Gaussian Curvature Estimates for the Convex Level Sets of p-Harmonic Functions

XI-NAN MA
University of Science and Technology of China

QIANZHONG OU
Hezhou University

AND

WEI ZHANG
University of Science and Technology of China

Abstract

We give a positive lower bound for the Gaussian curvature of the convex level sets of p-harmonic functions with the Gaussian curvature of the boundary and the norm of the gradient on the boundary. Combining the deformation process, this estimate gives a new approach to studying the convexity of the level sets of the p-harmonic function. © 2010 Wiley Periodicals, Inc.

1 Introduction

The convexity of the level sets of the solutions of elliptic partial differential equations has been studied for a long time. Using conformal mapping, Caratheodory obtained that the level curves of the Green's function on a simply connected convex domain in the plane are convex Jordan curves. For the minimal annulus whose boundary consists of two closed convex curves in parallel planes P_1 and P_2, in 1956 Shiffman [20] proved that the intersection of the surface with any parallel plane P between P_1 and P_2 is a convex Jordan curve. For elliptic partial differential equations on domains in $\mathbb{R}^n$, the convexity of the level sets of solutions was first considered by Gabriel [8] in 1957. He proved that the level sets of the Green function on a three-dimensional bounded convex domain are strictly convex. Later, in 1977, Lewis [12] extended Gabriel's result to p-harmonic functions in higher dimensions and obtained the following theorem.

THEOREM 1.1 (Gabriel [8] and Lewis [12]) *Let u satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1, \end{cases} \tag{1.1}$$

Communications on Pure and Applied Mathematics, Vol. LXIII, 0935–0971 (2010)

where $1 < p < +\infty$ *and* Ω_0 *and* Ω_1 *are bounded convex domains in* $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. (*We say that* u *satisfies the homogeneous Dirichlet boundary conditions in the convex ring* $\Omega = \Omega_0 \setminus \bar{\Omega}_1$.) *Then all the level sets of* u *are strictly convex with respect to the normal* ∇u.

In 1982, Caffarelli and Spruck [5] generalized the Lewis [12] results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli and Friedman [3], Korevaar [11] gave a new proof of Theorem 1.1 using the following observation: if the level sets of the solution of (1.1) are convex with respect to the gradient direction ∇u, then the rank of the second fundamental form of the level sets is constant in the domain. For more recent related extensions, please see the papers by Bianchini, Longinetti, and Salani [2] and Bian, Guan, Ma, and Xu [1]. A survey of this subject is given by Kawohl [10].

The aforementioned results are of a qualitative nature. This naturally leads us to the question of quantitative results, that is, estimates for the curvature of the level sets of the solutions of such elliptic problems. This is the topic of the present paper. The quantitative study in partial differential equations is very important in many problems; for example, see the survey by Lin [13].

For a two-dimensional harmonic function defined on a convex ring with homogeneous Dirichlet boundary conditions, by the theorem of Lewis [12], the level sets of this function are strictly convex. In 1983 Longinetti [14] proved that the curvature of the level sets of such a two-dimensional harmonic function attains its minimum on the boundary (see also Talenti [22] for some related results). Later, in 1987, Longinetti [15] obtained a similar theorem for minimal surfaces, where the convexity of the level sets follows from the theorem of Shiffman [20]. Recently Jost, Ma, and Ou [9] proved that the Gaussian curvature (i.e., the product of all the principal curvatures; see [21]) of the convex level sets of three-dimensional harmonic functions attains its minimum on the boundary. Ma, Ye, and Ye [17] got a sharp lower bound for the principal curvature of the level sets of harmonic functions and minimal graphs defined on convex rings in $\mathbb{R}^3$ with homogeneous Dirichlet boundary conditions. For the other related results and its application to the free boundary problem, please see the papers by Rosay and Rudin [18] and Dolbeault and Monneau [7].

In this paper, using the strong minimum principle, we obtain a lower bound on the Gaussian curvature of the convex level sets of higher-dimension p-harmonic functions. Our estimates depend on the Gaussian curvature of the boundary of the domain and the norm of the boundary gradient of the p-harmonic functions. From our estimates and combining the deformation process, we can give a new approach to studying the convexity of the level sets of p-harmonic functions.

Now we state our theorems.

THEOREM 1.2 *Let* Ω *be a bounded smooth domain in* $\mathbb{R}^n$, $n \geq 2$, *and* $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ *be a* p*-harmonic function in* Ω, *i.e.,*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \textit{in } \Omega. \tag{1.2}$$

Assume $1 < p < +\infty$ *and* $|\nabla u| \neq 0$ *in* $\bar{\Omega}$, *and let* K *be the Gaussian curvature of the level sets. If the level sets of* u *are strictly convex with respect to the normal* ∇u, *then we have the following statements*:

Case 1. *For* $n \geq 2$, $1 < p < +\infty$, *the function*

$$|\nabla u|^{n+1-2p} K$$

attains its minimum on the boundary.

Case 2. *For* $n = 2$, $1 < p < +\infty$, *and for* $n \geq 3$, $1 + \frac{2}{n} \leq p \leq n$, *the function*

$$|\nabla u|^{1-p} K$$

attains its minimum on the boundary.

Case 3. *For* $n = 2$, $\frac{3}{2} \leq p \leq 3$, *or* $n = 3$, $2 \leq p < \infty$, *or* $n \geq 4$, $p = \frac{n+1}{2}$, *the function* K *attains its minimum on the boundary.*

If u is a solution for (1.1), then we shall prove a useful fact that the norm of the gradient $|\nabla u|$ attains its maximum and minimum on the boundary in Proposition 4.1. Combining this fact, Theorem 1.1, and Theorem 1.2, we have the following consequences:

COROLLARY 1.3 *Let* u *satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1, \end{cases} \tag{1.3}$$

where $1 < p < +\infty$, Ω_0 *and* Ω_1 *are bounded smooth convex domains in* $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. *Let* K *be the Gaussian curvature of the level sets. Then we have the following estimates*:

Case 1a. *For* $1 < p \leq \frac{n+1}{2}$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right)^{n+1-2p}. \tag{1.4}$$

Case 1b. *For* $\frac{n+1}{2} < p < +\infty$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right)^{2p-n-1}. \tag{1.5}$$

Case 2. *For* $n = 2$, $1 < p < +\infty$, *and for* $n \geq 3$, $1 + \frac{2}{n} \leq p \leq n$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right)^{p-1}. \tag{1.6}$$

Case 3. *For $n = 2$, $\frac{3}{2} \le p \le 3$, or $n = 3$, $2 \le p < \infty$, or $n \ge 4$, $p = \frac{n+1}{2}$, we have*

$$\min_{\Omega} K \ge \min_{\partial\Omega} K. \tag{1.7}$$

Remark 1.4. In Theorem 1.2, we choose $\psi(x) = |\nabla u|^{1-p} K(x)$ for $n = 2$, $1 < p < +\infty$, and for $n \ge 3$, $1 + \frac{2}{n} \le p \le n$. Now we give an example to explain our choice for ψ.

Let $u(x)$ be the p-Green function of the ball $B_R(0) \subset \mathbb{R}^n$, i.e.,

$$u(x) = \begin{cases} |x|^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}} & \text{for } 1 < p < n, \\ -\log|x| + \log R & \text{for } p = n. \end{cases} \tag{1.8}$$

Then

$$|\nabla u|(x) = \begin{cases} \frac{n-p}{p-1}|x|^{\frac{1-n}{p-1}} & \text{for } 1 < p < n, \\ \frac{1}{|x|} & \text{for } p = n, \end{cases} \tag{1.9}$$

and the Gaussian curvature of the level set at x is

$$K(x) = |x|^{1-n}.$$

Hence

$$\psi(x) = |\nabla u|^{1-p} K(x) = \begin{cases} (\frac{n-p}{p-1})^{1-p} & \text{for } 1 < p < n, \\ 1 & \text{for } p = n. \end{cases} \tag{1.10}$$

From the above calculations, we know our $\psi(x) = |\nabla u|^{1-p} K(x)$ is sharp in some sense.

Remark 1.5. Our function $\psi(x) = |\nabla u|^{1-p} K$ in Theorem 1.2 is partly motivated by the works in Talenti [22]. Let u be a two-dimensional harmonic function with no critical points in the domain. Let k be the curvature of the level curve of u. In [22] Talenti proved $|\nabla u|^{-1} k$ is a harmonic function. From this observation one can also get the upper bound estimates on the curvature of the convex level curve of a two-dimensional harmonic function with boundary data.

Let K be the Gaussian curvature of the convex level sets, and let $\psi(x) = |\nabla u|^{2\theta} K$. For a suitable choice of θ we shall show that

$$\varphi = \log \psi(x) = \log K(x) + \theta \log |\nabla u|^2$$

satisfies the following elliptic differential inequality:

$$\sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} \varphi_{\alpha\beta} \le 0 \mod \nabla\varphi \quad \text{in } \Omega, \tag{1.11}$$

where

$$F^{\alpha\beta}(\nabla u) = |\nabla u|^2 \delta_{\alpha\beta} + (p-2) u_\alpha u_\beta.$$

In (1.11), we have suppressed the terms containing the gradient of φ with locally bounded coefficients; then we apply the strong minimum principle to obtain the results.

In Section 2, we first give a brief definition on the convexity of the level sets, and then obtain the curvature matrix (a_{ij}) of the level sets of a function, which appeared in [1, 9]. We prove the main theorem in Section 3, and then the corollary and some remarks in Section 4. The technique in the proof of these theorems consists in rearranging the terms in the second and third derivatives using the equation and the first-derivative condition of φ. The key idea is Pogorelov's method in a priori estimates in fully nonlinear elliptic equations.

2 The Curvature Matrix of Level Sets

In this section, we shall give a brief definition of the convexity of the level sets, then introduce the curvature matrix (a_{ij}) of the level sets of a function, which appeared in [1].

We first recall some fundamental notation in classical surface theory and give the definition of the convexity of a graph in Euclidean space with respect to the upward normal. Then we introduce the definition of the convexity of the level sets of a function u, and we derive the curvature matrix for the level sets of u.

2.1 Classical Differential Geometry of a Graph and Its Convexity

First we recall some fundamental notation in classical surface theory as in [6]. Assume a surface $\Sigma \subset \mathbb{R}^n$ is given by the graph of a function v in a domain in $\mathbb{R}^{n-1}$:

$$x_n = v(x'), \quad x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

The first fundamental form for the graph of $x_n = v(x')$ is given by

$$g_{ij} = \delta_{ij} + v_i v_j.$$

The upward normal direction $\vec{\nu}$ and the second fundamental form for the graph $x_n = v(x')$ are given by

$$\vec{\nu} = \frac{1}{W}(-v_1, -v_2, \dots, -v_{n-1}, 1), \qquad b_{ij} = \frac{v_{ij}}{W},$$

where $1 \le i, j \le n-1$ and $W = (1 + |\nabla v|^2)^{1/2}$.

Now we recall the definition of a convex graph in classical differential geometry [6].

DEFINITION 2.1 We define the graph of a function $x_n = v(x')$ to be *convex with respect to the upward normal*

$$\vec{\nu} = \frac{1}{W}(-v_1, -v_2, \dots, -v_{n-1}, 1) \tag{2.1}$$

if the second fundamental form $b_{ij} = v_{ij}/W$ of the graph $x_n = v(x')$ is nonnegative definite.

The principal curvature $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ of the graph of v, being the eigenvalues of the second fundamental form relative to the first fundamental form, satisfies

$$\det(b_{ij} - \kappa_l g_{ij}) = 0.$$

Equivalently, the κ_l satisfy

$$\det(a_{ij} - \kappa_l \delta_{ij}) = 0,$$

where

$$(a_{ij}) = (g^{il})^{\frac{1}{2}}(b_{lk})(g^{kj})^{\frac{1}{2}},$$

(g^{ij}) is the inverse matrix to (g_{ij}), and $(g^{ij})^{1/2}$ is the positive square root of (g^{ij}). They are given explicitly by

$$g^{ij} = \delta_{ij} - \frac{v_i v_j}{W^2}, \quad (g^{ij})^{\frac{1}{2}} = \delta_{ij} - \frac{v_i v_j}{W(1+W)}.$$

Then we have the following well-known formula:

LEMMA 2.2 ([4]) *The principal curvature of the graph $x_n = v(x')$ with respect to the upward normal* (2.1) *are the eigenvalues of the symmetric curvature matrix*

$$a_{il} = \frac{1}{W}\left\{v_{il} - \frac{v_i v_j v_{jl}}{W(1+W)} - \frac{v_l v_k v_{ki}}{W(1+W)} + \frac{v_i v_l v_j v_k v_{jk}}{W^2(1+W)^2}\right\}, \tag{2.2}$$

where the summation convention over repeated indices is employed.

2.2 Convexity of Level Sets of a Function

Now we give the definition of the convex level sets of a function u. Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$; its level sets can usually be defined in the following sense.

DEFINITION 2.3 Assuming $|\nabla u| \neq 0$ in Ω, we define the *level set* of u passing through the point $x_0 \in \Omega$ as $\Sigma^{u(x_0)} = \{x \in \Omega \mid u(x) = u(x_0)\}$.

Now we shall locally work near the point x_0 where $|\nabla u(x_0)| \neq 0$. We first state the definition of the convexity for the level set $\Sigma^{u(x_0)}$ in this special case. Without loss of generality we assume $u_n(x_0) \neq 0$ and work on the small neighborhood of x_0.

By the implicit function theorem, locally the level set $\Sigma^{u(x_0)}$ can be represented as a graph

$$x_n = v(x'), \quad x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1},$$

and $v(x')$ satisfies the equation

$$u(x_1, x_2, \dots, x_{n-1}, v(x_1, x_2, \dots, x_{n-1})) = u(x_0).$$

It follows that

$$u_i + u_n v_i = 0; \tag{2.3}$$

hence

$$v_i = -\frac{u_i}{u_n}. \tag{2.4}$$

From (2.4), the first fundamental form of the level set is

$$g_{ij} = \delta_{ij} + \frac{u_i u_j}{u_n^2},$$

and

$$W = (1 + |\nabla v|^2)^{\frac{1}{2}} = \frac{|\nabla u|}{|u_n|}. \tag{2.5}$$

Using (2.1) and (2.5), it follows that the upward normal direction of the level set is

$$\vec{\nu} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, \ldots, u_{n-1}, u_n). \tag{2.6}$$

Now we differentiate (2.3) again; we have

$$u_{ij} + u_{in} v_j + u_{nj} v_i + u_{nn} v_i v_j + u_n v_{ij} = 0,$$

then

$$\begin{aligned} v_{ij} &= -\frac{1}{u_n}(u_{ij} + u_{in} v_j + u_{nj} v_i + u_{nn} v_i v_j) \\ &= -\frac{1}{u_n^3}(u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}). \end{aligned}$$

If we set

$$h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \tag{2.7}$$

it follows that

$$v_{ij} = -\frac{h_{ij}}{u_n^3}. \tag{2.8}$$

From (2.5) and (2.8), with respect to the upward normal direction (2.1), the second fundamental form of the level set of function u is

$$b_{ij} = \frac{v_{ij}}{W} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}. \tag{2.9}$$

From Definition 2.1, we now give the definition of the convexity for the level set $\Sigma^{u(x_0)} = \{x \in \Omega \mid u(x) = u(x_0)\}$ of function $u(x)$ where $|\nabla u|(x) \neq 0$ in Ω.

Definition 2.4 For the function $u \in C^2(\Omega)$ we assume $|\nabla u| \neq 0$ in Ω. Without loss of generality we can let $u_n(x_0) \neq 0$ for $x_0 \in \Omega$. We define locally the level set $\Sigma^{u(x_0)} = \{x \in \Omega \mid u(x) = u(x_0)\}$ to be convex with respect to the upward normal direction

$$\vec{\nu} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, \ldots, u_{n-1}, u_n)$$

if the second fundamental form

$$b_{ij} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}$$

is nonnegative definite.

Remark 2.5. If we let ∇u be the upward normal of the level set $\Sigma^{u(x_0)}$ at x_0, then $u_n(x_0) > 0$ by (2.6). From Definition 2.4, if the level set $\Sigma^{u(x_0)}$ is convex with respect to the normal ∇u, then the matrix $(h_{ij}(x_0))$ is nonpositive definite.

2.3 Curvature Matrix of Level Sets of a Function

Now we obtain the representation of the curvature matrix (a_{ij}) of the level sets of a function u with the derivative of the function u. We work locally on $\Sigma^{u(x_0)} = \{x \in \Omega \mid u(x) = u(x_0)\}$. Let ∇u be the upward normal of the level set $\Sigma^{u(x_0)}$ at x_0; then $u_n(x_0) > 0$.

From (2.2), (2.5), and (2.9), it follows that the symmetric curvature matrix (a_{ij}) becomes

$$\begin{aligned} a_{ij} = \frac{1}{|\nabla u| u_n^2}\Bigg\{ -h_{ij} &+ \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} \\ &+ \frac{u_j u_l h_{il}}{W(1+W)u_n^2} - \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}\Bigg\}. \end{aligned} \tag{2.10}$$

From now on we denote

$$\begin{aligned} B_{ij} &= \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} + \frac{u_j u_l h_{il}}{W(1+W)u_n^2}, \\ C_{ij} &= \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}, \end{aligned} \tag{2.11}$$

and

$$A_{ij} = -h_{ij} + B_{ij} - C_{ij}; \tag{2.12}$$

then the symmetric curvature matrix of the level sets of u can be represented as

$$a_{ij} = \frac{1}{|\nabla u| u_n^2}[-h_{ij} + B_{ij} - C_{ij}] = \frac{1}{|\nabla u| u_n^2} A_{ij}. \tag{2.13}$$

With the above notation, we end this section with the following Codazzi-type formula, which will be used in the next section:

PROPOSITION 2.6 (See [1].) *Denote*

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x_k}$$

for $1 \le i, j, k \le n-1$; *then at the point where* $u_n = |\nabla u| > 0$, $u_i = 0$, $a_{ij,k}$ *commutes in* i, j, *and* k, *i.e.,*

$$a_{ij,k} = a_{ik,j}.$$

PROOF: Direct calculation shows

$$a_{ij,k} = -u_n^{-1}u_{ijk} + u_n^{-2}(u_{ij}u_{kn} + u_{ik}u_{jn} + u_{jk}u_{in}). \tag{2.14}$$

The right-hand side of (2.14) obviously commutes in i, j, and k. □

3 Proof of the Main Theorem

In this section, we shall consider the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

and we shall prove Theorem 1.2 by the strong minimum principle.

From now on, we employ the convention that the indices $1 \le \alpha, \beta, \gamma \le n$ and $1 \le i, j, k \le n-1$.

Denote

$$F^{\alpha\beta}(\nabla u) = |\nabla u|^2\delta_{\alpha\beta} + (p-2)u_\alpha u_\beta. \tag{3.2}$$

Then equation (3.1) is equivalent to

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{\alpha\beta} = 0. \tag{3.3}$$

PROOF OF THEOREM 1.2: Since the level sets of u are strictly convex with respect to the normal ∇u, the curvature matrix (a_{ij}) of the level sets is positive definite in Ω. We set $\psi(x) = |\nabla u|^{2\theta}K(x)$, and let

$$\varphi = \log\psi(x) = \log K(x) + \theta\log|\nabla u|^2,$$

where $K(x) = \det(a_{ij})$ is the Gaussian curvature of the level sets. In this section, for a suitable choice of θ we will derive the following elliptic inequality:

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} \le 0 \mod \nabla\varphi \quad \text{in } \Omega, \tag{3.4}$$

where we modify the terms of $\nabla\varphi$ with locally bounded coefficients. Then by the standard strong minimum principle, we get the result immediately.

In order to prove (3.4) at an arbitrary point $x_0 \in \Omega$, as in Caffarelli and Friedman [3], we choose the normal coordinate at x_0 by rotating the coordinate system suitably by T_{x_0}; we may assume that $u_i(x_0) = 0$, $1 \le i \le n-1$, and $u_n(x_0) = |\nabla u| > 0$. We can further assume that the matrix $(u_{ij}(x_0))$ $(1 \le i, j \le n-1)$ is diagonal and $u_{ii}(x_0) < 0$. We also choose T_{x_0} to vary smoothly with x_0. If we can establish (3.4) at x_0 under the above assumptions, then going back to the original coordinates we find that (3.4) remains valid with new locally bounded coefficients on $\nabla\varphi$ in (3.4) depending smoothly on the independent variables. Thus it suffices to establish (3.4) under the above assumptions.

From now on, all the calculation will be done at the fixed point x_0. In the following, we shall prove the theorem in three steps.

Step 1. We first compute formula (3.26).

Taking the first derivative of φ, we get

$$\varphi_\alpha = \sum_{1\le i,j\le n-1} a^{ij} a_{ij,\alpha} + \theta|\nabla u|^{-2}\,|\nabla u|^2_\alpha; \tag{3.5}$$

it follows that

$$\sum_{1\le i\le n-1} a^{ii} a_{ii,\alpha} = -2\theta u_n^{-1} u_{n\alpha} + \varphi_\alpha. \tag{3.6}$$

Taking the derivative of equation (3.5) once more, we have

$$\begin{aligned}\varphi_{\alpha\beta} = &\sum_{1\le i\le n-1} a^{ii} a_{ii,\alpha\beta} - \sum_{1\le i,j\le n-1} a^{ii} a^{jj} a_{ij,\alpha} a_{ij,\beta} \\ &- \theta|\nabla u|^{-4}\,|\nabla u|^2_\alpha\,|\nabla u|^2_\beta + \theta|\nabla u|^{-2}\,|\nabla u|^2_{\alpha\beta}.\end{aligned}$$

So

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{3.7}$$

where

$$\begin{aligned}\mathrm{I} &= \sum_{1\le i\le n-1} a^{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} a_{ii,\alpha\beta},\\ \mathrm{II} &= -\sum_{1\le i,j\le n-1}\ \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} a^{ii} a^{jj} a_{ij,\alpha} a_{ij,\beta},\\ \mathrm{III} &= -\theta|\nabla u|^{-4} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} |\nabla u|^2_\alpha |\nabla u|^2_\beta,\\ \mathrm{IV} &= \ \theta|\nabla u|^{-2} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} |\nabla u|^2_{\alpha\beta}.\end{aligned}$$

In the rest of this section, we will deal with the four terms above. For the term II, from (3.2) we have

$$\begin{aligned} F^{ii} &= |\nabla u|^2 = u_n^2, \quad 1\le i\le n-1,\\ F^{nn} &= |\nabla u|^2 + (p-2)u_n^2 = (p-1)u_n^2,\\ F^{\alpha\beta} &= 0 \quad \text{for } \alpha\ne\beta.\end{aligned} \tag{3.8}$$

Then

$$\mathrm{II} = -u_n^2 \sum_{1\le i,j,k\le n-1} a^{ii} a^{jj} a^2_{ij,k} - (p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii} a^{jj} a^2_{ij,n}. \tag{3.9}$$

Next we treat terms III and IV. At the considered point x_0, equation (3.3) becomes

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{\alpha\beta} = u_n^2 \Delta u + (p-2)u_n^2 u_{nn} = 0,$$

i.e.,

$$(p-1)u_{nn} = -\sum_{1\le i\le n-1} u_{ii}.$$

By (2.10) and (2.7), we get

$$u_{ii} = -u_n a_{ii} \quad \text{and} \quad h_{ii} = -u_n^3 a_{ii}, \tag{3.10}$$

so

$$u_{nn} = \frac{1}{p-1} u_n \sigma_1 \quad \text{where } \sigma_1 = \sum_{1\le i\le n-1} a_{ii}. \tag{3.11}$$

Now we have the formulas

$$\begin{aligned} \sum_{1\le\alpha\le n} F^{\alpha\alpha} u_{n\alpha}^2 &= F^{nn} u_{nn}^2 + \sum_{1\le i\le n-1} F^{ii} u_{ni}^2 \\ &= \frac{1}{p-1} u_n^4 \sigma_1^2 + u_n^2 \sum_{1\le i\le n-1} u_{ni}^2 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \sum_{1\le\alpha,\beta\le n} F^{\alpha\alpha} u_{\alpha\beta}^2 &= F^{nn} u_{nn}^2 + F^{nn} \sum_{1\le i\le n-1} u_{ni}^2 + \sum_{1\le i\le n-1} F^{ii} u_{ni}^2 \\ &\quad + \sum_{1\le i\le n-1} F^{ii} u_{ii}^2 \\ &= \frac{1}{p-1} u_n^4 \sigma_1^2 + p u_n^2 \sum_{1\le i\le n-1} u_{ni}^2 + u_n^4 \sum_{1\le i\le n-1} a_{ii}^2. \end{aligned} \tag{3.13}$$

Differentiating (3.3) with respect to x_n, we have

$$F^{\alpha\beta}_{,n} = 2u_n u_{nn} \delta_{\alpha\beta} + (p-2) u_{\alpha n} u_\beta + (p-2) u_\alpha u_{\beta n}; \tag{3.14}$$

together with (3.2) and (3.3), we can get

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha\beta} = -\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,n} u_{\alpha\beta} = -2(p-2) u_n \sum_{1\le j\le n-1} u_{nj}^2. \tag{3.15}$$

From (3.8), (3.11), and (3.12), it follows that

$$\begin{aligned} \mathrm{III} &= -\theta |\nabla u|^{-4} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} |\nabla u|^2_\alpha \, |\nabla u|^2_\beta \\ &= -4\theta u_n^{-2} \sum_{1\le\alpha\le n} F^{\alpha\alpha} u_{n\alpha}^2 \\ &= -4\theta \sum_{1\le j\le n-1} u_{nj}^2 - \frac{4}{p-1} \theta u_n^2 \sigma_1^2. \end{aligned} \tag{3.16}$$

In an analogous way, we treat term IV. By (3.10), (3.11), (3.13), and (3.15),

$$
\begin{aligned}
\mathrm{IV} &= \theta|\nabla u|^{-2} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}|\nabla u|^2_{\alpha\beta} \\
&= \theta u_n^{-2} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\Big(2u_n u_{n\alpha\beta} + 2\sum_{1\le\gamma\le n} u_{\gamma\alpha}u_{\gamma\beta}\Big) \\
&= 2\theta u_n^{-1} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{n\alpha\beta} + 2\theta u_n^{-2} \sum_{1\le\alpha,\beta\le n} F^{\alpha\alpha}u^2_{\alpha\beta} \\
&= \frac{2}{p-1}\,\theta u_n^2\sigma_1^2 + (8-2p)\theta \sum_{1\le j\le n-1} u^2_{nj} + 2\theta u_n^2 \sum_{1\le j\le n-1} a^2_{jj}.
\end{aligned} \tag{3.17}
$$

Combining (3.7), (3.9), (3.16), and (3.17), it follows that

$$
\begin{aligned}
&\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} \\
&\quad= \sum_{1\le i\le n-1} a^{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}a_{ii,\alpha\beta} - u_n^2 \sum_{1\le i,j,k\le n-1} a^{ii}a^{jj}a^2_{ij,k} \\
&\qquad - (p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii}a^{jj}a^2_{ij,n} + (4-2p)\theta \sum_{1\le j\le n-1} u^2_{nj} \\
&\qquad - \frac{2}{p-1}\,\theta u_n^2\sigma_1^2 + 2\theta u_n^2 \sum_{1\le j\le n-1} a^2_{jj}.
\end{aligned} \tag{3.18}
$$

Next we deal with the term

$$
\mathrm{I} = \sum_{1\le i\le n-1} a^{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}a_{ii,\alpha\beta}.
$$

By (2.13),

$$
a_{ij} = \frac{A_{ij}}{|\nabla u|u_n^2};
$$

it follows that

$$
A_{ii} = a_{ii}D \quad \text{where } D = |\nabla u|u_n^2. \tag{3.19}
$$

Taking the first and second derivatives of (3.19), we get

$$
\begin{aligned}
A_{ii,\alpha} &= a_{ii,\alpha}D + a_{ii}D_\alpha, \\
A_{ii,\alpha\beta} &= a_{ii,\alpha\beta}D + a_{ii,\alpha}D_\beta + a_{ii,\beta}D_\alpha + a_{ii}D_{\alpha\beta};
\end{aligned}
$$

then

$$
a_{ii,\alpha\beta} = \frac{1}{u_n^3}[A_{ii,\alpha\beta} - a_{ii,\alpha}D_\beta - a_{ii,\beta}D_\alpha - a_{ii}D_{\alpha\beta}].
$$

Hence

$$
\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3, \tag{3.20}
$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} A_{ii,\alpha\beta} \Big), \\
\mathrm{I}_2 &= -\frac{n-1}{u_n^3} \Big(\sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} D_{\alpha\beta} \Big), \\
\mathrm{I}_3 &= -\frac{2}{u_n^3} \sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} \Big(\sum_{1 \le i \le n-1} a^{ii} a_{ii,\beta} \Big) D_\alpha .
\end{aligned}
$$

Since

$$
\begin{aligned}
D_\alpha &= 3u_n^2 u_{n\alpha}, \\
D_{\alpha\beta} &= 5u_n u_{n\alpha} u_{n\beta} + 3u_n^2 u_{n\alpha\beta} + u_n \sum_{1 \le \gamma \le n} u_{\gamma\alpha} u_{\gamma\beta},
\end{aligned}
\tag{3.21}
$$

from (3.8), (3.12)–(3.15), and (3.21), we obtain

$$
\begin{aligned}
\mathrm{I}_2 &= -\frac{n-1}{u_n^3} \Big(\sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} D_{\alpha\beta} \Big) \\
&= -\frac{n-1}{u_n^3} \Big(5u_n \sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} u_{n\alpha} u_{n\beta} + 3u_n^2 \sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} u_{n\alpha\beta} \\
&\qquad + u_n \sum_{1 \le \alpha, \beta, \gamma \le n} F^{\alpha\beta} u_{\gamma\alpha} u_{\gamma\beta} \Big) \\
&= (n-1)(5p-17) \sum_{1 \le j \le n-1} u_{nj}^2 - (n-1) u_n^2 \\
&\quad \times \sum_{1 \le j \le n-1} a_{jj}^2 - \frac{6(n-1)}{p-1} u_n^2 \sigma_1^2 .
\end{aligned}
\tag{3.22}
$$

By (3.6) and (3.12), we have

$$
\begin{aligned}
\mathrm{I}_3 &= -\frac{2}{u_n^3} \sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} \Big(\sum_{1 \le i \le n-1} a^{ii} a_{ii,\beta} \Big) D_\alpha \\
&= 12\theta u_n^{-2} \sum_{1 \le \alpha \le n} F^{\alpha\alpha} u_{n\alpha}^2 - \frac{2}{u_n^3} \sum_{1 \le \alpha \le n} F^{\alpha\alpha} D_\alpha \varphi_\alpha \\
&= 12\theta \sum_{1 \le j \le n-1} u_{nj}^2 + \frac{12}{p-1} \theta u_n^2 \sigma_1^2 \\
&\quad - 6u_n \sum_{1 \le j \le n-1} u_{nj} \varphi_j - 6u_n^2 \sigma_1 \varphi_n .
\end{aligned}
\tag{3.23}
$$

For the term I_1, recalling the definition of A_{ij}, i.e., (2.11) and (2.12), at x_0 we have

$$C_{ii,\alpha\beta} = 0,$$

hence

$$A_{ii,\alpha\beta} = -h_{ii,\alpha\beta} + B_{ii,\alpha\beta}.$$

It follows that

$$\begin{aligned} \mathrm{I}_1 &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} A_{ii,\alpha\beta} \Big) \\ &= \mathrm{I}_{11} + \mathrm{I}_{12} \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \mathrm{I}_{11} &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} (-h_{ii,\alpha\beta}) \Big), \\ \mathrm{I}_{12} &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} B_{ii,\alpha\beta} \Big). \end{aligned}$$

Now we work on term I_{12}. By (2.11), (3.8), and (3.10),

$$\begin{aligned} \mathrm{I}_{12} &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} B_{ii,\alpha\beta} \Big) \\ &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} \left(\frac{2 \sum_{1 \le l \le n-1} u_i u_l h_{il}}{W(1+W) u_n^2} \right)_{\alpha\beta} \\ &= \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} \left(\frac{2 u_{i\alpha} u_{i\beta} h_{ii}}{u_n^2} \right) \\ &= -2u_n^2 \sum_{1 \le j \le n-1} a_{jj}^2 - 2(p-1) \sum_{1 \le j \le n-1} u_{nj}^2. \end{aligned} \tag{3.25}$$

Combining (3.18), (3.20), (3.22), (3.23), and (3.25), it follows that

$$\begin{aligned} \sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} \varphi_{\alpha\beta} = {} & \frac{1}{u_n^3} \sum_{1 \le i \le n-1} a^{ii} \Big(\sum_{1 \le \alpha,\beta \le n} F^{\alpha\beta} (-h_{ii,\alpha\beta}) \Big) \\ & - u_n^2 \sum_{1 \le i,j,k \le n-1} a^{ii} a^{jj} a_{ij,k}^2 \\ & - (p-1) u_n^2 \sum_{1 \le i,j \le n-1} a^{ii} a^{jj} a_{ij,n}^2 \\ & + (4-2p)\theta \sum_{1 \le j \le n-1} u_{nj}^2 - \frac{2}{p-1} \theta u_n^2 \sigma_1^2 + \end{aligned}$$

$$
\begin{aligned}
&+ 2\theta u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 + (n-1)(5p-17) \sum_{1\le j\le n-1} u_{nj}^2 \\
&- (n-1)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 - \frac{6(n-1)}{p-1} u_n^2\sigma_1^2 \\
&+ 12\theta \sum_{1\le j\le n-1} u_{nj}^2 + \frac{12}{p-1}\theta u_n^2\sigma_1^2 \\
&- 6u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j - 6u_n^2\sigma_1\varphi_n - 2u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 \\
&- 2(p-1) \sum_{1\le j\le n-1} u_{nj}^2.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} \\
&\quad = \frac{1}{u_n^3} \sum_{1\le i\le n-1} a^{ii}\Bigl(\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})\Bigr) \\
&\qquad - u_n^2 \sum_{1\le i,j,k\le n-1} a^{ii}a^{jj}a_{ij,k}^2 - (p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii}a^{jj}a_{ij,n}^2 \\
&\qquad + (2\theta - n - 1)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 \\
&\qquad + [(16-2p)\theta + 5pn - 17n - 7p + 19] \sum_{1\le j\le n-1} u_{nj}^2 \\
&\qquad + \Bigl[\frac{10}{p-1}\theta - \frac{6(n-1)}{p-1}\Bigr] u_n^2\sigma_1^2 - 6u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j - 6u_n^2\sigma_1\varphi_n.
\end{aligned}
\tag{3.26}
$$

Step 2. In this step we calculate the following term in (3.26),

$$
\mathrm{I}_{11} = \frac{1}{u_n^3} \sum_{1\le i\le n-1} a^{ii}\Bigl(\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})\Bigr);
$$

we shall get formula (3.46).

By differentiating (2.7) twice, we have

$$
\begin{aligned}
-h_{ii,\alpha} = {} & -u_n^2 u_{ii\alpha} - 2u_n u_{n\alpha} u_{ii} - u_{nn\alpha} u_i^2 - 2u_i u_{i\alpha} u_{nn} \\
& + 2u_n u_i u_{ni\alpha} + 2u_n u_{i\alpha} u_{ni} + 2u_{n\alpha} u_i u_{ni}
\end{aligned}
$$

and

$$
\begin{aligned}
-h_{ii,\alpha\beta} = & -u_n^2 u_{ii\alpha\beta} - 2u_n u_{n\alpha} u_{ii\beta} - 2u_n u_{n\beta} u_{ii\alpha} + 2u_n u_{i\alpha} u_{ni\beta} \\
& + 2u_n u_{i\beta} u_{ni\alpha} + 2u_n u_{i\alpha\beta} u_{ni} - 2u_n u_{n\alpha\beta} u_{ii} + 2u_{n\alpha} u_{i\beta} u_{ni} \\
& + 2u_{n\beta} u_{i\alpha} u_{ni} - 2u_{nn} u_{i\beta} u_{i\alpha} - 2u_{n\beta} u_{n\alpha} u_{ii}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta}) \\
& = -u_n^2 \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{ii\alpha\beta} - 4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha} u_{ii\beta} \\
& \quad + 4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\alpha} u_{ni\beta} + 2u_n u_{ni} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\alpha\beta} \\
& \quad - 2u_n u_{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha\beta} + 4u_{ni} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha} u_{i\beta} \\
& \quad - 2u_{nn} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\beta} u_{i\alpha} - 2u_{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\beta} u_{n\alpha} \\
& = J_1 + J_2 + J_3 + J_4,
\end{aligned} \tag{3.27}
$$

where

$$
\begin{aligned}
J_1 &= -u_n^2 \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{ii\alpha\beta}, \\
J_2 &= -4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha} u_{ii\beta} + 4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\alpha} u_{ni\beta}, \\
J_3 &= 2u_n u_{ni} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\alpha\beta} - 2u_n u_{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha\beta}, \\
J_4 &= 4u_{ni} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha} u_{i\beta} - 2u_{nn} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\beta} u_{i\alpha} \\
& \quad - 2u_{ii} \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\beta} u_{n\alpha}.
\end{aligned}
$$

We first treat term J_2. By (3.8),

$$
\begin{aligned}
J_2 &= -4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{n\alpha} u_{ii\beta} + 4u_n \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{i\alpha} u_{ni\beta} \\
&= -4u_n^3 \sum_{1\le j\le n-1} u_{nj} u_{iij} - 4(p-1)u_n^3 u_{nn} u_{nii} + 4u_n^3 u_{ii} u_{nii} \\
& \quad + 4(p-1)u_n^3 u_{ni} u_{nni}.
\end{aligned} \tag{3.28}
$$

From (3.15) and its analogy

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{i\alpha\beta} = -\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,i}u_{\alpha\beta} = -2(p-2)u_n u_{ni}u_{ii}, \tag{3.29}$$

together with (3.10), we have

$$\begin{aligned} J_3 &= 2u_n u_{ni}\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{i\alpha\beta} - 2u_n u_{ii}\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{n\alpha\beta} \\ &= -4(p-2)u_n^2u_{ni}^2u_{ii} + 4(p-2)u_n^2u_{ii}\sum_{1\le j\le n-1}u_{nj}^2 \\ &= 4(p-2)u_n^3u_{ni}^2a_{ii} - 4(p-2)u_n^3a_{ii}\sum_{1\le j\le n-1}u_{nj}^2. \end{aligned} \tag{3.30}$$

For term J_4, taking advantage of (3.8)–(3.11),

$$\begin{aligned} J_4 &= 4u_{ni}\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{n\alpha}u_{i\beta} - 2u_{nn}\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{i\beta}u_{i\alpha} \\ &\quad - 2u_{ii}\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{n\beta}u_{n\alpha} \\ &= 4u_n^2u_{ni}^2u_{ii} + 2(p-1)u_n^2u_{ni}^2u_{nn} - 2u_n^2u_{nn}u_{ii}^2 \\ &\quad - 2(p-1)u_n^2u_{ii}u_{nn}^2 - 2u_n^2u_{ii}\sum_{1\le j\le n-1}u_{nj}^2 \\ &= -4u_n^3u_{ni}^2a_{ii} + 2u_n^3u_{ni}^2\sigma_1 - \frac{2}{p-1}u_n^5a_{ii}^2\sigma_1 + \frac{2}{p-1}u_n^5a_{ii}\sigma_1^2 \\ &\quad + 2u_n^3a_{ii}\sum_{1\le j\le n-1}u_{nj}^2. \end{aligned} \tag{3.31}$$

Finally, we deal with term J_1. By differentiating (3.3) twice with respect to x_i, we have

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}u_{\alpha\beta ii} = -\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,ii}u_{\alpha\beta} - 2\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,i}u_{\alpha\beta i}.$$

From (3.2), we have

$$F^{\alpha\beta}_{,i} = 2u_n u_{ni}\delta_{\alpha\beta} + (p-2)u_{\alpha i}u_\beta + (p-2)u_\alpha u_{\beta i} \tag{3.32}$$

and

$$\begin{aligned} F^{\alpha\beta}_{,ii} &= 2\sum_{1\le\gamma\le n}u_{i\gamma}^2\delta_{\alpha\beta} + 2u_n u_{iin}\delta_{\alpha\beta} + (p-2)u_{ii\alpha}u_\beta \\ &\quad + (p-2)u_\alpha u_{ii\beta} + 2(p-2)u_{i\alpha}u_{i\beta}. \end{aligned} \tag{3.33}$$

Therefore,

$$
\begin{aligned}
J_1 &= -u_n^2 \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta} u_{ii\alpha\beta} \\
&= u_n^2 \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,ii} u_{\alpha\beta} + 2u_n^2 \sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}_{,i} u_{\alpha\beta i} \\
&= 2u_n^2 \Delta u \sum_{1\le\alpha\le n} u_{i\alpha}^2 + 2u_n^3 \Delta u u_{nii} + 2(p-2)u_n^3 \sum_{1\le\alpha\le n} u_{\alpha ii} u_{n\alpha} \\
&\quad + 2(p-2)u_n^2 \sum_{1\le\alpha\le n} u_{\alpha i} u_{\beta i} u_{\alpha\beta} + 4u_n^3 u_{ni} (\Delta u)_i \\
&\quad + 4(p-2)u_n^3 \sum_{1\le\alpha\le n} u_{\alpha i} u_{ni\alpha}.
\end{aligned}
\tag{3.34}
$$

From (3.27), (3.28), (3.30), (3.31), and (3.34), let us rewrite the terms in

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})$$

as

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta}) = L_1 + L_2, \tag{3.35}$$

where

$$
\begin{aligned}
L_1 &= 2u_n^3 \Delta u u_{nii} + 2(p-2)u_n^3 \sum_{1\le\alpha\le n} u_{\alpha ii} u_{n\alpha} + 4u_n^3 u_{ni} (\Delta u)_i \\
&\quad + 4(p-2)u_n^3 \sum_{1\le\alpha\le n} u_{\alpha i} u_{ni\alpha} - 4u_n^3 \sum_{1\le j\le n-1} u_{nj} u_{iij} \\
&\quad - 4(p-1)u_n^3 u_{nn} u_{nii} + 4u_n^3 u_{ii} u_{nii} + 4(p-1)u_n^3 u_{ni} u_{nni}
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
L_2 &= 2u_n^2 \Delta u \sum_{1\le\alpha\le n} u_{i\alpha}^2 + 2(p-2)u_n^2 \sum_{1\le\alpha,\beta\le n} u_{\alpha i} u_{\beta i} u_{\alpha\beta} \\
&\quad + 4(p-2)u_n^3 u_{ni}^2 a_{ii} - 4(p-2)u_n^3 a_{ii} \sum_{1\le j\le n-1} u_{nj}^2 - 4u_n^3 u_{ni}^2 a_{ii} \\
&\quad + 2u_n^3 u_{ni}^2 \sigma_1 - \frac{2}{p-1} u_n^5 a_{ii}^2 \sigma_1 + \frac{2}{p-1} u_n^5 a_{ii} \sigma_1^2 \\
&\quad + 2u_n^3 a_{ii} \sum_{1\le j\le n-1} u_{nj}^2.
\end{aligned}
\tag{3.37}
$$

So we have

$$\begin{aligned} \mathrm{I}_{11} &= \frac{1}{u_n^3} \sum_{1\le i\le n-1} a^{ii}\Big(\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})\Big) \\ &= \frac{1}{u_n^3} \sum_{1\le i\le n-1} a^{ii}[L_1 + L_2]. \end{aligned} \tag{3.38}$$

In the following computation, we first concentrate on term L_1.

By (3.29),

$$(p-1)u_n^2 u_{nni} = -2(p-2)u_n u_{ni} u_{ii} - u_n^2 \sum_{1\le j\le n-1} u_{jji},$$

then

$$\begin{aligned} L_1 &= 8(p-1)u_n^3 u_{ni} u_{nni} + 4(p-1)u_n^3 u_{ii} u_{nii} - 4(p-1)u_n^3 u_{nn} u_{iin} \\ &\quad + (2p-8)u_n^3 \sum_{1\le j\le n-1} u_{iij} u_{nj} + 4u_n^3 u_{ni} \sum_{1\le j\le n-1} u_{jji} \\ &= -4u_n^3 u_{ni} \sum_{1\le j\le n-1} u_{jji} + 4(p-1)u_n^3 u_{ii} u_{nii} \\ &\quad - 4(p-1)u_n^3 u_{nn} u_{iin} + (2p-8)u_n^3 \sum_{1\le j\le n-1} u_{iij} u_{nj} \\ &\quad - 16(p-2)u_n^2 u_{ni}^2 u_{ii}. \end{aligned} \tag{3.39}$$

By (2.12), (2.11), and (2.7), we have

$$A_{ii,\alpha} = -h_{ii,\alpha} = -2u_n u_{n\alpha} u_{ii} - u_n^2 u_{ii\alpha} + 2u_n u_{i\alpha} u_{in};$$

on the other hand by (2.13),

$$A_{ii,\alpha} = (a_{ii}|\nabla u|u_n^2)_\alpha = u_n^3 a_{ii,\alpha} + 3u_n^2 u_{n\alpha} a_{ii};$$

therefore

$$u_{ii\alpha} = -u_n a_{ii,\alpha} + 2u_n^{-1} u_{ni} u_{i\alpha} + u_n^{-1} u_{n\alpha} u_{ii}. \tag{3.40}$$

Now making use of (3.40) and recalling (3.10) and (3.11), we can write L_1 as

$$\begin{aligned} L_1 &= 4u_n^4 u_{ni} \sum_{1\le j\le n-1} a_{jj,i} - 4(p-1)u_n^4 u_{ii} a_{ii,n} + 4(p-1)u_n^4 u_{nn} a_{ii,n} \\ &\quad + (8-2p)u_n^4 \sum_{1\le j\le n-1} a_{ii,j} u_{nj} - 16(p-2)u_n^2 u_{ni}^2 u_{ii} \\ &\quad + (12p-32)u_n^2 u_{ni}^2 u_{ii} + (4-4p)u_n^2 u_{ni}^2 u_{nn} + 4u_n^2 u_{ii} u_{nn} \sum_{1\le j\le n-1} u_{jj} \\ &\quad + 4(p-1)u_n^2 u_{nn} u_{ii}^2 + (2p-8)u_n^2 u_{ii} \sum_{1\le j\le n-1} u_{nj}^2 = \end{aligned}$$

$$
\begin{aligned}
&= 4u_n^4 u_{ni} \sum_{1\le j\le n-1} a_{jj,i} + 4(p-1)u_n^5 a_{ii}a_{ii,n} + 4u_n^5\sigma_1 a_{ii,n} \\
&\quad + (8-2p)u_n^4 \sum_{1\le j\le n-1} a_{ii,j}u_{nj} + 4pu_n^3u_{ni}^2 a_{ii} - 4u_n^3\sigma_1 u_{ni}^2 \\
&\quad + \frac{4}{p-1}u_n^5\sigma_1^2 a_{ii} + 4u_n^5\sigma_1 a_{ii}^2 + (8-2p)u_n^3 a_{ii} \sum_{1\le j\le n-1} u_{nj}^2 \\
&= L_{11} + L_{12},
\end{aligned}
$$

where

$$
\begin{aligned}
L_{11} &= 4u_n^4 u_{ni} \sum_{1\le j\le n-1} a_{jj,i} + 4(p-1)u_n^5 a_{ii}a_{ii,n} + 4u_n^5\sigma_1 a_{ii,n} \\
&\quad + (8-2p)u_n^4 \sum_{1\le j\le n-1} a_{ii,j}u_{nj}, \\
L_{12} &= 4pu_n^3u_{ni}^2 a_{ii} - 4u_n^3\sigma_1 u_{ni}^2 + \frac{4}{p-1}u_n^5\sigma_1^2 a_{ii} + 4u_n^5\sigma_1 a_{ii}^2 \\
&\quad + (8-2p)u_n^3 a_{ii} \sum_{1\le j\le n-1} u_{nj}^2.
\end{aligned}
$$

By (3.6),

$$
\begin{aligned}
&\frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}\cdot L_{11} \\
&\quad = 4_n \sum_{1\le i,j\le n-1} a^{ii}u_{ni}a_{jj,i} + 4(p-1)u_n^2 \sum_{1\le i\le n-1} a_{ii,n} \\
&\qquad + (4p-16)\theta \sum_{1\le j\le n-1} u_{nj}^2 - \frac{8}{p-1}u_n^2\sigma_1^2 \\
&\qquad + 4u_n^2\sigma_1\varphi_n + (8-2p)u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j
\end{aligned}
\tag{3.41}
$$

and

$$
\begin{aligned}
\frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}\cdot L_{12} &= (-2pn+8n+6p-8)\sum_{1\le j\le n-1} u_{nj}^2 \\
&\quad - 4\sigma_1 \sum_{1\le i\le n-1} a^{ii}u_{ni}^2 + \frac{4p+4n-8}{p-1}n^2\sigma_1^2.
\end{aligned}
\tag{3.42}
$$

Now we consider the term L_2. Set

$$L_2 = L_{21} + L_{22},$$

where

$$
\begin{aligned}
L_{21} &= 2u_n^2 \Delta u \sum_{1\le\alpha\le n} u_{i\alpha}^2 + 2(p-2)u_n^2 \sum_{1\le\alpha,\beta\le n} u_{\alpha i}u_{\beta i}u_{\alpha\beta},\\
L_{22} &= 4(p-2)u_n^3u_{ni}^2a_{ii} - 4(p-2)u_n^3a_{ii} \sum_{1\le j\le n-1} u_{nj}^2 - 4u_n^3u_{ni}^2a_{ii}\\
&\quad + 2u_n^3u_{ni}^2\sigma_1 - \frac{2}{p-1}u_n^5a_{ii}^2\sigma_1 + \frac{2}{p-1}u_n^5a_{ii}\sigma_1^2\\
&\quad + 2u_n^3a_{ii} \sum_{1\le j\le n-1} u_{nj}^2.
\end{aligned}
$$

Then by (3.10) and (3.11),

$$
\begin{aligned}
\frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}\cdot L_{21} = \frac{4-2p}{p-1}u_n^2\sigma_1^2 - 2(p-2)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2\\
- 4(p-2)\sum_{1\le j\le n-1} u_{nj}^2.
\end{aligned}
\tag{3.43}
$$

We also have

$$
\begin{aligned}
\frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}\cdot L_{22} = (-4pn+8p+10n-22)\sum_{1\le j\le n-1} u_{nj}^2\\
+ 2\sigma_1 \sum_{1\le i\le n-1} a^{ii}u_{ni}^2 + \frac{2n-4}{p-1}u_n^2\sigma_1^2.
\end{aligned}
\tag{3.44}
$$

From (3.38), we have

$$
\begin{aligned}
\mathrm{I}_{11} &= \frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}\Bigg(\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})\Bigg)\\
&= \frac{1}{u_n^3}\sum_{1\le i\le n-1} a^{ii}[L_{11}+L_{12}+L_{21}+L_{22}],
\end{aligned}
\tag{3.45}
$$

In view of (3.41)–(3.44), we have

$$
\begin{aligned}
\mathrm{I}_{11} &= \frac{1}{u_n^3} \sum_{1\le i\le n-1} a^{ii}\Big(\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}(-h_{ii,\alpha\beta})\Big) \\
&= 4u_n \sum_{1\le i,j\le n-1} a^{ii}u_{ni}a_{jj,i} + 4(p-1)u_n^2 \sum_{1\le i\le n-1} a_{ii,n} \\
&\quad - 2\sigma_1 \sum_{1\le i\le n-1} a^{ii}u_{ni}^2 + \Big(-\frac{8}{p-1}\theta + \frac{2p+6n-8}{p-1}\Big)u_n^2\sigma_1^2 \\
&\quad + (4-2p)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 \\
&\quad + [(4p-16)\theta - 6pn + 10p + 18n - 22] \sum_{1\le j\le n-1} u_{nj}^2 \\
&\quad + 4u_n^2\sigma_1\varphi_n + (8-2p)u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j.
\end{aligned} \tag{3.46}
$$

Step 3. Now we combine (3.26) and (3.46). It follows that

$$
\begin{aligned}
\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} &= -u_n^2 \sum_{1\le i,j,k\le n-1} a^{ii}a^{jj}a_{ij,k}^2 \\
&\quad - (p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii}a^{jj}a_{ij,n}^2 \\
&\quad + 4u_n \sum_{1\le i,j\le n-1} a^{ii}u_{ni}a_{jj,i} \\
&\quad + 4(p-1)u_n^2 \sum_{1\le i\le n-1} a_{ii,n} - 2\sigma_1 \sum_{1\le i\le n-1} a^{ii}u_{ni}^2 \\
&\quad + [2p\theta + (n-3)(1-p)] \sum_{1\le j\le n-1} u_{nj}^2 \\
&\quad + \Big(\frac{2}{p-1}\,\theta + 2\Big)u_n^2\sigma_1^2 \\
&\quad + (2\theta - n - 2p + 3)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2 \\
&\quad + (2-2p)u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j - 2u_n^2\sigma_1\varphi_n.
\end{aligned} \tag{3.47}
$$

Now we shall concentrate on formula (3.47), and we shall treat it by applying (3.6). In the following calculation, without loss of generality, we shall isolate the $i = 1$ terms in (3.6) so that we can obtain the sharp upper bounds of the quadratic form on $a_{ij,\alpha}$ in (3.47). Through very careful calculation, we get formula (3.62).

Then we choose a suitable θ such that we can include the sharp case as explained in Remark 1.4.

Rewrite (3.47) as

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} = M_1 + M_2 + M_3, \tag{3.48}$$

where

$$\begin{aligned}
M_1 &= -u_n^2 \sum_{1\le i,j,k\le n-1} a^{ii}a^{jj}a_{ij,k}^2 + 4u_n \sum_{1\le i,j\le n-1} a^{jj}u_{nj}a_{ii,j},\\
M_2 &= -(p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii}a^{jj}a_{ij,n}^2 + 4(p-1)u_n^2 \sum_{1\le i\le n-1} a_{ii,n},\\
M_3 &= -2\sigma_1 \sum_{1\le i\le n-1} a^{ii}u_{ni}^2 + [2p\theta + (n-3)(1-p)] \sum_{1\le j\le n-1} u_{nj}^2\\
&\quad + \left(\frac{2}{p-1}\theta + 2\right)u_n^2\sigma_1^2 + (2\theta - n - 2p + 3)u_n^2 \sum_{1\le j\le n-1} a_{jj}^2\\
&\quad + (2-2p)u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j - 2u_n^2\sigma_1\varphi_n.
\end{aligned}$$

For the term M_1, we have

$$M_1 = M_{11} + M_{12} + M_{13} + M_{14}, \tag{3.49}$$

where

$$\begin{aligned}
M_{11} &= -u_n^2 \sum_{\substack{1\le i,j,k\le n-1\\ i\ne j, j\ne k, k\ne i}} a^{ii}a^{jj}a_{ij,k}^2, \qquad M_{12} = -u_n^2 \sum_{1\le i\le n-1} (a^{ii}a_{ii,i})^2,\\
M_{13} &= -u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} (a^{ii}a_{ii,j})^2 - 2u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} a^{ii}a^{jj}a_{ij,j}^2,\\
M_{14} &= 4u_n \sum_{1\le i,j\le n-1} a^{jj}u_{nj}a_{ii,j}.
\end{aligned}$$

By (3.6),

$$a^{11}a_{11,\alpha} = -\sum_{2\le i\le n-1} a^{ii}a_{ii,\alpha} - 2\theta u_n^{-1}u_{n\alpha} + \varphi_\alpha; \tag{3.50}$$

hence

$$
\begin{aligned}
M_{12} &= -u_n^2(a^{11}a_{11,1})^2 - u_n^2 \sum_{2\le i\le n-1}(a^{ii}a_{ii,i})^2\\
&= -u_n^2\Big(\sum_{2\le i\le n-1} a^{ii}a_{ii,1}\Big)^2 - 4\theta u_n u_{n1}\sum_{2\le i\le n-1} a^{ii}a_{ii,1}\\
&\quad - 4\theta^2 u_{n1}^2 + 2u_n^2\varphi_1 \sum_{2\le i\le n-1} a^{ii}a_{ii,1} + 4\theta u_n u_{n1}\varphi_1 - u_n^2\varphi_1^2\\
&\quad - u_n^2 \sum_{2\le i\le n-1}(a^{ii}a_{ii,i})^2,
\end{aligned}
\tag{3.51}
$$

and

$$
\begin{aligned}
M_{13} &= -u_n^2 \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})\cdot(a^{11}a_{11,j})^2\\
&\quad - u_n^2 \sum_{\substack{2\le i\le n-1\\ 1\le j\le n-1\\ i\ne j}}(1+2a_{ii}a^{jj})\cdot(a^{ii}a_{ii,j})^2\\
&= -u_n^2 \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})\cdot\Big(\sum_{2\le i\le n-1} a^{ii}a_{ii,j}\Big)^2\\
&\quad - 4\theta^2 \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})u_{nj}^2\\
&\quad - 4\theta u_n \sum_{2\le i,j\le n-1}(1+2a_{11}a^{jj})u_{nj}a^{ii}a_{ii,j}\\
&\quad + 2u_n^2 \sum_{2\le i,j\le n-1}(1+2a_{11}a^{jj})a^{ii}a_{ii,j}\varphi_j\\
&\quad + 4\theta u_n \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})u_{nj}\varphi_j\\
&\quad - u_n^2 \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})\varphi_j^2\\
&\quad - u_n^2 \sum_{2\le i\le n-1}(1+2a_{ii}a^{11})\cdot(a^{ii}a_{ii,1})^2\\
&\quad - u_n^2 \sum_{\substack{2\le i,j\le n-1\\ i\ne j}}(1+2a_{ii}a^{jj})\cdot(a^{ii}a_{ii,j})^2.
\end{aligned}
\tag{3.52}
$$

Making use of (3.50) again,

$$
\begin{aligned}
M_{14} &= 4u_n \sum_{1\le i\le n-1} a^{ii}u_{ni}a_{11,i} + 4u_n \sum_{\substack{1\le i\le n-1\\ 2\le j\le n-1}} a^{ii}u_{ni}a_{jj,i} \\
&= 4u_nu_{n1} \sum_{2\le i\le n-1} (a^{11}-a^{ii})a_{ii,1} \\
&\quad - 4u_n \sum_{2\le i,j\le n-1} a_{11}a^{ii}a^{jj}u_{ni}a_{jj,i} \\
&\quad + 4u_n \sum_{2\le i,j\le n-1} a^{ii}u_{ni}a_{jj,i} + 4u_na_{11} \sum_{2\le i\le n-1} a^{ii}u_{ni}\varphi_i \\
&\quad - 8\theta \sum_{1\le i\le n-1} a_{11}a^{ii}u_{ni}^2.
\end{aligned}
\tag{3.53}
$$

By (3.49) and (3.51)–(3.53),

$$
\begin{aligned}
M_1 = &- \sum_{2\le i\le n-1} (1+2a_{ii}a^{11})\cdot(u_na^{ii}a_{ii,1})^2 - \Big(\sum_{2\le i\le n-1} u_na^{ii}a_{ii,1}\Big)^2 \\
&+ 4u_{n1} \sum_{2\le i\le n-1} (a_{ii}a^{11}-1-\theta)\cdot(u_na^{ii}a_{ii,1}) \\
&+ \sum_{2\le j\le n-1} \Big\{ -(1+2a_{11}a^{jj})\cdot\Big(\sum_{2\le i\le n-1} u_na^{ii}a_{ii,j}\Big)^2 \\
&\qquad - \sum_{\substack{2\le i\le n-1\\ i\ne j}} 2a_{ii}a^{jj}\cdot(u_na^{ii}a_{ii,j})^2 - \sum_{2\le i\le n-1} (u_na^{ii}a_{ii,j})^2 \\
&+ 4u_{nj} \sum_{2\le i\le n-1} [a_{ii}a^{jj}-\theta-(1+2\theta)a_{11}a^{jj}]\cdot(u_na^{ii}a_{ii,j}) \Big\} \\
&- u_n^2 \sum_{\substack{1\le i,j,k\le n-1\\ i\ne j, j\ne k, k\ne i}} a^{ii}a^{jj}a_{ij,k}^2 - (4\theta^2+8\theta)u_{n1}^2 \\
&- \sum_{2\le j\le n-1} (4\theta^2+8\theta^2a_{11}a^{jj}+8\theta a_{11}a^{jj})u_{nj}^2 \\
&+ 2u_n^2\varphi_1 \sum_{2\le i\le n-1} a^{ii}a_{ii,1} + 4\theta u_nu_{n1}\varphi_1 \\
&+ 2u_n^2 \sum_{2\le i,j\le n-1} (1+2a_{11}a^{jj})a^{ii}a_{ii,j}\varphi_j \\
&+ 4\theta u_n \sum_{2\le j\le n-1} (1+2a_{11}a^{jj})u_{nj}\varphi_j + 4u_na_{11} \sum_{2\le j\le n-1} a^{jj}u_{nj}\varphi_j \\
&- u_n^2\varphi_1^2 - u_n^2 \sum_{2\le j\le n-1} (1+2a_{11}a^{jj})\varphi_j^2.
\end{aligned}
\tag{3.54}
$$

For the term M_2, we write

$$M_2 = M_{21} + M_{22},$$

where

$$M_{21} = 4(p-1)u_n^2 \sum_{1\le i\le n-1} a_{ii,n},$$
$$M_{22} = -(p-1)u_n^2 \sum_{1\le i,j\le n-1} a^{ii}a^{jj}a_{ij,n}^2.$$

By (3.50), together with (3.11), we obtain

$$\begin{aligned} M_{21} &= 4(p-1)u_n^2 a_{11,n} + 4(p-1)u_n^2 \sum_{2\le i\le n-1} a_{ii,n} \\ &= -4(p-1)u_n^2 \sum_{2\le i\le n-1} a_{11}a^{ii}a_{ii,n} - 8\theta u_n^2 a_{11}\sigma_1 \\ &\quad + 4(p-1)u_n^2 a_{11}\varphi_n + 4(p-1)u_n^2 \sum_{2\le i\le n-1} a_{ii,n}. \end{aligned} \tag{3.55}$$

Also by (3.50) and (3.11),

$$\begin{aligned} M_{22} &= -(p-1)u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} a^{ii}a^{jj}a_{ij,n}^2 \\ &\quad - (p-1)u_n^2(a^{11}a_{11,n})^2 - (p-1)u_n^2 \sum_{2\le i\le n-1}(a^{ii}a_{ii,n})^2 \\ &= -(p-1)u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} a^{ii}a^{jj}a_{ij,n}^2 \\ &\quad - (p-1)u_n^2\Big(\sum_{2\le i\le n-1} a^{ii}a_{ii,n}\Big)^2 - \frac{4}{p-1}\theta^2u_n^2\sigma_1^2 \\ &\quad - 4\theta u_n^2\sigma_1 \sum_{2\le i\le n-1} a^{ii}a_{ii,n} + 2(p-1)u_n^2\varphi_n \sum_{2\le i\le n-1} a^{ii}a_{ii,n} \\ &\quad + 4\theta u_n^2\sigma_1\varphi_n - (p-1)u_n^2\varphi_n^2 - (p-1)u_n^2 \sum_{2\le i\le n-1}(a^{ii}a_{ii,n})^2. \end{aligned} \tag{3.56}$$

Combining (3.55)–(3.56), we get

$$
\begin{aligned}
M_2 = &-(p-1)u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} a^{ii}a^{jj}a_{ij,n}^2 \\
&-(p-1)u_n^2\Big(\sum_{2\le i\le n-1} a^{ii}a_{ii,n}\Big)^2 \\
&-(p-1)u_n^2 \sum_{2\le i\le n-1} (a^{ii}a_{ii,n})^2 \\
&+4(p-1)u_n^2 \sum_{2\le i\le n-1} (1-a_{11}a^{ii})a_{ii,n} \\
&-4\theta u_n^2\sigma_1 \sum_{2\le i\le n-1} a^{ii}a_{ii,n} \\
&-\frac{4}{p-1}\theta^2u_n^2\sigma_1^2 - 8\theta u_n^2 a_{11}\sigma_1 + 4(p-1)u_n^2 a_{11}\varphi_n \\
&+2(p-1)u_n^2\varphi_n \sum_{2\le i\le n-1} a^{ii}a_{ii,n} + 4\theta u_n^2\sigma_1\varphi_n - (p-1)u_n^2\varphi_n^2.
\end{aligned}
\tag{3.57}
$$

After inserting (3.54) and (3.57) into (3.48), we regroup the terms in (3.48) in a natural way: N_1, the terms involving $a_{ii,n}$ $(2\le i\le n-1)$; N_2, the terms involving $a_{ii,1}$ $(2\le i\le n-1)$; N_3, the terms involving $a_{ii,j}$, $(2\le i,j\le n-1)$; N_4, the terms involving the gradient of φ; and N_5, all the rest of the terms. More precisely,

$$
\begin{aligned}
N_1 = (p-1)\Big\{&-\sum_{2\le i\le n-1}(u_n a^{ii}a_{ii,n})^2 - \Big(\sum_{2\le i\le n-1} u_n a^{ii}a_{ii,n}\Big)^2 \\
&+4u_n\sum_{2\le i\le n-1}\Big(a_{ii}-a_{11}-\frac{1}{p-1}\theta\sigma_1\Big)\cdot(u_n a^{ii}a_{ii,n})\Big\},
\end{aligned}
$$

$$
\begin{aligned}
N_2 = &-\sum_{2\le i\le n-1}(1+2a_{ii}a^{11})\cdot(u_n a^{ii}a_{ii,1})^2 - \Big(\sum_{2\le i\le n-1} u_n a^{ii}a_{ii,1}\Big)^2 \\
&+4u_{n1}\sum_{2\le i\le n-1}(a_{ii}a^{11}-1-\theta)\cdot(u_n a^{ii}a_{ii,1}),
\end{aligned}
$$

and

$$N_3 = \sum_{2\le j\le n-1}\Bigg\{-(1+2a_{11}a^{jj})\cdot\Big(\sum_{2\le i\le n-1} u_n a^{ii} a_{ii,j}\Big)^2 - \sum_{\substack{2\le i\le n-1\\ i\ne j}} 2a_{ii}a^{jj}\cdot(u_n a^{ii}a_{ii,j})^2 - \sum_{2\le i\le n-1}(u_n a^{ii}a_{ii,j})^2 + 4u_{nj}\sum_{2\le i\le n-1}[a_{ii}a^{jj}-\theta-(1+2\theta)a_{11}a^{jj}]\cdot(u_n a^{ii}a_{ii,j})\Bigg\}.$$

For the terms involving the gradient of φ, we have

$$\begin{aligned} N_4 = {} & 4\theta u_n u_{n1}\varphi_1 + 2u_n^2\varphi_1 \sum_{2\le i\le n-1} a^{ii}a_{ii,1} + (2-2p)u_n \sum_{1\le j\le n-1} u_{nj}\varphi_j \\ & + 2u_n^2 \sum_{2\le i,j\le n-1}(1+2a_{11}a^{jj})a^{ii}a_{ii,j}\varphi_j \\ & + 4\theta u_n \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})u_{nj}\varphi_j \\ & + 4u_n a_{11}\sum_{2\le j\le n-1} a^{jj}u_{nj}\varphi_j + 2(p-1)u_n^2\varphi_n \sum_{2\le i\le n-1} a^{ii}a_{ii,n} \\ & + (4\theta-2)u_n^2\sigma_1\varphi_n + 4(p-1)u_n^2 a_{11}\varphi_n - u_n^2\varphi_1^2 \\ & - u_n^2 \sum_{2\le j\le n-1}(1+2a_{11}a^{jj})\varphi_j^2 - (p-1)u_n^2\varphi_n^2, \end{aligned}$$

and the remaining terms are

$$\begin{aligned} N_5 = {} & \Big[-\frac{4}{p-1}\theta^2\sigma_1^2 - 8\theta a_{11}\sigma_1 + \Big(\frac{2}{p-1}\,\theta+2\Big)\sigma_1^2 \\ & + (2\theta-n-2p+3)\sum_{1\le j\le n-1} a_{jj}^2\Big]u_n^2 \\ & + \sum_{2\le j\le n-1}\big[-4\theta^2 - 8\theta^2 a_{11}a^{jj} - 8\theta a_{11}a^{jj} - 2\sigma_1 a^{jj} \\ & \qquad + 2p\theta + (n-3)(1-p)\big]u_{nj}^2 \\ & + \big[-4\theta^2 - 8\theta - 2\sigma_1 a^{11} + 2p\theta + (n-3)(1-p)\big]u_{n1}^2 \\ & - u_n^2 \sum_{\substack{1\le i,j,k\le n-1\\ i\ne j, j\ne k, k\ne i}} a^{ii}a^{jj}a_{ij,k}^2 - (p-1)u_n^2 \sum_{\substack{1\le i,j\le n-1\\ i\ne j}} a^{ii}a^{jj}a_{ij,n}^2. \end{aligned}$$

It follows that

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} = N_1 + N_2 + N_3 + N_4 + N_5. \tag{3.58}$$

We shall maximize the terms N_1, N_2, and N_3 via Lemma 3.1 for a different choice of parameters.

First, let us examine the term N_1. For $2 \le i \le n-1$, set $X_i = u_n a^{ii} a_{ii,n}$, $\lambda = 1$, $\mu = u_n$, $b_i = 1$, and $c_i = a_{ii} - a_{11} - \frac{1}{p-1}\theta\sigma_1$. By Lemma 3.1, we have

$$\begin{aligned} N_1 &\le 4(p-1)u_n^2\Big[\sum_{2\le i\le n-1} c_i^2 - \frac{1}{n-1}\Big(\sum_{2\le i\le n-1} c_i\Big)^2\Big] \\ &= \Big[4(p-1)\sum_{1\le j\le n-1} a_{jj}^2 + \frac{4(n-2)}{(n-1)(p-1)}\theta^2\sigma_1^2 \\ &\qquad - \frac{8}{n-1}\theta\sigma_1^2 - \frac{4(p-1)}{n-1}\sigma_1^2 + 8\theta a_{11}\sigma_1\Big]u_n^2. \end{aligned} \tag{3.59}$$

For the term N_2, set $X_i = u_n a^{ii} a_{ii,1}$, $\lambda = 1$, $\mu = u_{n1}$, $b_i = 1 + 2a_{ii}a^{11}$, and $c_i = a_{ii}a^{11} - 1 - \theta$ where $2 \le i \le n-1$. By Lemma 3.1, we have

$$N_2 \le 4u_{n1}^2\Gamma_1,$$

where

$$\Gamma_1 = \sum_{2\le i\le n-1}\frac{c_i^2}{b_i} - \Big(1 + \sum_{2\le i\le n-1}\frac{1}{b_i}\Big)^{-1}\Big(\sum_{2\le i\le n-1}\frac{c_i}{b_i}\Big)^2.$$

Next we shall simplify Γ_1. By denoting

$$\beta_i = \frac{1}{b_i},$$

we have

$$a_{ii}a^{11} = \frac{1}{2\beta_i} - \frac{1}{2}, \qquad c_i = \frac{1}{2\beta_i} - \frac{3}{2} - \theta.$$

Hence

$$\begin{aligned} \Gamma_1 &= \sum_{2\le i\le n-1}\beta_i\Big(\frac{1}{2\beta_i} - \frac{3}{2} - \theta\Big)^2 \\ &\quad - \Big(1 + \sum_{2\le i\le n-1}\beta_i\Big)^{-1}\Big[\sum_{2\le i\le n-1}\beta_i\Big(\frac{1}{2\beta_i} - \frac{3}{2} - \theta\Big)\Big]^2 \\ &= \frac{1}{4}\sum_{2\le i\le n-1}\frac{1}{\beta_i} - \Big(1 + \sum_{2\le i\le n-1}\beta_i\Big)^{-1}\Big(\frac{n+1}{2} + \theta\Big)^2 + \Big(\frac{3}{2} + \theta\Big)^2. \end{aligned}$$

Since

$$1 \leq 1 + \sum_{2 \leq i \leq n-1} \beta_i \leq n-1,$$

it follows that

$$\begin{aligned}\Gamma_1 &\leq \frac{1}{4} \sum_{2 \leq i \leq n-1} \frac{1}{\beta_i} - \frac{1}{n-1}\left(\frac{n+1}{2} + \theta\right)^2 + \left(\frac{3}{2} + \theta\right)^2 \\ &= \frac{n-2}{n-1}(1+\theta)^2 + \frac{1}{4}(2\sigma_1 a^{11} - 2).\end{aligned}$$

Therefore,

$$N_2 \leq \left[\frac{4(n-2)}{n-1}(1+\theta)^2 + 2\sigma_1 a^{11} - 2\right] u_{n1}^2. \tag{3.60}$$

Now we will deal with the term N_3. For every $j \geq 2$ fixed and $2 \leq i \leq n-1$, set $X_i = u_n a^{ii} a_{ii,j}$, $\lambda = 1 + 2a_{11}a^{jj}$, $\mu = u_{nj}$, $b_i = 1 + 2a_{ii}a^{jj}$ $(i \neq j)$, $b_j = 1$, and $c_i = a_{ii}a^{jj} - \theta - (1+2\theta)a_{11}a^{jj}$. By Lemma 3.1, we have

$$N_3 \leq 4 \sum_{2 \leq j \leq n-1} \Gamma_j u_{nj}^2,$$

where

$$\Gamma_j = c_j^2 + \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \frac{c_i^2}{b_i} - \left(\frac{1}{\lambda} + 1 + \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \frac{1}{b_i}\right)^{-1} \left(c_j + \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \frac{c_i}{b_i}\right)^2.$$

Also, denoting

$$\beta_i = \frac{1}{b_i} \quad (i \neq j),$$

we have

$$a_{ii}a^{jj} = \frac{1}{2\beta_i} - \frac{1}{2}, \qquad c_i = \frac{1}{2\beta_i} - \delta,$$

where

$$\delta = \frac{1}{2} + \theta + (1+2\theta)a_{11}a^{jj}.$$

Notice that

$$c_j = \frac{3}{2} - \delta, \qquad \frac{\delta}{\lambda} = \frac{1}{2} + \theta.$$

We obtain

$$\begin{aligned}\Gamma_j = c_j^2 &+ \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \beta_i \left(\frac{1}{2\beta_i} - \delta\right)^2 \\ &- \left(\frac{1}{\lambda} + 1 + \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \beta_i\right)^{-1} \left[c_j + \sum_{\substack{2 \leq i \leq n-1 \\ i \neq j}} \beta_i \left(\frac{1}{2\beta_i} - \delta\right)\right]^2 =\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4} \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \frac{1}{\beta_i} - \Big(\frac{1}{\lambda} + 1 + \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \beta_i\Big)^{-1} \Big(\frac{n}{2} + \frac{\delta}{\lambda}\Big)^2 + \frac{9}{4} + \frac{\delta^2}{\lambda} \\
&= \frac{1}{4} \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \frac{1}{\beta_i} - \Big(\frac{1}{\lambda} + 1 + \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \beta_i\Big)^{-1} \Big(\frac{n+1}{2} + \theta\Big)^2 \\
&\quad + \frac{9}{4} + \Big(\frac{1}{2} + \theta\Big)\delta.
\end{aligned}
$$

Obviously,

$$
1 \le \frac{1}{\lambda} + 1 + \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \beta_i \le n - 1;
$$

hence

$$
\begin{aligned}
\Gamma_j &\le \frac{1}{4} \sum_{\substack{2 \le i \le n-1 \\ i \ne j}} \frac{1}{\beta_i} - \frac{1}{n-1}\Big(\frac{n+1}{2} + \theta\Big)^2 + \frac{9}{4} + \Big(\frac{1}{2} + \theta\Big)\delta \\
&= \frac{n-2}{n-1}\,\theta^2 - \frac{2}{n-1}\,\theta + \frac{n-3}{2(n-1)} + \frac{1}{2}\,\sigma_1 a^{jj} + 2\theta^2 a_{11} a^{jj} + 2\theta a_{11} a^{jj}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
N_3 \le \sum_{2 \le j \le n-1} \Big(&\frac{4n-8}{n-1}\,\theta^2 - \frac{8}{n-1}\,\theta + \frac{2n-6}{n-1} \\
&+ 2\sigma_1 a^{jj} + 8\theta^2 a_{11} a^{jj} + 8\theta a_{11} a^{jj}\Big) u_{nj}^2.
\end{aligned}
\tag{3.61}
$$

If we let

$$
\begin{aligned}
q_1(\theta) &= 2\theta + 2p - n - 1, \\
q_2(\theta) &= \frac{1}{(n-1)(p-1)}[-4\theta^2 + (2n - 8p + 6)\theta + 2(p-1)(n - 2p + 1)], \\
q_3(\theta) &= -\frac{4}{n-1}\theta^2 + \Big(2p - \frac{8}{n-1}\Big)\theta + n - 1 - p(n-3) - \frac{4}{n-1},
\end{aligned}
$$

then collecting (3.58)–(3.61), we finally obtain

$$
\begin{aligned}
\sum_{1 \le \alpha, \beta \le n} F^{\alpha\beta} \varphi_{\alpha\beta} \le \Big[&q_1(\theta) \sum_{1 \le j \le n-1} a_{jj}^2 + q_2(\theta)\sigma_1^2\Big] u_n^2 \\
&+ q_3(\theta) \sum_{1 \le j \le n-1} u_{nj}^2 \mod \nabla\varphi,
\end{aligned}
\tag{3.62}
$$

where we have suppressed the terms containing the gradient of φ with locally bounded coefficients.

By a simple observation, a sufficient condition to guarantee

$$\sum_{1\le\alpha,\beta\le n} F^{\alpha\beta}\varphi_{\alpha\beta} \le 0 \mod \nabla\varphi$$

is

$$\begin{cases} q_1(\theta)+q_2(\theta)\le 0,\\ q_1(\theta)+(n-1)q_2(\theta)\le 0,\\ q_3(\theta)\le 0. \end{cases} \tag{3.63}$$

By solving $q_1(\theta)+q_2(\theta)\le 0$, we have

- $1<p<2$: $\theta\le\frac{(n-3)(p-1)}{2}$ or $\theta\ge\frac{n+1}{2}-p$;
- $p=2$: $-\infty<\theta<+\infty$;
- $2<p<+\infty$: $\theta\le\frac{n+1}{2}-p$ or $\theta\ge\frac{(n-3)(p-1)}{2}$,

while $q_1(\theta)+(n-1)q_2(\theta)\le 0$ implies

- $1<p<n$: $\theta\le\frac{1-p}{2}$ or $\theta\ge\frac{n+1}{2}-p$;
- $p=n$: $-\infty<\theta<+\infty$;
- $n<p<+\infty$: $\theta\le\frac{n+1}{2}-p$ or $\theta\ge\frac{1-p}{2}$.

Also, from $q_3(\theta)\le 0$, we get

- $1<p<2$: $\theta\le\frac{(n-1)(p-1)}{2}-1$ or $\theta\ge\frac{n-3}{2}$;
- $p=2$: $-\infty<\theta<+\infty$;
- $2<p<+\infty$: $\theta\le\frac{n-3}{2}$ or $\theta\ge\frac{(n-1)(p-1)}{2}-1$.

Therefore, by solving the inequalities in (3.63), we can obtain the following solutions:

For $n=2$:

- $1<p<2$: $\theta\le\frac{p-3}{2}$ or $-\frac{1}{2}\le\theta\le\frac{1-p}{2}$ or $\theta\ge\frac{3}{2}-p$;
- $p=2$: $-\infty<\theta<+\infty$;
- $2<p<+\infty$: $\theta\le\frac{3}{2}-p$ or $\frac{1-p}{2}\le\theta\le-\frac{1}{2}$ or $\theta\ge\frac{p-3}{2}$.

For $n=3$:

- $1<p<2$: $\theta\le\min(\frac{1-p}{2},p-2)$ or $\theta\ge 2-p$;
- $p=2$: $\theta\le-\frac{1}{2}$ or $\theta\ge 0$;
- $2<p<3$: $\theta\le\frac{1-p}{2}$ or $\theta=2-p$ or $\theta=0$ or $\theta\ge p-2$;
- $3\le p<+\infty$: $\theta\le 2-p$ or $\theta=0$ or $\theta\ge p-2$.

For $n\ge 4$:

- $1<p<2$: $\theta\le\min(\frac{1-p}{2},\frac{(n-1)(p-1)}{2}-1)$ or $\theta\ge\frac{n+1}{2}-p$;
- $p=2$: $\theta\le-\frac{1}{2}$ or $\theta\ge\frac{n-3}{2}$;
- $2<p<n$: $\theta\le\frac{1-p}{2}$ or $\theta=\frac{n+1}{2}-p$ or $\theta\ge\frac{(n-1)(p-1)}{2}-1$;
- $n\le p<+\infty$: $\theta\le\frac{n+1}{2}-p$ or $\theta\ge\frac{(n-1)(p-1)}{2}-1$.

Combining the above results we arrive at the following conclusions:

Case 1. For $n \geq 2$, $1 < p < +\infty$, set $\theta = \frac{n+1}{2} - p$. Then for $\psi(x) = |\nabla u|^{n+1-2p} K(x)$ and $\varphi = \log \psi(x) = \log K(x) + (\frac{n+1}{2} - p) \log |\nabla u|^2$, inequality (3.62) turns into

$$\begin{aligned}\sum_{1\leq\alpha,\beta\leq n} F^{\alpha\beta}\varphi_{\alpha\beta} &\leq -\frac{2(n+1)}{n-1}(p-2)^2 \sum_{1\leq j\leq n-1} u_{nj}^2 \\ &\leq 0 \mod \nabla\varphi.\end{aligned}$$

Case 2. For $n = 2$, $1 < p < +\infty$, and for $n \geq 3$, $1 + \frac{2}{n} \leq p \leq n$, we set $\theta = \frac{1-p}{2}$. Then for $\psi(x) = |\nabla u|^{1-p} K(x)$ and $\varphi = \log \psi(x) = \log K(x) + \frac{1-p}{2} \log |\nabla u|^2$, (3.62) turns into

$$\begin{aligned}\sum_{1\leq\alpha,\beta\leq n} F^{\alpha\beta}\varphi_{\alpha\beta} &\leq (p-n)\Bigg(\sum_{1\leq j\leq n-1} a_{jj}^2 - \frac{1}{n-1}\sigma_1^2\Bigg) \\ &\qquad - \frac{n}{n-1}(p-1-\frac{2}{n})(p+n-4) \sum_{1\leq j\leq n-1} u_{nj}^2 \\ &\leq 0 \mod \nabla\varphi.\end{aligned}$$

Case 3. Now we set $\theta = 0$. Then for

$$n = 2,\ \frac{3}{2} \leq p \leq 3 \quad \text{or} \quad n = 3,\ 2 \leq p < \infty \quad \text{or} \quad n \geq 4,\ p = \frac{n+1}{2},$$

and $\varphi = \log \psi(x) = \log K(x)$, (3.62) turns into

$$\begin{aligned}\sum_{1\leq\alpha,\beta\leq n} F^{\alpha\beta}\varphi_{\alpha\beta} &\leq (2p-n-1)\Bigg(\sum_{1\leq j\leq n-1} a_{jj}^2 - \frac{2}{n-1}\sigma_1^2\Bigg) \\ &\qquad + (n-3)\left(\frac{n+1}{n-1} - p\right) \sum_{1\leq j\leq n-1} u_{nj}^2 \\ &\leq 0 \mod \nabla\varphi.\end{aligned}$$

In the above formulas we have suppressed the terms containing the gradient of φ with locally bounded coefficients.

From the minimum principle we complete our proof. □

Now we state the following elementary calculus lemma.

LEMMA 3.1 *Let $\lambda \geq 0$, $\mu \in \mathbb{R}$, $b_i > 0$, and $c_i \in \mathbb{R}$ for $2 \leq i \leq n-1$. Define the quadratic polynomial*

$$Q(X_2, \ldots, X_{n-1}) = -\sum_{2\leq i\leq n-1} b_i X_i^2 - \lambda\Bigg(\sum_{2\leq i\leq n-1} X_i\Bigg)^2 + 4\mu \sum_{2\leq i\leq n-1} c_i X_i.$$

Then we have

$$Q(X_2, \dots, X_{n-1}) \le 4\mu^2 \Gamma,$$

where

$$\Gamma = \sum_{2\le i\le n-1} \frac{c_i^2}{b_i} - \lambda\Big(1 + \lambda \sum_{2\le i\le n-1} \frac{1}{b_i}\Big)^{-1}\Big(\sum_{2\le i\le n-1} \frac{c_i}{b_i}\Big)^2.$$

PROOF: We shall maximize the quadratic polynomial $Q(X_2, \dots, X_{n-1})$. At the maximum point, we have

$$\frac{\partial Q}{\partial X_k} = 0 \quad \text{for } k = 2, 3, \dots, n-1,$$

i.e.,

$$X_k = 2\mu\frac{c_k}{b_k} - \frac{\lambda}{b_k}\sum_{2\le i\le n-1} X_i \quad \text{for } k = 2, 3, \dots, n-1. \tag{3.64}$$

Summing (3.64) with respect to $2 \le k \le n-1$, we obtain

$$\sum_{2\le k\le n-1} X_k = 2\mu\Big(1 + \lambda \sum_{2\le k\le n-1}\frac{1}{b_k}\Big)^{-1} \sum_{2\le k\le n-1}\frac{c_k}{b_k}. \tag{3.65}$$

Hence

$$\begin{aligned}
&Q(X_2, \dots, X_{n-1}) \\
&\quad \le -\sum_{2\le i\le n-1} b_i\Big(2\mu\frac{c_i}{b_i} - \frac{\lambda}{b_i}\sum_{2\le j\le n-1} X_j\Big)^2 - \lambda\Big(\sum_{2\le i\le n-1} X_i\Big)^2 \\
&\qquad + 4\mu\sum_{2\le i\le n-1} c_i\Big(2\mu\frac{c_i}{b_i} - \frac{\lambda}{b_i}\sum_{2\le j\le n-1} X_j\Big) \\
&\quad = 4\mu^2\sum_{2\le i\le n-1}\frac{c_i^2}{b_i} - \lambda\Big(1 + \lambda\sum_{2\le i\le n-1}\frac{1}{b_i}\Big)\Big(\sum_{2\le j\le n-1} X_j\Big)^2 \\
&\quad = 4\mu^2\Big[\sum_{2\le i\le n-1}\frac{c_i^2}{b_i} - \lambda\Big(1 + \lambda\sum_{2\le i\le n-1}\frac{1}{b_i}\Big)^{-1}\Big(\sum_{2\le i\le n-1}\frac{c_i}{b_i}\Big)^2\Big].
\end{aligned}$$

□

4 Proof of the Corollary and Some Remarks

In this section, we first prove a proposition on the monotonicity of the norm of the gradient along the gradient direction, which also appeared in [14, 16]. Using this observation, we prove Corollary 1.3. We mention that the combination of Corollary 1.3 and a deformation process gives a new approach to studying the convexity of the level sets of the solution to the p-harmonic equation on a convex ring with homogeneous Dirichlet boundary conditions.

PROPOSITION 4.1 *Let u satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1, \end{cases} \tag{4.1}$$

where $1 < p < +\infty$, Ω_0 and Ω_1 are bounded smooth convex domains in $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. Then $|\nabla u|$ strictly increases in the direction ∇u. It follows that $|\nabla u|$ attains its minimum on $\partial\Omega_0$ and attains its maximum on $\partial\Omega_1$.

PROOF: By the Gabriel-Lewis theorem, Theorem 1.1, the level set of u is strictly convex with respect to the normal direction of ∇u. At any fixed point x_0 in Ω, we may let $u_n = |\nabla u| > 0$ and $u_i = 0$ $(1 \leq i \leq n-1)$ by rotation. Let H be the mean curvature of the level set with respect to the normal direction ∇u. Then (4.1) implies

$$(p-1)u_{nn} = -\sum_{1 \leq i \leq n-1} u_{ii} = u_n H; \tag{4.2}$$

hence

$$\sum_{1 \leq \alpha \leq n} (|\nabla u|^2)_\alpha u_\alpha = 2u_n^2 u_{nn} = \frac{2}{p-1} u_n^3 H > 0 \tag{4.3}$$

where the last inequality is due to the strict convexity of the level set. □

Now we combine Theorem 1.2 and the above proposition to give a proof of Corollary 1.3.

PROOF OF COROLLARY 1.3: If u is the smooth solution of equation (4.1), then from the Gabriel-Lewis theorem, Theorem 1.1, we know that the level sets of u are strictly convex with respect to the normal ∇u. Since $|\nabla u|$ attains its minimum on $\partial\Omega_0$ and attains its maximum on $\partial\Omega_1$, from Theorem 1.2, we can get the estimates in Corollary 1.3. □

Remark 4.2. From Corollary 1.3, we can combine the deformation process to give a new proof of the Gabriel-Lewis theorem when Ω_0 and Ω_1 are bounded, smooth, strictly convex domains.

Let $0 \in \Omega_1$. At the initial time we let the domain be the standard ball ring $U = B_R(0) \setminus \bar{B}_r(0)$ $(0 < r < R)$, and we let

$$\Omega_t = (1-t)U + t\Omega, \quad 0 \leq t \leq 1,$$

where the sum is the Minkowski vector sum and $\Omega = \Omega_0 \setminus \bar{\Omega}_1$. So the domain Ω_t is a family of smooth, strictly convex rings (see Schneider [19]). We assume the p-harmonic function u_t satisfies the homogeneous Dirichlet boundary conditions in the convex ring Ω_t (see (1.1)). By the maximum principle $|\nabla u_t| \neq 0$ in Ω_t (Kawohl [10]) and by standard elliptic theory, we have the uniform estimates on $\|u_t\|_{C^3(\Omega_t)}$ with the bound depending only on the geometry of Ω.

PROOF OF THEOREM 1.1: If Theorem 1.1 is not true, then there exist $0 < t_0 < 1$; it is the first time that the Gaussian curvature of the level sets of u_{t_0} becomes 0 at some point $x_{t_0} \in \Omega_{t_0}$. Take a sequence $\{t_i\}$ such that $t_i \to t_0$ $(0 < t_i < t_0)$; then from estimates (1.4) and (1.5), we get uniformly positive lower bounds for the Gaussian curvature of the level sets of u_{t_i}. Since we have the uniform estimates on $\|u_t\|_{C^3(\Omega_t)}$, we can take the limit and get positive lower bounds on the Gaussian curvature of the level sets of u_{t_0}. This contradicts the initial assumption. Then we complete the proof of Theorem 1.1 on the strictly convex domain case. □

For the general convex domain we can first use the approximation with a strictly convex domain to get the convexity of the level sets; then we use the constant rank theorem of the level sets by Korevaar [11] to get its strict convexity.

Acknowledgment. The first author would like to thank Prof. P. Guan for useful discussions on this subject, and he would also like to thank the encouragement of Prof. Xu-jia Wang. The research of the first author was supported by National Science Foundation of China Grant No. 10671186 and the BaiRen Program in the Chinese Academy of Sciences.

Bibliography

[1] Bian, B. J.; Guan, P.; Ma, X. N.; Xu, L. A microscopic convexity principle for the level sets of soluion for nonlinear elliptic partial differential equations. Preprint.

[2] Bianchini, C.; Longinetti, M.; Salani, P. Quasiconcave solutions to elliptic problems in convex rings. *Indiana Univ. Math. J*. **58** (2009), no. 4, 1565–1590.

[3] Caffarelli, L. A.; Friedman, A. Convexity of solutions of some semilinear elliptic equations. *Duke Math. J*. **52** (1985), no. 2, 431–455.

[4] Caffarelli, L.; Nirenberg, L.; Spruck, J. Nonlinear second order elliptic equations. IV. Starshaped compact Weingarten hypersurfaces. *Current topics in partial differential equations*, 1–26. Kinokuniya, Tokyo, 1986.

[5] Caffarelli, L.; Spruck, J. Convexity properties of solutions to some classical variational problems. *Comm. Partial Differential Equations* **7** (1982), no. 11, 1337–1379.

[6] do Carmo, M. P. Differential geometry of curves and surfaces. Prentice-Hall, Englewood Cliffs, N.J., 1976.

[7] Dolbeault, J.; Monneau, R. Convexity estimates for nonlinear elliptic equations and application to free boundary problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **19** (2002), no. 6, 903–926.

[8] Gabriel, R. M. A result concerning convex level surfaces of 3-dimensional harmonic functions. *J. London Math. Soc*. **32** (1957), 286–294.

[9] Jost, J.; Ma, X. N.; Ou, Q. Curvature estimates in dimensions 2 and 3 for the level sets of p-harmonic functions in convex rings. Preprint.

[10] Kawohl, B. Rearrangements and convexity of level sets in PDE. Lectures Notes in Mathematics, 1150. Springer, Berlin, 1985.

[11] Korevaar, N. J. Convexity of level sets for solutions to elliptic ring problems. *Comm. Partial Differential Equations* **15** (1990), no. 4, 541–556.

[12] Lewis, J. L. Capacitary functions in convex rings. *Arch. Rational Mech. Anal*. **66** (1977), no. 3, 201–224.

[13] Lin, F. H. Complexity of solutions of partial differential equations. *Handbook of geometric analysis*, no. 1, 229–258. Advanced Lectures in Mathematics (ALM), 7. International, Somerville, Mass., 2008.

[14] Longinetti, M. Convexity of the level lines of harmonic functions. *Boll. Un. Mat. Ital. A (6)* **2** (1983), no. 1, 71–75.

[15] Longinetti, M. On minimal surfaces bounded by two convex curves in parallel planes. *J. Differential Equations* **67** (1987), no. 3, 344–358.

[16] Longinetti, M.; Salani, P. On the Hessian matrix and Minkowski addition of quasiconvex functions. *J. Math. Pures Appl. (9)* **88** (2007), no. 3, 276–292.

[17] Ma, X. N.; Ye, J.; Ye, Y. Principal curvature estimates for the level sets of harmonic functions and minimal graph in $\mathbb{R}^3$. Preprint.

[18] Rosay, J.-P.; Rudin, W. A maximum principle for sums of subharmonic functions, and the convexity of level sets. *Michigan Math. J*. **36** (1989), no. 1, 95–111.

[19] Schneider, R. *Convex bodies: the Brunn-Minkowski theory*. Encyclopedia of Mathematics and Its Applications, 44. Cambridge University Press, Cambridge, 1993.

[20] Shiffman, M. On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. *Ann. of Math. (2)* **63** (1956), 77–90.

[21] Spivak, M. *A comprehensive introduction to differential geometry*. Vol. IV. 2nd ed. Publish or Perish, Wilmington, Del., 1979.

[22] Talenti, G. On functions, whose lines of steepest descent bend proportionally to level lines. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **10** (1983), no. 4, 587–605.

XI-NAN MA
University of Science and Technology of China
Department of Mathematics
Hefei 230026 Anhui Province
P.R. CHINA
E-mail: xinan@ustc.edu.cn

QIANZHONG OU
Hezhou University
Department of Mathematics
Hezhou 542800 Guangxi Province
P.R. CHINA
E-mail: ouqzh@163.com

WEI ZHANG
University of Science and Technology of China
Department of Mathematics
Hefei 230026 Anhui Province
P.R. CHINA
E-mail: zhwmath@mail.ustc.edu.cn

Received September 2009.