k}} f(\frac{X}{|X|})$ is convex in $\mathbb{R}^{n+1} \setminus \{0\}$; then W is positive definite. □

We now use Theorem 4.2 to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is the same as in the proof of Theorem 1.1 by the method of continuity; here we make use of Theorem 4.2. The openness and uniqueness have already treated in the proof of Theorem 1.1. The closeness follows from a priori estimates in Lemma 2.3 and quasi-linear elliptic theory in the case of $k = 1$ and the a priori estimates in Lemma 4.1 in the case of $k = 2$, and the preservation of convexity in Theorem 4.2. ■

If set $F(\kappa) = -S_k^{-frac1k}(\kappa)$, $\phi = |X|^{\frac{n+1}{k}} f^{-\frac{1}{k}}(\frac{X}{|X|})\langle X, N\rangle^{-\frac{1}{k}}$. Equation (2.7 can be written as

$$F(\kappa) = \phi, \quad \text{on } M. \tag{4.12}$$

Without the extra factor $\langle X, N\rangle^{-\frac{1}{k}}$, Theorem 4.2 would follow Theorem 1.2 in [4]. Here we cannot apply Theorem 1.2 directly. The proof of Theorem 4.2 is similar to the proof of Theorem 1.2 in [4], it relies on the following deformation lemma. A similar lemma was also proved for spherical hessian equations in [12] and for curvature equations in [11] (only with different homogeneity on the right side of the equation).

Lemma 4.3. Assume M_o is a piece of C^4 hypersurface M; M is the solution of Equation (2.7) and the matrix $W = \{h_{ij}\}$ is semipositive definite. Suppose there is a positive constant $C_o > 0$, such that for a fixed integer, $(n-1) \geq l \geq k, \forall X \in M_o, S_l(W(X)) \geq C_o$. Let $\phi(X) = S_{l+1}(W(X))$ and let $\tau(X)$ be the largest eigenvalue of $\{-(F^{-\frac{1}{k}})_{X_A X_B}(X, e_{n+1})\}$, where the differential is ordinary differential in $\mathbb{R}^{n+1}$. Then, there are constant C depending only $||X||_{C^3}, ||F||_{C^2}$ and C_o, so the following differential inequality holds at each point $X \in M_o$,

$$\sum_{\alpha,\beta} F^{\alpha\beta}\phi_{\alpha\beta} \leq k(n-l)F^{\frac{k+1}{k}} S_l(W)\tau\langle X, e_{n+1}\rangle + C(|\nabla\phi| + \phi),$$

where $F^{\alpha\beta} = \frac{\partial S_k(W)}{\partial w_{\alpha\beta}}$. □

Proof. A proof was already given in [11] for the following prescribed curvature equation:

$$S_k(\kappa_1, \kappa_2, \ldots, \kappa_n)(X) = F(X), \quad \text{on} \quad M. \tag{4.13}$$

Since we are treating a different homogeneity here, we will make a minor change in the last step of the proof in [11]. We follow the same notations as in [11] (see also [12]). For any $z \in M_o$, let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of W at z. Since $S_l(W) \geq C_o > 0$ and $M \in C^3$, for any $z \in M$, there is a positive constant $C > 0$ depending only on $||X||_{C^3}$, $||F||_{C^2}$, n and C_o, such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \geq C$. Let $G = \{1, 2, \ldots, l\}$ and $B = \{l+1, \ldots, n\}$. As $\phi = S_{l+1}(W)$ and $\phi_\alpha = \sum_{i,j} S^{ij} h_{ij\alpha}$, there is $C > 0$ such that

$$C\phi(z) \geq \sum_{i\in B} h_{ii}(z), \quad C(\phi(z) + |\phi_\alpha(z)|) \geq \sum_{i\in B} h_{ii\alpha}(z). \tag{4.14}$$

By (2.21) in [11], there is $c > 0$ such that

$$\sum_{\alpha=1}^{n} F^{\alpha\alpha}\phi_{\alpha\alpha} \leq cS_l(G)\sum_{i\in B}\left[f_{ii} - \frac{k+1}{k}\frac{f_i^2}{f}\right]. \tag{4.15}$$

Since

$$f(X, e_{n+1}) = F(X)\langle X, e_{n+1}\rangle,$$

use (2.2), so that for $\forall \quad i \in \{1, 2, \ldots, n\}$,

$$
\begin{aligned}
f_i &= \sum_{A=1}^{n+1} F_{X_A} e_i^A \langle X, e_{n+1} \rangle + F(X) h_{ii} \langle X, e_i \rangle, \\
f_{ii} &= \sum_{A,C=1}^{n+1} F_{X_A X_C} e_i^A e_i^C \langle X, e_{n+1} \rangle + \sum_{A=1}^{n+1} F_{X_A} X_{ii}^A \langle X, e_{n+1} \rangle \\
&\quad + 2 \sum_{A=1}^{n+1} F_{X_A} e_i^A h_{ii} \langle X, e_i \rangle + F(X) \left[\sum_{j=1}^{n} h_{iij} \langle X, e_j \rangle + h_{ii} - h_{ii}^2 \langle X, e_{n+1} \rangle \right].
\end{aligned}
$$

From (2.2) and (4.14), for $i \in B$ we get

$$
f_i = \sum_{A=1}^{n+1} F_{X_A} e_i^A \langle X, e_{n+1} \rangle, \quad f_{ii} = \sum_{A,C=1}^{n+1} F_{X_A X_C} e_i^A e_i^C \langle X, e_{n+1} \rangle.
$$

It follows that for $\forall \quad i \in B$,

$$
f_{ii} - \frac{k+1}{k} \frac{f_i^2}{f} \le C \sum_{A,C=1}^{n+1} \left[F_{AC} - \frac{k+1}{k} \frac{F_A F_C}{F} \right] e_i^A e_i^C \langle x, e_{n+1} \rangle. \tag{4.16}
$$

So, the lemma follows from (4.15) and (4.16). The proof of the lemma is complete.

Proof of Theorem 4.2. By the Evans–Krylov theorem and Schauder theorem, $X, e_{n+1} \in C^{4,\alpha}$. If $W = \{h_{ij}\}$ is not of full rank at some point x_o, then there is $n-1 \ge l \ge k$ such that $S_l(W(x)) > 0$, $\forall x \in M$ and $\phi(x_o) = S_{l+1}(W(x_o)) = 0$. By Lemma 4.3 and the condition on F,

$$
\sum_{\alpha,\beta}^{n} F^{\alpha\beta} \phi_{\alpha\beta}(X) \le C_1 |\nabla \phi(X)| + C_2 \phi(X). \tag{4.17}
$$

The strong minimum principle implies $\phi = S_{l+1}(W) \equiv 0$. On the other hand, M is star-shaped with respect to origin, so $\langle X, e_{n+1} \rangle > 0$, where $\langle , \rangle$ is ordinary inner product in $\mathbb{R}^{n+1}$. Since M is compact, there is a point $x \in M$ such that all the principal curvatures of M at x are positive. This is a contradiction. ■

5 The Question of C^2 Estimates for Admissible Solutions of Curvature Equations

A large part of the study of curvature measures has been carried out for convex bodies. There are some generalizations of these curvature measures to other classes of subsets in R^{n+1}. The notion of $(n-k)$th curvature measure can be naturally extended to k-convex bodies (i.e., the principal curvatures of the boundary $(\kappa_1, \kappa_2, \dots, \kappa_n) \in \Gamma_k$ at every point). Since $k < n$, an admissible solution of (1.2) is not convex in general. It is important to study the existence of admissible solutions of Equation (1.2). For $k = n$, the answer is affirmative by the solution of the Alexandrov problem. For $k = 1$, Equation (1.2) is quasi-linear, which can be solved using the method of continuity. C^1 estimates in Section 2 implies $C^{2,\alpha}$ estimates by the standard quasi-linear elliptic theory and the Schauder theorem. Lemma 2.5 guarantees the openness. The following is proved in [10]; it answers the corresponding Christoffel problem for the mean curvature measure in the admissible case. The issue of the convexity has been addressed in Theorem 1.2.

Theorem 5.1. Suppose $k = 1$, $f(x) \in C^1(\mathbb{S}^n)$, $f > 0$, then there exists a unique admissible hypersurface $M \in C^{2,\alpha}, \alpha \in (0,1)$ with positive mean curvature, such that it satisfies (1.2). □

For the intermediate cases $1 < k < n$, the existence of admissible solutions of Equation (1.2) depends on the establishment of C^2 a priori estimates which *is still open.* At this point, we would like to raise a
Question: For a given smooth positive function $g(x, t, p)$, do there exist a priori second derivative estimates for admissible solutions of equation for the general equation (1.3) on $\mathbb{S}^n$?

It is known that if $g^{1/k}(x, t, p)$ is convex in $p \in \mathbb{R}^n$, such C^2 estimates exist. This is a quite restrictive condition. The function on the right-hand side of the prescribed curvature measure equation (1.2) does not satisfy this condition. In the rest of this section, we prove C^2 estimates for Equation (1.3) in the case $k = n$. We consider the following equation:

$$\det h_{ij} = G(x, \rho, \nabla\rho), \tag{5.1}$$

where ρ is the radial function of a hypersurface $M \subset \mathbb{R}^{n+1}$, that is ρ is a positive function on $\mathbb{S}^n$. We assume C^0 estimates for this equation, our purpose is to get the C^2 estimates.

Theorem 5.2. Suppose M is a convex hypersurface and satisfies Equation (5.1), where the prescribed function $G(x, \rho, \nabla\rho)$ is a positive smooth function. Suppose $\rho(x)$ is an admissible solution with $0 < C_1 \le \rho(x) \le C_2 < \infty, x \in \mathbb{S}^n$. Then there exist a positive constant C that depends only on the constant $C_1, C_2, |G|_{C^2}$ and the lower bound depends on G, such that we have the following C^2 estimates:

$$|\rho|_{C^2} \leq C. \qquad \square$$

Since any function in $(x, \rho, \nabla\rho)$ can be written in terms of (X, N), when $k = n$, Equation (1.3) can be expressed as

$$F(h_{ij}) = \log\det h_{ij} = \log G(x, \rho, \nabla\rho) = g(X, N), \tag{5.2}$$

where $F^{ij} = \frac{\partial F}{\partial h_{ij}}, F^{ij,rs} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{rs}}$. Pick a local orthonormal frame in $\mathbf{R}^{n+1}$ such that $\{e_1, e_2, \ldots, e_n\}$ are tangent to M. We use the following two different types of indices:

$$1 \leq i, j, k \leq n, 1 \leq A, B, C \leq n+1.$$

For any point $X_o \in M$, if we choose a local orthonormal frame with diagonal second fundamental form $(h_{ij}(X_o))$, then at this point we have the following:

$$D_l g = g_{X^A} {e_l}^A + g_{N^B}(N^B)_l = g_{X^A} {e_l}^A + g_{N^B} h_{ll} {e_l}^B, \tag{5.3}$$

where $D_l g$ is the covariant derivative with respect to e_l, and $g_{X^A} = \frac{\partial g}{\partial X^A}, g_{N^A} = \frac{\partial g}{\partial N^A}$, etc. As in last section at X_o, we get

$$\begin{aligned} D_{11} g &= \left(g_{X^A} {e_1}^A\right)_1 + \left(g_{N^B}(N^B)_1\right)_1 \\ &= g_{X^A X^B} {e_1}^A {e_1}^B + 2 g_{X^A N^B} {e_1}^A (N^B)_1 + g_{X^A}\left({e_1}^A\right)_1 + g_{N^A N^B}(N^A)_1 (N^B)_1 + g_{N^B}(N^B)_{11} \\ &= g_{X^A X^B} {e_1}^A {e_1}^B + +2 h_{11} g_{X^A N^B} {e_1}^A {e_1}^B - h_{11} g_{X^A} N^A + h_{11}^2 g_{N^A N^B} {e_1}^A {e_1}^B \\ &\quad + g_{N^B}\left(h_{11j} {e_j}^B - h_{11}^2 N^B\right) = h_{11}^2\left[g_{N^A N^B} {e_1}^A {e_1}^B - g_{N^B} N^B\right] \\ &\quad + h_{11}\left[2 g_{X^A N^B} {e_1}^A {e_1}^B - g_{X^A} N^A\right] + g_{X^A X^B} {e_1}^A {e_1}^B + g_{N^B} h_{11j} {e_j}^B. \end{aligned} \tag{5.4}$$

Differentiating Equation (5.2), we have

$$F^{ij} h_{ijl} = D_l g, \tag{5.5}$$

$$F^{ij} h_{ij11} = D_{11} g - F^{ij,rs} h_{ij1} h_{rs1}. \tag{5.6}$$

Proof of Theorem 5.2. We only need to establish the curvature estimates. Let

$$P(X,\xi) = \log(h_{kl}\xi_k\xi_l)(X) - \alpha\langle X, N\rangle,$$

where $\xi \in \mathbb{S}^n$ and α is a positive constant that will be determined later. Suppose that $P(X,\xi)$ attains its maximum at some $X_o \in M$ and $\xi_0 \in \mathbb{S}^n$. We may assume ξ_o is e_1 and the other directions $e_2, \ldots, e_n$ can be chosen such that $(e_1, e_2, \ldots, e_n)$ is a local orthonormal frame near X_o and $h_{ij}(X_o)$ is diagonal. Then the function

$$P(x) = \log h_{11} - \alpha\langle X, N\rangle \tag{5.7}$$

attains its maximum at $X_o \in M$. At the point X_o, we have

$$P_i(X) = \frac{h_{11i}}{h_{11}} - \alpha\langle X, N\rangle_i.$$

So at X_o,

$$\frac{h_{11i}}{h_{11}} = \alpha h_{ii}\langle X, e_i\rangle, \quad P_{ii}(X) = \frac{h_{11ii}}{h_{11}} - \frac{h_{11i}^2}{h_{11}^2} - \alpha\langle X, N\rangle_{ii}. \tag{5.8}$$

By (5.4) and (5.6),

$$\begin{aligned}
h_{11}\sum_i F^{ii}P_{ii}(X) &= F^{ii}\big[h_{ii11} + h_{ii}h_{11}^2 - h_{11}h_{ii}^2\big] - \frac{1}{h_{11}}F^{ii}h_{11i}^2 \\
&\quad -\alpha h_{11}F^{ii}\big[h_{iij}\langle X, e_j\rangle + h_{ii} - h_{ii}^2\langle X, N\rangle\big] \\
&= D_{11}g - \alpha h_{11}F^{ii}h_{iij}\langle X, e_j\rangle + [\alpha\langle X, N\rangle - 1]Hh_{11} \\
&\quad + h_{11}^2 - n\alpha h_{11} - F^{ij,rs}h_{ij1}h_{rs1} - \frac{1}{h_{11}}F^{ii}h_{11i}^2 \\
&= [\alpha\langle X, N\rangle - 1]Hh_{11} + h_{11}^2[1 + g_{N^AN^B}{e_1}^A{e_1}^B - g_{N^B}N^B] \\
&\quad + h_{11}[2g_{X^AN^B}{e_1}^A{e_1}^B - g_{X^A}N^A - n\alpha] + g_{X^AX^B}{e_1}^A{e_1}^B \\
&\quad + g_{N^B}h_{11j}{e_j}^B - \alpha h_{11}F^{ii}h_{iij}\langle X, e_j\rangle \\
&\quad - F^{ij,rs}h_{ij1}h_{rs1} - \frac{1}{h_{11}}F^{ii}h_{11i}^2 \\
&\le 0.
\end{aligned} \tag{5.9}$$

From (5.3) and (5.8),

$$\begin{aligned} g_{N^B} h_{11j} e_j{}^B - \alpha h_{11} F^{ii} h_{iij} \langle X, e_j \rangle &= g_{N^B} h_{11j} e_j{}^B - \alpha h_{11} D_j g \langle X, e_j \rangle \\ &= g_{N^B} h_{11j} e_j{}^B - \alpha h_{11} [g_{X^A} e_j{}^A + g_{N^B} h_{jj} e_j{}^B] \langle X, e_j \rangle \\ &= g_{N^B} e_j{}^B [h_{11j} - \alpha h_{11} h_{jj} \langle X, e_j \rangle] - \alpha h_{11} g_{X^A} e_j{}^A \langle X, e_j \rangle \\ &= -\alpha h_{11} g_{X^A} e_j{}^A \langle X, e_j \rangle. \end{aligned} \tag{5.10}$$

For $F = \log\det$,

$$-F^{ij,rs} h_{ij1} h_{rs1} - \frac{1}{h_{11}} F^{ii} h_{11i}^2 \geq 0. \tag{5.11}$$

Combining (5.9)–(5.11), we get the following crucial inequality:

$$\begin{aligned} &[\alpha \langle X, N \rangle - 1] H h_{11} + h_{11}^2 [1 + g_{N^A N^B} e_1{}^A e_1{}^B - g_{N^B} N^B] \\ &\quad + h_{11} [2 g_{X^A N^B} e_1{}^A e_1{}^B - g_{X^A} N^A - n\alpha - \alpha g_{X^A} e_j{}^A \langle X, e_j \rangle] + g_{X^A X^B} e_1{}^A e_1{}^B \leq 0. \end{aligned} \tag{5.12}$$

By the assumption $0 < C_1 \leq \rho(x) \leq C_2 < \infty, \forall x \in \mathbb{S}^n$, we have

$$0 < C_3 \leq \langle X, N \rangle \leq C_4 < \infty.$$

(5.12) yields $h_{11}(X_o) \leq C$, if α large enough. ■

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