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A Brunn–Minkowski inequality for the Hessian eigenvalue in three-dimensional convex domain ☆

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Abstract

We use the deformation methods to obtain the strictly log concavity of solution of a class Hessian equation in bounded convex domain in $\mathbb{R}^3$, as an application we get the Brunn–Minkowski inequality for the Hessian eigenvalue and characterize the equality case in bounded strictly convex domain in $\mathbb{R}^3$.

Keywords: Constant rank theorem; Hessian equation; Eigenvalue; Brunn–Minkowski inequality

1. Introduction

The convexity is an issue of interest for a long time in partial differential equation. It connects the geometric properties to analysis inequalities. In 1976, Brascamp and Lieb [2] establish the log-concavity of the fundamental solution of diffusion equation with convex potential in bounded convex domain in $\mathbb{R}^n$. As a consequence, they proved the log-concavity of the first eigenfunction

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of Laplace equation in convex domains. At the same time, they obtained the Brunn–Minkowski inequality for the first eigenvalue as following:

$$\lambda\big((1-t)K_0+tK_1\big)^{-\frac{1}{2}} \geqslant (1-t)\lambda(K_0)^{-\frac{1}{2}}+t\lambda(K_1)^{-\frac{1}{2}}, \tag{1.1}$$

where $t\in[0,1]$, K_0, K_1 are nonempty convex bodies in $\mathbb{R}^n$. In fact, in that paper they proved that this inequality holds for all compact connected domain having sufficiently regular boundary. One always is interesting on the equality case. For example, Jerison [7] pointed out that it is related to uniqueness of the solution for the Minkowski problem about λ. In [5], Colesanti provides a new proof of (1.1) for convex bodies, and essentially tells us that equality holds if and only if K_0 is homothetic to K_1. At the same time, he asked whether the same kind of result holds for a class of fully nonlinear elliptic operator called Hessian operators other than Laplace operator.

If we consider the following eigenvalue problems for bounded strict convex domain $K\subset R^n$ with C^∞ boundary,

$$\begin{cases} S_k\big(D^2u\big)=\lambda(K)(-u)^k, & u<0 \quad \text{in } K,\\ u=0 \quad \text{on } \partial K, \end{cases} \tag{1.2}$$

where S_k is the so-called Hessian operators. For $k=1,\dots,n$, and a C^2 function u, the k-th Hessian operator $S_k(D^2u)$ is the k-th elementary symmetric function of the eigenvalues of the Hessian matrix of u. Also if u satisfies $S_i(D^2u)>0$ for all $1\leqslant i\leqslant k$, then we call u an admissible solution of (1.2) (see for example [4]).

Equivalently we can define

$$\lambda(K)=\inf\left\{-\frac{\int_K uS_k(D^2u)\,dx}{\int_K |u|^{k+1}\,dx}\right\}, \tag{1.3}$$

where the inf is taken over the functions $u\in C^2(K)\cap C(\bar{K})$, admissible and $u=0$ on ∂K. Obviously this functional $\lambda(K)$ is homogeneous of order $-2k$.

Note that when $k=1$, the last equation (1.2) is corresponding to the Laplace operator, the Brunn–Minkowski inequality for $\lambda(K)$ is just (1.1). When $k=n$, Eq. (1.2) is corresponding to the Monge–Ampère operator, the Brunn–Minkowski inequality of the $\lambda(K)$ had obtained in Salani [14], i.e. $\lambda^{-\frac{1}{2n}}(K)$ is concave in K, and the equality holds if and only if K_0 is homothetic to K_1.

Wang [16] proved for $1<k<n$, up to a positive factor, for Eq. (1.2) exists a unique negative admissible solution $u\in C^\infty(K)\cap C^{1,1}(\bar{K})$. Furthermore Eq. (1.2) exists exactly one positive eigenvalue in a convex domain with smooth boundary (in fact, the result by Wang is true for a larger class of domains). In this paper, we use Wang's result to deal with the case $k=2$ in 3-dimensional convex domain.

Theorem 1.1. *Suppose K is a bounded smooth strict convex domain in $\mathbb{R}^3$, and $u\in C^\infty(K)\cap C^{1,1}(\bar{K})$ is the unique (up to a positive factor) admissible solution of*

$$\begin{cases} S_2\big(D^2u\big)=\lambda(K)(-u)^2, & u<0 \quad \textit{in } K,\\ u=0 \quad \textit{on } \partial K, \end{cases} \tag{1.4}$$

then $v=-\log(-u)$ is a strictly convex function in K.

With this result we will prove the Brunn–Minkowski inequality for the positive eigenvalue of S_2 operator. Our method is from Colesanti [5] and Salani [14].

Theorem 1.2. *Suppose K_0, K_1 are bounded smooth strict convex domains in $\mathbb{R}^3$, and $t \in [0,1]$, then the functional λ satisfies the inequality:*

$$\lambda\big((1-t)K_0 + tK_1\big)^{-\frac{1}{4}} \geqslant (1-t)\lambda(K_0)^{-\frac{1}{4}} + t\lambda(K_1)^{-\frac{1}{4}}. \tag{1.5}$$

Moreover equality holds if and only if K_0 is homothetic to K_1.

The plan of the paper is as follows. In Section 2, we prove the Hessian of v has constant rank if the function v in Theorem 1.1 is convex, this is essential for our paper. In Section 3, combing the boundary estimates we use the deformation process to get the function v is strict convex, then we complete the proof of Theorem 1.1. In the last section, we prove the Brunn–Minkowski inequality and characterize the equality case.

2. A constant rank theorem

In this section we establish a constant rank theorem for the convex solution of the related nonlinear elliptic equation.

In what follows, $\mathcal{S}^n$ denotes the set of the symmetric $n \times n$ matrices, and $\mathcal{S}^n_+ (\mathcal{S}^n_{++})$ is the subset of the semipositive (positive) definite matrices.

Let $K \subset \mathbb{R}^3$ be any bounded domain. Note that if let $v = -\log(-u)$, then Eq. (1.4) is equivalent to

$$\begin{cases} S_2\big(D^2 v\big) - Tr\big(P(Dv)D^2 v\big) = \lambda(K) & \text{in } K, \\ v(x) \to +\infty, \quad x \to \partial K, \end{cases} \tag{2.1}$$

where $Tr(A)$ denotes the trace of matrix A, and $P(\nabla v) = (P_{ij})$ is a matrix with

$$P_{ij} = |\nabla v|^2 \delta_{ij} - v_i v_j, \quad i, j = 1, 2, 3.$$

It follows that P is semipositive definite, and $Tr(PD^2 v) = \sum_{i,j=1}^3 P_{ij} v_{ij}$. By a simple observation, if u is an admissible solution for Eq. (1.4), then v is an admissible solution for Eq. (2.1).

One of main ingredients of the proof of Theorem 1.1 is a constant rank theorem, which states as follows (see e.g. [3] and [9]). Some related results have been obtained in [11].

Theorem 2.1. *Suppose v is a $C^4(K)$ admissible solution of* (2.1)*, and the Hessian matrix $\{v_{ij}\}$ of v is semipositive definite in $K \subset \mathbb{R}^3$, then $\{v_{ij}\}$ must have constant rank in K, where K is any domain.*

Proof. Let v be an admissible solution of Eq. (2.1). Let $W = \{v_{ij}\}$ denote the Hessian matrix of v. Since v is admissible, from Eq. (2.1) the rank of W can be only 2 or 3.

Suppose now $z_0 \in K$ is a point in which $W(z_0)$ is of minimal rank 2, we shall show W is of constant rank 2 in K.

Let $\phi(z) = \det W(z)$, and define $U = \{z \in K \mid \phi(z) = 0\}$. We will show that $U = K$, which means W is of constant rank 2 in K. First, U is not empty, since clearly $z_0 \in U$. Note that, by the definition of U, U is a closed set in K. We shall show that there is a small open neighborhood O of z_0 in U, and $\phi(z) \equiv 0$ in O which implies the set U is an open set in K. Since K is connected, we must have $U = K$.

Following the notations in [3] and [9], for two functions h, k defined in an open set $O \subset K$. Let $y \in O$, we say that $h(y) \lesssim k(y)$ if there exist positive constants c_1 and c_2 such that

$$(h-k)(y) \leqslant \big(c_1|\nabla\phi| + c_2\phi\big)(y). \tag{2.2}$$

We write $h(y) \sim k(y)$ if both $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$ hold. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constant c_1, and c_2 independent of y in this neighborhood. Finally, say $h \sim k$ if $h \lesssim k$ and $k \lesssim h$.

Now let $F = F(Dv, D^2v) := S_2(D^2v) - Tr(P(Dv)D^2v)$, and we shall use the following notations:

$$F^{ij} = \frac{\partial F}{\partial v_{ij}}, \qquad F_{v_l} = \frac{\partial F}{\partial v_l},$$

$$F^{ij,rs} = \frac{\partial^2 F}{\partial v_{ij}\partial v_{rs}}, \qquad F^{ij}_{v_l} = \frac{\partial^2 F}{\partial v_{ij}\partial v_l}, \qquad F_{v_k v_l} = \frac{\partial^2 F}{\partial v_k \partial v_l}.$$

We shall show that

$$\sum_{i,j=1}^{3} F^{ij}\phi_{ij} \lesssim 0, \tag{2.3}$$

in an open small neighborhood O of z_0.

Since $\phi \geqslant 0$ in K and $\phi(z_0) = 0$, it then follows from the strong minimum principle that $\phi(z) \equiv 0$ in O. In order to prove (2.3) at an arbitrary point $z \in O$, as in Caffarelli–Friedman [3], we choose the normal coordinate, i.e. we perform a rotation T_z about z so that in the new coordinates W is diagonal at z and $v_{11} \geqslant v_{22} \geqslant v_{33}$ at z. Consequently we can choose T_z to vary smoothly with z. If we can establish (2.3) at z under the assumption that W is diagonal at z, then go back to the original coordinates we find that (2.3) remains valid with new coefficients c_1, c_2 in (2.2), depending smoothly on the independent variable. Thus it remains to establish (2.3) under the assumption that W is diagonal at z.

Since rank is at least 2, there exists a positive constant C, which depends only on $\|v\|_{C^4}$, such that $v_{11} \geqslant v_{22} \geqslant C$ at z. In the following, all calculations are working at the point z using the notation "$\lesssim$", with the understanding that the constants in (2.3) are under control.

Next we compute ϕ, its first and second derivatives in the directions e_i, e_j, we find

$$0 \sim \phi \sim v_{33}, \tag{2.4}$$

$$0 \sim \phi_i \sim v_{33i}, \tag{2.5}$$

and

$$\phi_{ij} \sim v_{11}v_{22}v_{33ij} - 2v_{11}v_{23i}v_{23j} - 2v_{22}v_{13i}v_{13j}. \tag{2.6}$$

Differentiating Eq. (2.1) along e_3 once we get

$$F^{ij}v_{ij3}+F_{v_l}v_{l3}=0. \tag{2.7}$$

In this paper, the summation convention over repeated indices will be employed.
From (2.4)–(2.5), and since (v_{ij}) is diagonal at z, one can see

$$F^{11}v_{113}+F^{22}v_{223}+2F^{12}v_{123}\sim 0. \tag{2.8}$$

Differentiating Eq. (2.1) along e_3 twice to get

$$F^{ij}v_{ij33}+F^{ij,rs}v_{ij3}v_{rs3}+2F^{ij}_{v_l}v_{ij3}v_{l3}+F_{v_l}v_{l33}+F_{v_lv_s}v_{l3}v_{s3}=0, \tag{2.9}$$

it follows that

$$F^{ij}v_{ij33}\sim -F^{ij,rs}v_{ij3}v_{rs3},$$

which together with (2.6) imply

$$\frac{F^{ij}\phi_{ij}}{v_{11}v_{22}}\sim -F^{ij,rs}v_{ij3}v_{rs3}-\frac{2}{v_{11}}F^{ij}v_{13i}v_{13j}-\frac{2}{v_{22}}F^{ij}v_{23i}v_{23j}. \tag{2.10}$$

Now we compute the partial derivatives of F along (v_{ij}), it follows that

$$F^{11}\sim v_{22}-\left(v_2^2+v_3^2\right),\qquad F^{22}\sim v_{11}-\left(v_1^2+v_3^2\right),\qquad F^{12}=F^{21}\sim v_1v_2, \tag{2.11}$$

$$F^{11,22}=F^{22,11}=1,\qquad F^{12,21}=F^{21,12}=-1. \tag{2.12}$$

Hence we have

$$\begin{aligned}\frac{F^{ij}\phi_{ij}}{v_{11}v_{22}}\sim{}& -2v_{113}v_{223}+2v_{123}^2\\ &-\frac{2}{v_{11}}\left(F^{11}v_{113}^2+F^{22}v_{123}^2+2F^{12}v_{113}v_{123}\right)\\ &-\frac{2}{v_{22}}\left(F^{11}v_{123}^2+F^{22}v_{223}^2+2F^{12}v_{223}v_{123}\right).\end{aligned} \tag{2.13}$$

We want to prove

$$\frac{F^{ij}\phi_{ij}}{v_{11}v_{22}}\lesssim 0.$$

It comes out two possibilities.

Case 1. If $F^{12}v_{123}\neq 0$, put (2.8) into (2.13) to substitute $2F^{12}v_{123}$ above, then

$$\begin{aligned}
\frac{F^{ij}\phi_{ij}}{v_{11}v_{22}} &\sim -2v_{113}v_{223}+2v_{123}^2-\frac{2F^{11}}{v_{11}}v_{113}^2-\frac{2F^{22}}{v_{22}}v_{223}^2\\
&\quad-\left(\frac{2F^{22}}{v_{11}}+\frac{2F^{11}}{v_{22}}\right)v_{123}^2\\
&\quad+\left(\frac{2}{v_{11}}v_{113}+\frac{2}{v_{22}}v_{223}\right)\left(F^{11}v_{113}+F^{22}v_{223}\right)\\
&\sim 2\left(1-\frac{F^{22}}{v_{11}}-\frac{F^{11}}{v_{22}}\right)\left(v_{123}^2-v_{113}v_{223}\right)\\
&\sim -2\frac{v_{11}v_{22}-(v_1^2+v_3^2)v_{22}-(v_2^2+v_3^2)v_{11}}{v_{11}v_{22}}\left(v_{123}^2-v_{113}v_{223}\right)\\
&\sim \frac{-2\lambda}{v_{11}v_{22}}\left(v_{123}^2-v_{113}v_{223}\right). \qquad (2.14)
\end{aligned}$$

The last "$\sim$" comes from the original equation

$$\lambda=F\sim v_{11}v_{22}-\left(v_1^2+v_3^2\right)v_{22}-\left(v_2^2+v_3^2\right)v_{11}.$$

If $v_{113}v_{223}\leqslant 0$, then the proof of (2.3) is concluded.
While if $v_{113}v_{223}>0$, by using (2.8) and (2.11) again we have

$$\begin{aligned}
4v_1^2v_2^2v_{123}^2 &\sim \left(F^{11}v_{113}+F^{22}v_{223}\right)^2\\
&\gtrsim 4F^{11}F^{22}v_{113}v_{223}\\
&\gtrsim 4\left[\lambda+\left(v_2^2+v_3^2\right)\left(v_1^2+v_3^2\right)\right]v_{113}v_{223}\\
&\gtrsim 4v_1^2v_2^2v_{113}v_{223}. \qquad (2.15)
\end{aligned}$$

Recall (2.14), this implies (2.3) true.

Case 2. If $F^{12}v_{123}=0$, we still use (2.8) to substitute v_{223} in (2.13), then

$$\begin{aligned}
\frac{F^{ij}\phi_{ij}}{v_{11}v_{22}} &\sim 2\left(1-\frac{F^{22}}{v_{11}}-\frac{F^{11}}{v_{22}}\right)v_{123}^2\\
&\quad-\frac{2(F^{11}v_{11}+F^{22}v_{22}-v_{11}v_{22})F^{11}}{F^{22}v_{11}v_{22}}v_{113}^2\\
&\sim \frac{-2\lambda}{v_{11}v_{22}}\left(v_{123}^2+\frac{F^{11}}{F^{22}}v_{113}^2\right)\\
&\lesssim 0. \qquad (2.16)
\end{aligned}$$

So the proof of (2.3) is concluded. Then proof of Theorem 2.1 is completed. □

3. Proof of Theorem 1.1

In this section, we shall use the deformation process to prove Theorem 1.1. The constant rank theorem is a very powerful tool to produce strict convex solution for nonlinear elliptic equation, see for example Caffarelli and Friedman [3], Korevaar and Lewis [9] and Guan and Ma [6]. Here we follow the approach of Korevaar and Lewis [9] and Ma and Xu [11] to get the result. First we need understand the radial solution of Eq. (1.4) defined on ball. Then we study the geometrical properties of the solution near the convex boundary.

The following lemma is well known, for completeness we give the proof along the idea of McCuan [12] (see pp. 172–173 in [12]).

Lemma 3.1. *Let $B_R(o)$ be unit ball in R^3 with radius $R > 0$. Let $u \in C^\infty(B_R) \cap C^{1,1}(\bar{B_R})$ be the eigenfunction for Eq.* (1.4) *in $B_R(o)$. Then $v = -\log(-u)$ is a strict convex function in $B_R(o)$.*

Proof. By the uniqueness of solution for Eq. (1.4) up to a constant factor, we know the solution u is a radial function. We set

$$u(x) = \varphi\big(|x|\big) = \varphi(r), \quad \text{for } r = |x|,$$

where $r \in [0, R]$, $\varphi(r) < 0$ for $r \in [0, R)$. Then φ is an increasing function in $(0, R)$ and $\varphi'(0) = \varphi(R) = 0$.

Since

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r},$$
$$\frac{\partial^2 r}{\partial x_i \partial x_j} = -r^{-3} x_i x_j + r^{-1}\delta_{ij},$$

it follows that

$$u_{ij} = \big(\varphi'' r^{-2} - \varphi' r^{-3}\big) x_i x_j + \varphi' r^{-1} \delta_{ij},$$

and

$$S_2\big(D^2 u\big) = 2\varphi'\varphi'' r^{-1} + \big(\varphi'\big)^2 r^{-2}. \tag{3.1}$$

For $v = -\log(-\varphi)$, we have

$$\varphi' = e^{-v} v',$$
$$\varphi'' = e^{-v}\big[v'' - \big(v'\big)^2\big].$$

So Eq. (1.4) on u transforms to the following equation on v.

$$2rv'v'' - 2r\big(v'\big)^3 + \big(v'\big)^2 = \lambda r^2 \quad \text{for } 0 < r < R. \tag{3.2}$$

It follows that

$$v'(0)=0, \qquad v'(r)>0 \quad \text{for } 0<r<R,$$

and

$$v''(0)\geqslant 0. \tag{3.3}$$

Now let $L=\lim_{r\to 0^+}[\frac{v'(r)}{r}]$, then by L'Hopital rule provides that

$$L=\lim_{r\to 0^+} v''(r)=v''(0). \tag{3.4}$$

From (3.2)–(3.4), it follows that

$$v''(0)=\sqrt{\frac{\lambda}{3}}>0. \tag{3.5}$$

We assume by contradiction the existence of a smallest positive r_o for which $v''(r_o)=0$. We know $v'(r)>0$ for $0<r\leqslant R$ and $v''(r)>0$ for $0<r<r_o$. Differentiating (3.2) and evaluating at $r=r_o$, we obtain

$$v'''(r_o)=\frac{\lambda}{v'(r_o)}+\frac{(v'(r_o))^2}{r_o}>0, \tag{3.6}$$

which contradicts the sign of v''.

Hence v is strictly convex in $[0,R)$ and the lemma is proven. □

Now we state the following well-known boundary convexity lemma, see for example Caffarelli and Friedman [3] (p. 450, Lemma 4.3) or Korevaar [8] (p. 610, Lemma 2.4).

Lemma 3.2. *(See [3] or [8].) Let $\Omega\subset\mathbb{R}^n$ be smooth, bounded and strictly convex (i.e. all the principal curvature of $\partial\Omega$ are positive). Let $u\in C^\infty(\Omega)\cap C^{1,1}(\bar\Omega)$ satisfies*

$$u<0 \quad \text{in } \Omega, \qquad u=0 \quad \text{and} \quad Du\cdot\nu>0 \quad \text{on } \partial\Omega, \tag{3.7}$$

where ν is the exterior normal to $\partial\Omega$. Let

$$\Omega_\varepsilon=\{x\in\Omega\colon\, d(x,\partial\Omega)>\varepsilon\} \tag{3.8}$$

and let $v=f(u)$. Then for small enough $\varepsilon>0$ the function v is strictly convex in a boundary strip $\Omega\backslash\Omega_\varepsilon$ if f satisfies

$$\text{(i)} \quad f'>0, \qquad \text{(ii)} \quad f''>0, \qquad \text{(iii)} \quad \lim_{u\to 0^-}\frac{f'}{f''}=0. \tag{3.9}$$

Remark 3.3. In Korevaar [8], the function $u\in C^2(\bar\Omega)$. But we can follow the calculation in Caffarelli and Friedman [3] to know the similar result is true in our case $u\in C^\infty(\Omega)\cap C^{1,1}(\bar\Omega)$.

Now we use the deformation technique combined with Theorem 2.1 (constant rank theorem) to obtain the proof of Theorem 1.1 as in Korevaar and Lewis [9] and Ma and Xu [11]. For completeness we repeat partly their proof.

Proof of Theorem 1.1. Now if K is the ball $B_R(o)$, by Lemma 3.1, for the solution u of (1.4), we have $v = -\log(-u)$ is a strict convex function in $B_R(o)$. For an arbitrary bounded strict convex domain K, set $K_t = (1-t)B_R(o) + tK, 0 \leqslant t \leqslant 1$. Then from the theory of convex bodies (see for example Sections 1.7, 1.8 and 2.5 in the book [15], and Section 3.1 in the book [17]). We can deform $B_R(o)$ continuously into K by the family (K_t), $0 \leqslant t < 1$, of strictly convex domain in such a way that $\partial K_t \to \partial K_s$ as $t \to s$ in the sense of Hausdorff distance, whenever $0 \leqslant s \leqslant 1$. And the deformation also is chosen so that $\partial K_t, 0 \leqslant t < 1$, can be locally represented for some α, $0 < \alpha < 1$, by a function whose norm in the space $C^{2,\alpha}$ of functions with Hölder continuous second derivatives depends only on δ, whenever $0 < t \leqslant \delta < 1$.

Suppose $u_t \in C^\infty(K_t) \cap C^{1,1}(\bar{K}_t)$ is the admissible solution of (1.4), $v_t := -\log(-u_t)$ and H_t is the corresponding Hessian matrix of v_t. First H_0 is positive definite, and from the boundary estimates (Lemma 3.2) we have H_δ is positive definite in an ε neighborhood of ∂K_δ. From the a priori estimates of the solution u on the Hessian equation [16], we know this bounded depends only on the uniformly bounded geometry of K_t which depends on the geometry K and t. We conclude that if $v(.,s)$ is strictly convex for all $0 \leqslant s < t$, then $v(.,t)$ is convex.

So if for some $\delta, 0 < \delta < 1$, H_δ is positive semi-definite but not positive definite in K_δ, we say it is impossible by constant rank theorem (Theorem 2.1) and boundary estimates (Lemma 3.2). We conclude H_δ is positive definite. Then $v = -\log(-u)$ is strictly convex in K. □

4. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2.

Now we state some element propositions on the convexity of the matrix functions. First we recall Jensen's inequality for means (see [2]). If a, b are real positive numbers, $\alpha \in [-\infty, +\infty]$ and $\lambda \in (0,1)$, we define

$$m_\alpha(a,b,\lambda) = \begin{cases} [(1-\lambda)a^\alpha + \lambda b^\alpha]^{1/\alpha} & \text{if } \alpha \in (-\infty,0)\cup(0,+\infty), \\ \min(a,b) & \text{if } \alpha = -\infty, \\ a^{1-\lambda}b^\lambda & \text{if } \alpha = 0, \\ \max(a,b) & \text{if } \alpha = +\infty. \end{cases}$$

Jensen's inequality for means implies that

$$m_\alpha(a,b,\lambda) \leqslant m_\beta(a,b,\lambda) \quad \text{if } \alpha \leqslant \beta. \tag{4.1}$$

In particular, the *arithmetic–geometric mean inequality* holds

$$a^{1-\lambda}b^\lambda \leqslant (1-\lambda)a + \lambda b \quad \text{for every } a, b \geqslant 0,\ \lambda \in [0,1].$$

Lemma 4.1. *If $f(x)$ is a positive concave function in $\mathbb{R}^n$, then f^{-1} is convex.*

Proof. Note that the condition of f means

$$f\big((1-t)x+ty\big)\geqslant(1-t)f(x)+tf(y),\quad \forall x,y\in\mathbb{R}^n,\ t\in[0,1],\tag{4.2}$$

thus we have

$$\begin{aligned}f\big((1-t)x+ty\big)^{-1}&\leqslant\big[(1-t)f(x)+tf(y)\big]^{-1}\\&=\big[(1-t)\big(f(x)^{-1}\big)^{-1}+t\big(f(y)^{-1}\big)^{-1}\big]^{-1}\\&\leqslant(1-t)f(x)^{-1}+tf(y)^{-1},\end{aligned}\tag{4.3}$$

where the last inequality comes from Jensen's inequality above. This says that f^{-1} is convex. □

Remark 4.2. Note that in the above if f^{-1} is not strictly convex, i.e. in (4.3)

$$f\big((1-t)x+ty\big)^{-1}=(1-t)f(x)^{-1}+tf(y)^{-1},\tag{4.4}$$

then the two equalities in (4.3) must hold at the same time, which means f is not strictly concave then either

$$f\big((1-t)x+ty\big)=(1-t)f(x)+tf(y)\tag{4.5}$$

or

$$\big[(1-t)\big(f(x)^{-1}\big)^{-1}+t\big(f(y)^{-1}\big)^{-1}\big]^{-1}=(1-t)f(x)^{-1}+tf(y)^{-1}.\tag{4.6}$$

Proposition 4.3. *As in Section 2, we let $P(\nabla v)=(P_{ij})$ be a matrix with $P_{ij}=|\nabla v|^2\delta_{ij}-v_iv_j$, $i,j=1,2,3$, and we let $Tr(PA)=\sum_{i,j=1}^3 P_{ij}a_{ij}$ for $A=(a_{ij})$. If $|Dv|\neq 0$, then the function $f(A):=\frac{S_2(A^{-1})}{Tr(PA^{-1})}$ is convex in $A\in\mathcal{S}^3_{++}$, and $\frac{1}{Tr(PA^{-1})}$ is concave in $A\in\mathcal{S}^3_{++}$.*

Proof. $\frac{1}{Tr(PA^{-1})}$ is concave in $A\in\mathcal{S}^3_{++}$ from the appendix in [1]. Now we concentrate the proof of the first part.

By Lemma 4.1, it is sufficiently to prove that $f(A)^{-1}=\frac{Tr(PA^{-1})}{S_2(A^{-1})}$ is concave in A. Since $P\in\mathcal{S}^3_{+}$, so it can be written as $O\Delta_P O^T$ with $OO^T=I$ and Δ_P, the diagonal matrix of the eigenvalues of P. Then we have

$$\begin{aligned}\frac{Tr(PA^{-1})}{S_2(A^{-1})}&=\frac{Tr(O\Delta_P O^TA^{-1})}{S_2(A^{-1})}\\&=\frac{Tr(\Delta_P O^TA^{-1}O)}{S_2(O^TA^{-1}O)}\\&=\frac{Tr(\Delta_P\tilde{A}^{-1})}{S_2(\tilde{A}^{-1})}.\end{aligned}\tag{4.7}$$

Hence without loss of generality we may assume $P=\Delta_P$ is a diagonal matrix, in our case, which is diagonal$(|\nabla v|^2,|\nabla v|^2,0)$. We are going to show that

$$\frac{Tr(P(A+B)^{-1})}{S_2((A+B)^{-1})} \geqslant \frac{Tr(PA^{-1})}{S_2(A^{-1})} + \frac{Tr(PB^{-1})}{S_2(B^{-1})}, \tag{4.8}$$

for any $A, B \in \mathcal{S}^3_{++}$. In fact, it follows from a direct calculation that

$$\begin{aligned}\frac{Tr(PA^{-1})}{S_2(A^{-1})} &= \frac{P_{11}(A_{22}A_{33} - A_{23}A_{32}) + P_{22}(A_{11}A_{33} - A_{13}A_{31}) + P_{33}(A_{11}A_{22} - A_{12}A_{21})}{A_{11} + A_{22} + A_{33}} \\ &= \frac{P_{11}A_{22}A_{33} + P_{22}A_{11}A_{33} + P_{33}A_{11}A_{22}}{TrA} \\ &\quad - \frac{P_{11}A_{23}A_{32} + P_{22}A_{13}A_{31} + P_{33}A_{12}A_{21}}{TrA},\end{aligned}$$

then we have

$$\begin{aligned}&\frac{Tr(P(A+B)^{-1})}{S_2((A+B)^{-1})} - \frac{Tr(PA^{-1})}{S_2(A^{-1})} - \frac{Tr(PB^{-1})}{S_2(B^{-1})} \\ &\quad = \sum_{i=1}^{3} \frac{C_i[A_{ii}TrB - B_{ii}TrA]^2}{TrATrBTr(A+B)} + \frac{P_{11}[A_{23}TrB - B_{23}TrA]^2}{TrATrBTr(A+B)} \\ &\quad\quad + \frac{P_{22}[A_{13}TrB - B_{13}TrA]^2}{TrATrBTr(A+B)} + \frac{P_{33}[A_{12}TrB - B_{12}TrA]^2}{TrATrBTr(A+B)} \\ &\quad \geqslant 0,\end{aligned}$$

where $C_i = \frac{1}{2}\sum_{j=1}^{3} P_{jj} - P_{ii} \geqslant 0$. □

Remark 4.4. Note that actually we have $C_1 = C_2 = P_{33} = 0$, $C_3 = P_{11} = P_{22} = |\nabla v|^2 \neq 0$, and the equality in (4.8) holds if and only if

$$\frac{TrA}{TrB} = \frac{A_{33}}{B_{33}} = \frac{A_{13}}{B_{13}} = \frac{A_{23}}{B_{23}}. \tag{4.9}$$

Now we state another useful result.

Proposition 4.5. *$S_2(A^{-1})^{\frac{1}{2}}$ is convex in $A \in \mathcal{S}^3_{++}$.*

Proof. As a special case of Theorem 15.16 in [10]. □

Along the ideas of Colesanti [5] and Salani [14], now we will prove Theorem 1.2.

Proof of Theorem 1.2. For $i = 0, 1$, let K_i be a convex domain in $\mathbb{R}^3$, and let u_i be the solution of

$$\begin{cases} S_2(D^2 u_i) = \lambda(K_i)(-u_i)^2, & u_i < 0 \quad \text{in } \operatorname{int}(K_i), \\ u_i = 0 \quad \text{on } \partial K_i, \end{cases}$$

then the function $v_i(x) = -\log(-u_i(x))$ solves

$$\begin{cases} S_2\big(D^2 v_i\big) - Tr\big(P(\nabla v_i) D^2 v_i\big) = \lambda(K_i), & \text{in } \operatorname{int}(K_i), \\ v_i(x) \to +\infty, \quad x \to \partial K_i, \end{cases} \tag{4.10}$$

where $P = (P_{ij})$ with $P_{ij} = |\nabla v|^2 \delta_{ij} - v_i v_j$. In Section 2 we have proved v_i is strictly convex in K_i, so that

$$\det\big(D^2 v_i(x)\big) > 0, \quad \forall x \in \operatorname{int}(K_i). \tag{4.11}$$

By the boundary condition verified by v_i, we have

$$\nabla v_i\big(\operatorname{int}(K_i)\big) = \mathbb{R}^3. \tag{4.12}$$

Let us now consider the conjugate function v_i^* of v_i:

$$v_i^*(\rho) = \sup_{x \in K_i} \big[(x, \rho) - v_i(x)\big], \quad \rho \in \mathbb{R}^3.$$

For the basic properties of this function we refer to [13]; here we just put out some points connected with our concerns.

Note that v_i^* is defined on the image of K_i through the gradient map of v_i, which is, by (4.12), the whole $\mathbb{R}^3$. Moreover as v_i is strictly convex, $v_i \in C^1(\mathbb{R}^3)$, and ∇v_i^* is the inverse map of ∇v_i:

$$x = \nabla v_i^*\big(\nabla v_i(x)\big), \quad \forall x \in K_i.$$

In particular this identity and (4.11) imply that $v_i^* \in C^2(\mathbb{R}^3)$ and

$$D^2 v_i(x) = \big[D^2 v_i^*\big(\nabla v_i(x)\big)\big]^{-1}, \quad \forall x \in K_i. \tag{4.13}$$

Let $t \in [0, 1]$, we define $K_t = (1-t)K_0 + tK_1$. Now we introduce a new function w in K_t in the following:

$$w(z) = \min\big\{(1-t)v_0(x) + tv_1(y)\colon x \in K_0,\ y \in K_1,\ (1-t)x + ty = z\big\}, \tag{4.14}$$

w is called the *infimal convolution* of v_0 and v_1 in K_t. It is a strictly convex function, and from the boundary conditions in problem (4.10) it can be deduced that

$$\lim_{z \to \partial K_t} w(z) = +\infty. \tag{4.15}$$

Moreover w satisfies the following identity (see Theorem 16.4 in [13])

$$w^* = (1-t)v_0^* + tv_1^* \quad \text{in } \mathbb{R}^3. \tag{4.16}$$

Now (4.11), (4.13), and (4.16) imply that w^* is $C^2(\mathbb{R}^3)$, strictly convex and

$$D^2 w^* > 0 \quad \text{in } \mathbb{R}^3.$$

Consequently, $w \in C^2(\mathrm{int}(K_t))$. Let us fix $z \in K_t$, by the definition of w and the boundary conditions in (4.10), there exist unique $x \in \mathrm{int}(K_0)$ and $y \in \mathrm{int}(K_1)$ such that $z = (1-t)x + ty$ and

$$w(z) = (1-t)v_0(x) + tv_1(y). \tag{4.17}$$

By the Lagrange multipliers theorem one deduces immediately that

$$\nabla v_0(x) = \nabla v_1(y) := \rho. \tag{4.18}$$

On the other hand,

$$\nabla w^*(\rho) = (1-t)\nabla v_0^*(\rho) + t\nabla v_1^*(\rho) = (1-t)x + ty = z = \nabla w^*\big(\nabla w(z)\big). \tag{4.19}$$

Hence by the injectivity of ∇w we have

$$\nabla w(z) = \rho. \tag{4.20}$$

Therefore,

$$\begin{aligned} D^2 w(z) &= \big[D^2\big(w^*(\rho)\big)\big]^{-1} = \big[(1-t)D^2 v_0^*(\rho) + tD^2 v_1^*(\rho)\big]^{-1} \\ &= \big[(1-t)\big(D^2 v_0(x)\big)^{-1} + t\big(D^2 v_1(y)\big)^{-1}\big]^{-1}. \end{aligned} \tag{4.21}$$

Note that (4.18), (4.20) imply the matrix P satisfies

$$P(\nabla w) = P(\nabla v_0) = P(\nabla v_1) \equiv P(\rho) := P. \tag{4.22}$$

Now we state the following Claim A.

Claim A.

$$S_2\big(D^2 w(z)\big) - Tr\big(PD^2 w(z)\big) \leqslant \max_{i\in\{0,1\}} \lambda(K_i), \quad \textit{for all } z \in K_t. \tag{4.23}$$

If Claim A is true, then we define

$$\bar{u}(z) := -e^{-w(z)}, \quad z \in K_t,$$

so $\bar{u}(z)$ has the following properties:

$$\begin{cases} S_2\big(D^2\bar{u}\big) \leqslant \max\limits_{i\in\{0,1\}} \lambda(K_i)(-\bar{u})^2, & \bar{u} < 0 \quad \text{in } \mathrm{int}(K_t), \\ \bar{u} = 0 \quad \partial K_t. \end{cases} \tag{4.24}$$

By multiplying both sides of the inequality above by $-\bar{u}$ and then integrate over K_t to obtain

$$\max_{i\in\{0,1\}} \lambda(K_i) \geqslant -\frac{\int_{K_t} \bar{u} S_2(D^2\bar{u})\,dx}{\int_{K_t} |\bar{u}|^3\,dx} \geqslant \lambda(K_t), \tag{4.25}$$

where the last inequality follows from the definition of λ in (1.3). Hence

$$\max\{\lambda(K_0), \lambda(K_1)\} \geqslant \lambda\big((1-t)K_0 + tK_1\big), \quad \forall K_0, K_1,\ t \in [0,1]. \tag{4.26}$$

In order to get the Brunn–Minkowski inequality, we just replace K_0 with K_0', K_1 with K_1' and t with t' in which

$$K_0' = [\lambda(K_0)]^{1/4} K_0, \qquad K_1' = [\lambda(K_1)]^{1/4} K_1,$$
$$t' = \frac{t[\lambda(K_1)]^{-1/4}}{(1-t)[\lambda(K_0)]^{-1/4} + t[\lambda(K_1)]^{-1/4}}. \tag{4.27}$$

Proof of Claim A. Since the solution $w(z)$ is strictly convex in K_t, then there exists a unique point $z_o \in K_t$ such that $Dw(z_o) = 0$. So we divide two cases to prove Claim A.

Case 1. At the unique point $z_o \in K_t$ with $Dw(z_o) = 0$.

In this case, we know from (4.18)–(4.20), there are unique $x_o \in K_0$ and $y_o \in K_1$ such that $z_o = (1-t)x_o + ty_o$ and $Dv_0(x_o) = Dv_1(y_o) = 0$. Moreover by (4.10), we have

$$\lambda(K_0) = S_2(D^2 v_0)(x_o), \qquad \lambda(K_1) = S_2(D^2 v_1)(y_o).$$

In order to prove Claim A, (4.23), we only need to prove

$$S_2(D^2 w(z_o)) \leqslant \max\{S_2(D^2 v_0)(x_o), S_2(D^2 v_1)(y_o)\}. \tag{4.28}$$

Since we have (4.21), it follows that

$$D^2 w(z_o) = \big[(1-t)\big(D^2 v_0(x_o)\big)^{-1} + t\big(D^2 v_1(y_o)\big)^{-1}\big]^{-1}, \tag{4.29}$$

using Proposition 4.5, we have

$$S_2^{\frac{1}{2}}\big(D^2 w(z_o)\big) \leqslant (1-t) S_2^{\frac{1}{2}}\big(D^2 v_0(x_o)\big) + t S_2^{\frac{1}{2}}\big(D^2 v_1(y_o)\big), \tag{4.30}$$

then we have (4.28) and finish the proof of this case.

Case 2. At the point $z \in K_t$ with $Dw(z) \neq 0$.

In this case, we know from (4.18)–(4.20), there are unique $x \in K_0$ and $y \in K_1$ such that $z = (1-t)x + ty$ and $Dv_0(x) = Dv_1(y) \neq 0$.

Using Proposition 4.3 we have

$$\frac{S_2(D^2 w(z))}{Tr(PD^2 w(z))} \leqslant (1-t)\frac{S_2(D^2 v_0(x))}{Tr(PD^2 v_0(x))} + t\frac{S_2(D^2 v_1(y))}{Tr(PD^2 v_1(y))}. \tag{4.31}$$

Consequently it follows from (4.10) that

$$\begin{aligned}\frac{S_2(D^2w(z))}{Tr(PD^2w(z))} &\leqslant (1-t)\frac{\lambda(K_0)}{Tr(PD^2v_0(x))}+t\frac{\lambda(K_1)}{Tr(PD^2v_1(y))}+1\\ &\leqslant \max_{i\in\{0,1\}}\lambda(K_i)\left[(1-t)\frac{1}{Tr(PD^2v_0(x))}+t\frac{1}{Tr(PD^2v_1(y))}\right]+1\\ &\leqslant \max_{i\in\{0,1\}}\lambda(K_i)\frac{1}{Tr(PD^2w(z))}+1, \qquad (4.32)\end{aligned}$$

where the last inequality still comes from Proposition 4.3.

So we now get the following inequality

$$S_2\big(D^2w(z)\big)-Tr\big(PD^2w(z)\big)\leqslant \max_{i\in\{0,1\}}\lambda(K_i), \quad \text{for all } z\in K_t, \qquad (4.33)$$

and complete the proof of Claim A. □

Up to now, we complete the proof of the Brunn–Minkowski inequality.

Prescribing the equality case. Now we will deal with the equality case in Theorem 1.2.

If K_0 is homothetic to K_1, that is, if $K_0=K_1+\bar{\rho}$ for some $\bar{\rho}\in\mathbb{R}^3$, then the equality holds in (1.5) by the homogeneity of $\lambda(.)$ and by the invariance with respect to translation.

Conversely, if equality holds in (1.5), then the arguments in previous show that equality must hold in (4.26), up to a normalization of the involved sets. Namely, let K_0', K_1' and t' be as in (4.27) and let

$$K_t'=\big(1-t'\big)K_0'+t'K_1'.$$

Thanks to the homogeneity of λ we may assume

$$\lambda\big(K_t'\big)=\lambda\big(K_0'\big)=\lambda\big(K_1'\big)=1.$$

Hence reduce the equality to the case in which the bodies K_0, K_1 and K_t have the same eigenvalue 1.

We shall prove the following Claim B.

Claim B. *For $x\in K_0$ and $y\in K_1$ such that $z=(1-t)x+ty$ and $Dv_0(x)=Dv_1(y)=Dw(z)$, we have*

$$D^2v_0(x)=D^2v_1(y). \qquad (4.34)$$

If we have Claim B, then as in Colesanti (p. 129 in [5]), we conclude that

$$\begin{aligned}&D^2v_0(x)=D^2v_1(y)\Rightarrow D^2v_0^*(\rho)=D^2v_1^*(\rho) \quad \forall\rho\in\mathbb{R}^3\\ &\quad\Longrightarrow\quad \nabla v_0^*(\rho)=\nabla v_1^*(\rho)+\bar{\rho}, \quad \forall\rho \text{ and for some fixed } \bar{\rho}\in\mathbb{R}^3.\end{aligned}$$

Finally we have

$$K_0 = \nabla v_0^*\big(R^3\big) = \nabla v_1^*\big(R^3\big) + \bar{\rho} = K_1 + \bar{\rho},$$

then we complete the proof of Theorem 1.2.

Proof of Claim B. As in the above Claim A, now we also divide two cases to prove Claim B.

Case 1. At the unique point $z_o \in K_t$ with $Dw(z_o) = 0$.

In this case, we know from (4.18)–(4.20), there are unique $x_o \in K_0$ and $y_o \in K_1$ such that $z_o = (1-t)x_o + ty_o$ and $Dv_0(x_o) = Dv_1(y_o) = 0$. Since we have (4.21), it follows that

$$D^2 w(z_o) = \big[(1-t)\big(D^2 v_0(x_o)\big)^{-1} + t\big(D^2 v_1(y_o)\big)^{-1}\big]^{-1}, \tag{4.35}$$

using the equality case in Proposition 4.5 (Lieberman [10]), we have the following equality

$$S_2^{\frac{1}{2}}\big(D^2 w(z_o)\big) = (1-t)S_2^{\frac{1}{2}}\big(D^2 v_0(x_o)\big) + tS_2^{\frac{1}{2}}\big(D^2 v_1(y_o)\big), \tag{4.36}$$

if and only if

$$D^2 v_0(x_o) = D^2 v_1(y_o). \tag{4.37}$$

Case 2. At the point $z \in K_t$ with $Dw(z) \neq 0$.

In this case, we know from (4.18)–(4.20), there are $x \in K_0$ and $y \in K_1$ such that $z = (1-t)x + ty$ and $Dv_0(x) = Dv_1(y) \neq 0$. Then the calculations above show that all the inequalities in (4.31)–(4.32) become equalities, in particular from (4.31) we have

$$\frac{S_2([(1-t)(D^2v_0(x))^{-1} + t(D^2v_1(y))^{-1}]^{-1})}{Tr(P[(1-t)(D^2v_0(x))^{-1} + t(D^2v_1(y))^{-1}]^{-1})} = (1-t)\frac{S_2(D^2v_0(x))}{Tr(PD^2v_0(x))} + t\frac{S_2(D^2v_1(y))}{Tr(PD^2v_1(y))}.$$

For simplicity, let

$$A = \big(D^2 v_0(x)\big)^{-1}, \qquad B = \big(D^2 v_1(y)\big)^{-1},$$

then from the above equality and Remark 4.2 we must have

$$\frac{Tr(P((1-t)A + tB)^{-1})}{S_2([(1-t)A + tB]^{-1})} = (1-t)\frac{Tr(PA^{-1})}{S_2(A^{-1})} + t\frac{Tr(PB^{-1})}{S_2(B^{-1})}. \tag{4.38}$$

This is equivalent to the equality case in (4.8) for its homogeneity of degree 1. Now from Remark 4.4, we have

$$\frac{TrA}{TrB} = \frac{A_{33}}{B_{33}} = \frac{A_{13}}{B_{13}} = \frac{A_{31}}{B_{31}} = \frac{A_{23}}{B_{23}} = \frac{A_{32}}{B_{32}} := c. \tag{4.39}$$

Notice that the equality in (4.32) implies

$$\frac{1}{Tr(P[(1-t)A+tB]^{-1})}=(1-t)\frac{1}{Tr(PA^{-1})}+t\frac{1}{Tr(PB^{-1})}.$$

By homogeneity of degree 1, it is equivalent to

$$\frac{1}{Tr(PA^{-1})}+\frac{1}{Tr(PB^{-1})}-\frac{1}{Tr(P(A+B)^{-1})}=0. \tag{4.40}$$

With the help of the arguments in the proof of Proposition 4.3, we may assume P is a diagonalized matrix with $P_{11}=P_{22}=|\nabla v|^2>0$, $P_{33}=0$. Hence a simple calculation gives

$$\frac{1}{Tr(PA^{-1})}=\frac{\det A}{P_{11}(A_{22}A_{33}-A_{23}^2+A_{11}A_{33}-A_{13}^2)}. \tag{4.41}$$

Putting together (4.39)–(4.41), we have

$$(A_{12}-cB_{12})^2+(A_{22}-cB_{22})^2=0, \tag{4.42}$$

which implies

$$A_{12}=cB_{12}, \qquad A_{22}=cB_{22}. \tag{4.43}$$

From (4.39) and (4.43), we have $A=cB$ due to the symmetry of the involved matrix. Now the equality in (4.31) immediately implies $A=B$, i.e.

$$D^2v_0(x)=D^2v_1(y),$$

thanks to the homogeneity of degree -1 in $\frac{S_2(A^{-1})}{Tr(PA^{-1})}$.

Now we complete the proof of Claim B. □

This finishes the proof of Theorem 1.2. □

We suspect Theorem 1.2 should be true in high-dimensional case.

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