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The Convexity of Level Sets for Solutions to Partial Differential Equations

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Abstract

We review the methods and results in the study of convexity of level sets for solutions to partial differential equations. We also derive a sharp Gauss curvature estimate for the level sets of three dimensional minimal graph defined on a convex ring in $\mathbb{R}^3$ with boundary contained in two parallel hyperplanes.

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1 Introduction and some classical examples

Convexity is one of the basic properties in geometry and analysis. It can be expressed in term of curvatures when the object is smooth. Therefore convexity has not only been extensively studied in geometry, but also an interesting topic in analysis for many decades. For solutions of partial differential equations, we are concerned with both the convexity of solutions and the convexity of level sets of solutions. In this paper, we briefly review the study of convexity of level sets, and also prove some new results.

The study of convexity can be traced back to Morse [41, 42] in 1920', where he studied the critical points of solutions to partial differential equations. Clearly, critical points are closely related to the geometry of solutions. Pogorelov [43, 44] did some inaugurated works on convexity. He introduced new a priori estimates

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for the convexity of hypersurfaces. In 1956, Shiffman [46] proved several beautiful theorems concerning the geometry of a minimal annulus whose boundary consists of two closed convex Jordan curves contained in two parallel planes P_1, P_2, respectively. One of his theorems states that the intersection of the surface with any parallel plane P, between P_1 and P_2, is a convex Jordan curve, from which it follows in particular that this surface is embedded. To our knowledge, this is the first precise result on convexity of level sets.

In 1957, Gabriel [13] proved that the level sets of the Green function on a 3-dimension convex domain Ω are strictly convex. Gabriel introduced the following so called *quasi-concave function*:

$$Q(x,y) := u(\frac{x+y}{2}) - \min\{u(x), u(y)\},\ (x,y) \in \Omega \times \Omega. \tag{1.1}$$

Clearly, the level sets of u are all strictly convex if and only if $Q(x,y) > 0, \forall (x,y) \in \Omega \times \Omega, x \neq y$.

For the following boundary problem:

$$\begin{cases} \triangle u = -2 \text{ in } \Omega \subset \mathbb{R}^2, \\ u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Makar-Limanov [39] proved, in 1971, that $u^{\frac{1}{2}}$ is strictly concave on a bounded smooth convex domain Ω . This means that even though the solution itself may not be convex, but a function of the solution can be convex. This striking idea sparked further investigations on the convexity.

In 1976, Brascamp-Lieb [4] considered the following heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \triangle u & \text{in} \quad \Omega \times (0,\infty), \\ u = 0 & \text{on} \quad \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x) & \forall\, x \in \Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$, and u_0 is a given function vanishing on the boundary $\partial\Omega$. When $\log u_0$ is concave, they proved that $\forall\, t > 0$, $\log u$ is always concave (in x), too. This property also implies that the first eigenfunction to the following Laplace equation is log-concave:

$$\begin{cases} \triangle u = -\lambda_1 u & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In this paper, we first consider in Section 2 the convexity of level sets of solutions. Then as a continuation of Section 2, we discuss in Section 3 the convexity of the solution itself. In Section 4, we give some curvature formulas on the level sets of the hypersurfaces, and give another proof of a constant rank theorem on the second fundamental form of the convex level sets of immersed hypersurfaces in $\mathbb{R}^3$ via moving frame. In the last section, we prove a new result on the Gauss curvature estimates of the level sets of minimal graph defined on convex ring in $\mathbb{R}^3$ with boundary contained in two parallel hyperplanes. In this paper we will restrict ourselves to materials which we are interested in and familiar with. The readers are referred to the book [26] by Kawohl, which is still a good source on this subject.

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2 History of convexity of level set

Obviously, when a function is convex, it's level sets are all convex. But we emphasize on the convexity of level sets, and regard the convexity of function itself as an ingredient of it, although we will discuss the convexity of function itself separately in the next section.

For a function u defined in a domain Ω in $\mathbb{R}^n$, it's level set can be usually defined in the following four sense:

Definition 2.1. $S^t := \{x \in \Omega | u(x) > t\}$.

Definition 2.2. $S_t := \{x \in \Omega | u(x) < t\}$.

Definition 2.3. $S(t) := \{x \in \Omega | u(x) = t\}$.

Definition 2.4. $S(x,t) := \{(x,t) \in \mathbb{R}^{n+1} | u(x) = t,\, x \in \Omega\}$.

Where t is any given real constant.

Sometimes S^t in **Definition 2.1** may be called *super level set*, accordingly S_t called *sub level set*. One should take a suitable one of the four definitions contextually. If one look $G := \{(x, u(x)) | x \in \Omega\}$ as the graph of u, then one can also define the level set of u by cutting G with a horizontal plane $P_t := \{(x,t) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n\}$. For example, when P_t cuts G into two parts, S^t may be popularly said as the projection on Ω of the upper part, and $S(x,t)$ is just the kerf. To get a regular shape of the level set of u, one usually assume that P_t cuts G transversally. In particularly, if u is differentiable, then one usually assume $|\nabla u| \neq 0$. We say that the level set of u is convex, which means that $S^t(S_t)$ is a convex domain or that $S(t)(S(x,t))$ is a convex hypersurface.

Next, we will expose the history and methods in studying the convexity of level sets in three subsections.

2.1 Macroscopic maximum principle

In 1977, Lewis [33] extended the Gabriel's result to p-harmonic functions in higher dimensional case and obtained the following theorem:

Theorem 2.5. (Lewis [33]) *Let u satisfy*

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = 0 & in\ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on\ \partial\Omega_0, \\ u = 1 & on\ \partial\Omega_1, \end{cases} \tag{2.4}$$

where $1 < p < +\infty$, Ω_0 and Ω_1 are bounded convex domains in $\mathbb{R}^n, n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$, and $\Omega = \Omega_0 \backslash \bar{\Omega}_1$ is called convex ring (we call here and below u satisfies *the homogeneous Dirichlet boundary conditions in convex ring). Then all the level sets of u are strictly convex.*

In 1982, Caffarelli-Spruck [8] generalized the Gabriel-Lewis' method to a class of semilinear elliptic partial differential equations as follows:

$$\begin{cases} \triangle u = f(u) & \text{in } \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1. \end{cases} \tag{2.5}$$

Where $\Omega \subset \mathbb{R}^n$ is a convex ring as before, and f is a continuous function in $\mathbb{R}$. If f is monotonously nondecreasing, and $f(0) = 0$, then the level sets of the solution u of (2.5) are all $C^{1,\alpha}$ convex hypersurfaces.

Gabriel's method has got some more extensions, and the latest is such as the following theorem due to Greco [17]:

Theorem 2.6. (Greco) *Let $\Omega = \Omega_0 \backslash \bar{\Omega}_1$ be a convex ring in $\mathbb{R}^n$. Let u satisfy:*

$$\begin{cases} \triangle u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1, \\ 0 < u < 1 & \text{in } \Omega. \end{cases} \tag{2.6}$$

where $f(x, u, \nabla u)$ is a locally uniformly continuous function, and $s^2 f(sx, u, \frac{\nabla u}{s})$ is monotonous increasing in $s > 0$. Then $|\nabla u| \neq 0$. If assume in addition the following structure condition:

$$s^3 f(x, u, \frac{\nabla u}{s}) \text{ is convex in } (s, x) \in \mathbb{R}^+ \times \Omega. \tag{2.7}$$

Then the level sets of u are all convex.

Notice that the monotonicity of f in u is not required in this theorem.

Gabriel's method has been also extent to prove the convexity of the solution itself, for the references one can see [1, 8, 14, 15, 27, 28, 30]. People nowadays call it a *macroscopic method*. The key idea is to deduce a maximum principle on the *quasi-concave function* $Q(x, y)$. So it also be called the *concave maximum principle*. We will discuss it further in the next section. Besides, concave envelope is also an important *macroscopic method* in studying the convexity. Concave envelope has been used successfully in proving the convexity of the solution itself by many authors, while it did not become a story of proving the convexity of level set until recent years (Cuoghi-Salani [10]). Let u be a function in a domain $\Omega \subset \mathbb{R}^n$. The quasiconcave envelope of u is a function defined by:

$$u^*(x) = \max\left\{ \min\{u(x_1), \ldots, u(x_{n+1})\} \middle| x = \sum_{i=1}^{n+1} \lambda_i x_i,\ \sum_{i=1}^{n+1} \lambda_i = 1,\ x_i \in \bar{\Omega}, \lambda_i \geq 0 \right\}. \tag{2.8}$$

Roughly speaking, u^* is such a function as: the super level set of u^* is just the convex envelope of the sup level set of u. For the quasiconcave envelope u^*, Cuoghi-Salani [10] proved the following theorem:

Theorem 2.7. (Cuoghi-Salani) *Let* $\Omega = \Omega_0 \backslash \bar{\Omega}_1$ *be a convex ring in* $\mathbb{R}^n$. *Let* u *be an admissible solution of the following problem*:

$$\begin{cases} F(x, u, \nabla u, D^2 u) = 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega_0, \\ u = 1 & on \quad \partial\Omega_1, \end{cases} \tag{2.9}$$

Assume $\nabla u \neq 0$. *Then* u^* *is a viscosity subsolution of* (2.9), *provided* F *satisfying*:

(i) F *is continuous in all variables*;

(ii) $F(x, u, \nabla u, D^2 u)$ *monotonously decrease in* u;

(iii) $F(x, u, \nabla u, D^2 u)$ *is degenerate elliptic*;

(iv) *There exists a constant* $p_0 < 0$ *such that* $\forall\, p \leq p_0,\ \theta \in \mathbb{R}^n$:

$$(x, t, A) \to F(x, t^{1/p}, t^{1/p-1}\theta, t^{1/p-3}A) \text{ is a concave map in } \Omega \times (1, +\infty) \times \Gamma_F. \tag{2.10}$$

By the above theorem, if assume in addition that F satisfies the viscosity comparison principle, then $u^* \leq u$, and hence $u = u^*$, which means the level sets of u are all convex.

Remark 2.8. *Degenerate elliptic in Theorem* 2.7 *means*:

$$F(x, u, \nabla u, A) \leq F(x, u, \nabla u, B),\ \forall A, B \in \Gamma,\ \ A - B \in \mathcal{S}_n^+,$$

where $\mathcal{S}_n$ *is the set of* $n \times n$ *symmetric real matrices.* $\mathcal{S}_n^+ (\mathcal{S}_n^{++})$ *is the subset of* $\mathcal{S}_n$ *of the positive semidefinite* (*definite*) *matrices.* $\Gamma \subset \mathcal{S}_n$ *is a convex cone containing* $\mathcal{S}_n^{++}$ *with the vertex at origin, while* $\Gamma_F = \bigcup \Gamma$ *where the union* $\bigcup$ *is extended to every cone* Γ *such that* F *is degenerate elliptic in it. A solution* u *is called admissible if* $(D^2 u) \in \Gamma_F$.

The *macroscopic method* originated by Gabriel may have too more constraints in application, and so far has been just applied successfully to a special class of equations. It is also difficult to get the result of the strictly convexity by using this method, while the strictly convexity may usually be crucial in some geometry applications.

2.2 Microscopic maximum principle

The subject being considered need not be smooth while one characterizes the convexity of it. In other words, the macroscopic description of convexity doesn't involved with the conception of differentiation. But, the classical solutions of differential equations are always differentiable and differentiating is of cause the essential idea in differential geometry. So, it's natural to look insight the convexity in the microscopic point of view. For example, for a second differentiable function u, when the Hessian matrix $(D^2 u)$ is positive definite at some point, then u is locally convex at this point. For another example, when the level set is considered as a hypersurface, then it's locally strict convex at a point if the second fundamental form of the hypersurface is positive definite.

In fact, there has been a maturated idea to study convexity in the microscopic point of view, i.e., the deformation idea or the continuous method. Meanwhile, a powerful tool, constant rank theorem, gradually comes to maturity. In 1985, Caffarelli-Friedman [6] first proved a constant rank theorem on solutions to a class of semilinear elliptic equations, and hence got further result of strict convexity of the solutions. The similar idea also appeared in Singer-Wong-Yau-Yau [47]. We will give further discussion on this in the next section.

For the constant rank theorem on level sets, in 1990, Korevaar [31] first proved a remarkable theorem as following:

Theorem 2.9. (Korevaar) *Let Ω be a connected domain in $\mathbb{R}^n$. Let $u \in C^4(\Omega)$ solve*:

$$Lu := A(\Delta u - \frac{u_i u_j}{|\nabla u|^2} u_{ij}) + B(\frac{u_i u_j}{|\nabla u|^2} u_{ij}) = f, \tag{2.11}$$

where A, B, f are C^2 functions in $u, \mu = |\nabla u|$ satisfying the following structure conditions:

(i) $(\sqrt{\frac{A}{B}})_{\mu\mu} \geq 0$;

(ii) $(\frac{f}{B\mu^2})_{\mu\mu} \leq 0$.

Suppose that $|\nabla u| \neq 0$ and that the level sets of u are all locally convex. Then all the level sets of u have the second fundamental forms with (*the same*) *constant rank throughout Ω.*

The equations in Theorem 2.9 include p-Laplacian equations and mean curvature equations as the special cases, precisely:

p-Laplacian equations: $A = \mu^{p-2}$, $B = (p-1)\mu^{p-2}$,

mean curvature equations: $A = \frac{1}{\sqrt{1+\mu^2}}$, $B = \frac{1}{(1+\mu^2)^{3/2}}$.

By using the above constant rank theorem, Korevaar then got the strict convexity of level sets of solutions to the according equations under some additional conditions. In particularly, he could reproved the results of Lewis [33] and proved the strictly convexity of level sets of minimal surfaces with *homogeneous Dirichlet boundary conditions on convex ring*. Using this results, in section 5, we will prove a Gauss curvature estimates of the level sets of minimal graph in $\mathbb{R}^3$.

Korevaar proved Theorem 2.9 with some new idea and techniques. He discussed the problem on the graph of the solution. And hence by choosing optimal local coordinates, he deduced some invariant formulations on level sets.

In the view point of constant rank theorem, strict convexity is a nature result, that is, when the level sets of solution take the constant rank property, then we can get the strict convexity of them once we get the convexity of them through the deformation. In fact, the general microscopic technique is a strong maximum principle, while the macroscopic technique is just a weak maximum principle.

The structure conditions in Theorem 2.6 and Theorem 2.7 are in fact some extensions of those in Theorem 2.9. Recently, Xu [52] got an extension of Korevaar's theorem that the function f in the equation (2.11) also contains the coordinate variable x, and that the structure condition (ii) accordingly turns to:

$$\mu^3 \frac{f(x, u, \frac{\nabla u}{\mu})}{B(u, \frac{1}{\mu})} \quad \text{is convex in } (x, \mu).$$

At last Bian-Guan-Ma-Xu [3] could generalize the constant rank theorem to a class fully nonlinear elliptic equation.

A more recent extension to prescribed mean curvature hypersurface was given by Hu and the authors [22]. Let M^n be a smooth immersed hypersurface in $\mathbb{R}^{n+1}$ and $X : M \to \mathbb{R}^{n+1}$ be the immersion, satisfying

$$H = -f(X, N), \tag{2.12}$$

where H, N is the mean curvature and unit normal vector of M^n at X respectively. f is a smooth function in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Let ξ be a fixed unit vector in $\mathbb{R}^{n+1}$. Then the height function of M^n corresponding to ξ can be expressed as $u(X) = \langle X, \xi \rangle$, here $\langle \cdot, \cdot \rangle$ means the usual Euclidean product in $\mathbb{R}^{n+1}$. Now, the level set of M^n corresponding to ξ with height c is defined as

$$\Sigma_c = \{X \in M^n | u(X) = c\}. \tag{2.13}$$

Suppose u has no critical point on M^n, then Σ_c can be considered as a hypersurface in the hyperplane $\Pi = \{X \in \mathbb{R}^{n+1} | \langle X, \xi \rangle = c\}$ in $\mathbb{R}^{n+1}$.

With the notations as above, the constant rank theorem of level sets of prescribed mean curvature hypersurface is as following:

Theorem 2.10. *Let M^n be the prescribed mean curvature immersed hypersurface in $\mathbb{R}^{n+1}$ satisfying* (2.12)*. Assume that the height function u of M^n corresponding to ξ have no critical point, and that the level sets are all locally convex, i.e., their second fundamental forms are positive semidefinite. Then the second fundamental forms of all the level sets have (the same) constant rank, provided $f(X, N) = f(X) \geq 0$ and the matrix*

$$2f \frac{\partial^2 f}{\partial X_A \partial X_B} - 3 \frac{\partial f}{\partial X_A} \frac{\partial f}{\partial X_B}$$

is positive semidefinite, where $1 \leq A, B \leq n+1$, in other words, for $f > 0$, $f^{-\frac{1}{2}}$ is concave function in $\mathbb{R}^{n+1}$.

Remark 2.11. *For the general case $H = -f(X, N)$ in* (2.12)*, the structure conditions on f can be similarly obtained to ensure the result of Theorem* 2.10 (*see* [22]).

2.3 Some sharp estimates

To get the convexity of level sets of a function defined in a domain, one usually assume the convexity of level sets on the boundary. So, one can also look insight the convexity of level sets by taking a comparison of them in the interior and on the boundaries. In fact, some interesting quantitative results have been found on the convexity of level sets.

For a two dimensional harmonic function defined on a convex ring with homogeneous Dirichlet boundary conditions, by the theorem of Lewis [33], as explained, the level set of this function is strictly convex. In 1983, using the support function of the level curves and the maximum principle, Longinetti [36] proved that the curvature of the level curves of such a two dimensional harmonic function attains its minimum on the boundary (see also Talenti [49] for related results). Later, in 1987, Longinetti [37] used the same technique to obtain a similar theorem for minimal surfaces, where the convexity of the level sets follows from the theorem of Shiffman [46].

1989, Rosay-Rudin [45] raised a new measure of convexity of level sets, and then proved that under this measure, the level set convexity of harmonic solutions to convex ring problems is at least as good as that of the two ring's boundaries.

For a compact 2-dimensional minimal surface with boundaries, in 1992, Huang [23] deduced an elliptic equality on the curvature of level lines of the surface, and hence he could compare the level sets convexity in the interior with that on the boundaries.

Recently, Jost and the authors [25] gave a quantitative estimate on the Gauss curvature of level sets of p-harmonic functions in 2 and 3 dimensional Euclidian domains:

Theorem 2.12. *Let u be a p-harmonic function in a n-dimensional Euclidian domain Ω, i.e.,*

$$div(|\nabla u|^{p-2}\nabla u) = 0 \quad in \quad \Omega. \tag{2.14}$$

If $|\nabla u| \neq 0$ and the level sets of u are all strictly convex, then, the Gauss curvature of the level sets of u can not attain the minimum in Ω , unless it's a constant, provided: (i) $n = 2$, $\frac{3}{2} \leq p \leq 3$ *or* (ii) $n = 3$, $2 \leq p < +\infty$.

We can apply Theorem 2.12 to obtain the following version of Lewis's Theorem 2.5.

Corollary 2.13. *Let u be the solution of the following boundary value problem,*

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = 0 & in\ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on\ \partial\Omega_0, \\ u = 1 & on\ \partial\Omega_1, \end{cases} \tag{2.15}$$

where Ω is a convex ring as in theorem 2.5*. If $n = 2$ and $\frac{3}{2} \leq p \leq 3$, then the curvature of the level lines of u attains its minimum on $\partial\Omega$. If $n = 3$ and $2 \leq p < +\infty$, then the Gaussian curvature of the level sets of u attains its minimum on $\partial\Omega$.*

For the minimum hypersurface in $\mathbb{R}^4$, we also have the following quantitative estimate on it level sets, which will be proved in section 5:

Theorem 2.14. *Let M^3 be a hypersurface satisfying* (2.12) *with $f \equiv 0$, i.e., a minimal hypersurface in $\mathbb{R}^4$. Let the height function u of M^3 corresponding to a direction ξ have no critical point, and let the level sets of u be all strictly convex. Then, the Gauss curvature of the level sets of u can not attain the minimum in M^3 , unless it's a constant*

Combining this with the results of Korevaar [31] (see Theorem 2.9 and the text below it), we have:

Corollary 2.15. *Let u be the solution of the following boundary value problem,*

$$\begin{cases} \sum_{i=1}^3 D_i(\frac{u_i}{\sqrt{1+|Du|^2}}) = 0 & in\ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on\ \partial\Omega_0, \\ u = 1 & on\ \partial\Omega_1, \end{cases} \tag{2.16}$$

where $\Omega \subset \mathbb{R}^3$ is a convex ring as in theorem 2.5*. Then the Gauss curvature of the level sets of u attains its minimum on $\partial\Omega$.*

In 1983, Longinetti [36] and Talenti [49] found some harmonic or subharmonic functions related to the curvature of level lines of 2 dimensional harmonic functions. Similar results were found on 2 dimensional minimum surface by Huang and the authors [24]. These results combining with maximum principle imply the quantitative estimates of the corresponding level lines convexity. Furthermore, they can get the convexity of the level lines.

We need mention the result by Cabre-Chanillo [5] which also got the convexity of level sets in two dimension case for some elliptic equations. And we can find some important applications on the convexity of level sets to some geometry problems, for example the classification of singularities of mean curvature flow by Xu-Jia Wang [51].

Now, maybe we can summarize the approaches to the level sets convexity as the following three:

(1) Concave maximum principle (Gabriel [12, 13];
(2) Constant rank theorem (Korevaar [31]);
(3) Comparing with boundary convexity (Longinetti [36]).

3 History of convexity of solution itself

When a solution itself is a convex function, then of cause, all the level sets of it are convex. So the approaches to convexity of solution itself are also the (indirect) approaches to convexity of level sets. In fact, all the approaches borrow ideas each others. Moreover, recall that for the solution u of (1.2), Makar-Limanov [39] proved the concavity of $u^{\frac{1}{2}}$. Which means that the solution itself may not be a convex function (Kennington [28] pointed out in Remark 4.2.3 that the power $\frac{1}{2}$ is sharp), but some function in this solution can take some convexity. This is an initiation in the study of convexity. Nowadays, people usually use the so called α-*concave* conception as following(see Kennington [28]):

$\forall\alpha \in [-\infty, +\infty]$, one call $u \geq 0$ α-*concave* if

$$\begin{cases} u \quad \text{is constant} & \text{for} \quad \alpha = +\infty; \\ u^\alpha \quad \text{is concave} & \text{for} \quad \alpha \in (0, +\infty); \\ \ln u \quad \text{is concave} & \text{for} \quad \alpha = 0; \\ S^t \quad \text{is convex}\, \forall t \in \mathbb{R} & \text{for} \quad \alpha = -\infty. \end{cases} \tag{3.17}$$

Where we take the conventions that $\ln u = -\infty$ and $u^\alpha = +\infty$, $\forall\alpha \in (-\infty, 0)$ for $u = 0$. By Jensen's inequality, $\forall\, \alpha < \beta$, u is β-*concave* $\Rightarrow$ u is α-*concave* (but the inverse is fail). This means that to show level sets convexity , one needs only to show α-*concavity* for some $\alpha \in [-\infty, 1)$. The approaches to α-*concavity* can be roughly summarized as:

(1) Macroscopic maximum principle (Korevaar [29, 30]);
(2) Microscopic maximum principle (Caffarelli-Friedman [6]).

3.1 Macroscopic maximum principle

Similar to the function $Q(x, y)$ raised by Gabriel, Korevaar [29], while studying the convexity of capillary surface in 1983, introduced the following concavity function:

$$\varphi(x, y, \lambda) = u(z) - \lambda u(y) - (1 - \lambda)u(x), \tag{3.18}$$

where $z = \lambda y + (1 - \lambda)x, 0 \leq \lambda \leq 1, \forall x, y \in \Omega$. One can see that u is convex if and only if $\varphi \leq 0$ in $\Omega \times \Omega$.

If $u \in C^2(\Omega)$ is not convex, then there are two cases:
(i) $\varphi(x, y) > 0$ for $(x, y) \in \partial(\Omega \times \Omega)$, or
(ii) φ obtains a positive maximum in $\Omega \times \Omega$.

The first case can be treated by the boundary points lemma (see lemma 3.11 and lemma 3.12 in Kawhol [26]), while the second case may be treated by concavity maximum principle.

Korevaar's concavity maximum principle is (see [29]):

Theorem 3.16. (Korevaar) *Let* $u \in C^2(\Omega)$ *solve*

$$a^{ij}(\nabla u)u_{ij} = b(u, \nabla u) \quad in\, \Omega, \tag{3.19}$$

where $a^{ij} = a^{ji}$, $(a^{ij}) > 0$ *and* ∇u *is the gradient of* u *. Then the function* φ *defined by* (3.18) *can not attain a positive maximum in* Ω*, provided* b *satisfying*

$$\frac{\partial b}{\partial u} > 0, \qquad \frac{\partial^2 b}{\partial u^2} \leq 0. \tag{3.20}$$

Korevaar's concavity maximum principle has got many extensions [8, 14, 15, 27, 28, 30].

Similar to the quasiconcave envelope as in (2.8), concave envelope u^{**} of a function u is defined by:

$$u^{**}(x) = \max\left\{ \sum_i \lambda_i u(x_i) \middle| x = \sum_i \lambda_i x_i, \sum_i \lambda_i = 1, x_i \in \Omega, \lambda_i \geq 0 \right\}. \tag{3.21}$$

By using this concave envelope, Alvarez-Lasry-Lions [1] proved the following theorem:

Theorem 3.17. (Alvarez-Lasry-Lions) *Let* Ω *be a bounded convex domain in* $\mathbb{R}^n$*. Let* u *solve*

$$F(x, u, \nabla u, D^2u) = 0 \qquad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n. \tag{3.22}$$

Assume:
(a) *boundary condition is state constraint's type, such as* u *is constant* (*maybe infinity*) *or as* $\dfrac{\partial u}{\partial \nu} = +\infty$ (ν *is out normal*);
(b) F *satisfies the comparison principle*:
if $u \in C(\bar{\Omega})$ *is a subsolution of (3.22), and* $v \in LSC(\bar{\Omega})$ *is the supersolution, then* $u \le v$ *in* $\bar{\Omega}$;
(c) F *is elliptic, satisfying the concavity structure condition*:

$$(x, r, A) \mapsto -F(x, r, p, A^{-1}) \quad \text{is concave, } \forall\, p \in \mathbb{R}^n. \tag{3.23}$$

Then u *is convex.*

McCuan [40] and Colesanti-Salani [9] gave some further discussions on the application of concave envelope in the study of convexity.

3.2 Microscopic maximum principle

Constant rank theorem is the key step during the deformation process to getting convexity result. The deformation process (i.e., the continuous method) may be carried out briefly as: For a given differential equation in a convex domain $\Omega \subset \mathbb{R}^n$, assume it's solution is unique. First, one proves the solution of the same equation in the unit ball is strict convex (it's usually a radial solution at this time). Then one deforms the unit ball continuously to the considering domain Ω. If the convex solution is not strict convex at a moment during the deformation, then constant rank theorem confirms that the solution is not strict convex at every point throughout Ω. But the domain is still strict convex at this moment. A priori estimate shows that the solution is strict convex near the boundary. These leads to a contradiction.

Caffarelli-Friedman [6] proved a *constant rank theorem* for convex solutions of quasilinear elliptic equations in $\mathbb{R}^2$, a similar result was also discovered by Yau [47] at the same time. Korevaar-Lewis [32] generalized their results to $\mathbb{R}^n$, that is:

Theorem 3.18. (Caffarelli-Friedman, Korevaar-Lewis) *Let* Ω *be a connected domain in* $\mathbb{R}^n$. *Let* $u \in C^4(\Omega)$ *solve*

$$\triangle u = f(x, u, \nabla u) > 0.$$

Assume $f^{-1}(x, u, \nabla u)$ *is concave in* (x, u) *and* $(u_{ij}) \ge 0$, *i.e., the Hessian of* u *is positive semidefinite. Then* (u_{ij}) *takes (the same) constant rank throughout* Ω.

By using this constant rank theorem, they retrieved the result of Makar-Limanov [39] and got the strict convexity naturally at the same time.

In the recently years, constant rank theorem was found successively in the fully nonlinear equations derived from classical geometry problems, such as Christoffel-Minkowski problem and prescribed Weingarten curvature problem. Guan-Ma [19] extended constant rank theorem to the equation $S_k(u_{ij}) = f(x)$, and the structure condition is that $f^{-1/k}(x)$ is convex. Then Guan-Lin-Ma [18] generalize the similar results to curvature equations. Guan-Ma-Zhou [20] extended it further to the quotient equation $\frac{\sigma_k(u_{ij})}{\sigma_l(u_{ij})}(x) = f(x)$ provided $f^{-\frac{1}{k-l}}(x)$ is convex. Later Caffarelli-Guan-Ma [7] extended it to a more general equation as following:

$$F(D^2u(x)) = f(x, u(x), \nabla u(x)), \quad \forall x \in \Omega.$$

At last Bian-Guan [2] proved the following generally theorem.

Theorem 3.19. (Bian-Guan) *Suppose $F = F(r, p, u, x) \in C^{2,1}(S^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega)$, and satisfies the following elliptic condition*

$$\frac{\partial F}{\partial r_{\alpha\beta}}(D^2u(x), \nabla u(x), u(x), x) > 0,$$

and

$$F(A^{-1}, p, u, x) \quad \textit{is locally convex in} \quad (A, u, x) \quad \textit{for each } p \textit{ fixed.}$$

Let $u \in C^{2,1}(\Omega)$ is a convex solution of the following equation

$$F(D^2u(x), \nabla u(x), u(x), x) = 0, \quad \forall x \in \Omega,$$

Then the Hessian (u_{ij}) takes (the same) constant rank throughout the connected domain $\Omega \subset \mathbb{R}^n$.

In Ma-Xu [38] and Liu-Ma-Xu [35], they proved the constant rank theorem for some Hessian equations in lower dimensional cases, and hence treated the classical Brunn-Minkowski inequality successfully.

Recently Han-Ma-Wu [21] obtain some constant rank theorems for the k-convex solution for Poisson equation and give some applications to the existence of k-convex hypersurfaces with prescribed mean curvature in $\mathbb{R}^{n+1}$.

Although various manner of ways have been found to get so many results on convexity, people still have no a deep look insight into the convexity. For example, all the above results only contain the sufficient conditions, while people do not know the necessary and sufficient conditions. Not until the last few years, the necessary and sufficient conditions for ensuring the solutions to a class of linear parabolic equations preserving convexity were found by Janson-Tysk [48] and Lions-Musiela [34], separately, while the level sets version of that is still a secretary. Comparing to the convexity of solution itself, people indeed know much less about the convexity of level sets. Makar-Limanov [39] had proven the solution of (1.2) is $\frac{1}{2}$-concave for a long time. But so far people can not prove the level sets convexity of the solution directly in high dimensional case. Take the free boundary problem for another example:

$$\begin{cases} u_t = \Delta u & \textit{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u|_{t=0} = u_0 & u_0 \in L^1, \text{ supp}\, u_0 \text{ is compact.} \end{cases} \tag{3.24}$$

The existence and regularity of it have beset people for a long time, since these may depend on the convexity of $\operatorname{supp} u(t)$, which is just the obstruction people can not surmount, see [11] and the references there in.

4 A curvature formula of level sets of hypersurfaces

In this section we first state a curvature formula on the level sets of hypersurfaces in R^{n+1}, which was proved by Hu and the authors [22]. For the completeness we also give the details here. This formula will be used in the Gauss curvature estimates in next section.

Using this formula we also give another proof of a constant rank theorem of the curvature of convex level sets for prescribed mean curvature surfaces in $\mathbb{R}^3$ by Korevaar [31]. We will do the calculations by using the moving frame.

For a C^2 function u defined on a domain Ω in $\mathbb{R}^n$, let $\kappa_1, \ldots, \kappa_{n-1}$ be the principle curvature of the level sets of u. Then the k-th curvature of the level sets, denote by L_k, is the k-th elementary symmetric function of $\kappa_1, \ldots, \kappa_{n-1}$. Clearly, L_1 and L_{n-1} are mean curvature and Gauss curvature of the level sets respectively. If u has no critical point, i.e., $|\nabla u| \neq 0$, then Trudinger [50] (see also [16]) gave a formula of L_k as:

$$L_k = \frac{\partial \sigma_{k+1}(D^2u)}{\partial u_{ij}} u_i u_j \mid \nabla u \mid^{-k-2}, \tag{4.1}$$

where summation convention has been used for the repeated indices, and $\sigma_k(D^2u)$ is the *k-th* elementary symmetric function of the eigenvalues of the Hessian (D^2u).

Now, we give an analogous formula of (4.1) on hypersurface in $\mathbb{R}^{n+1}$. Precisely, we will prove(with the notations as in subsection2.2)

Proposition 4.1. *Let M^n be a hypersurface in $\mathbb{R}^{n+1}$. And let u, Σ_c be the height function and the level set of M^n, respectively, with respect to a fixed unit direction ξ, as given in subsection* 2.2. *Then the k-th curvature of the level set Σ_c is*

$$L_k = \frac{\partial \sigma_{k+1}(\mathbf{B})}{\partial h_{ij}} u_i u_j |\nabla u|^{-(k+2)}, \tag{4.2}$$

where $\mathbf{B} = (h_{ij})$ is the second fundamental form of M^n, $\sigma_k(\mathbf{B})$ is the k-th elementary symmetric function of the eigenvalues of (h_{ij}), and $u_i (1 \leq i \leq n)$ is the first derivative of u computed in any orthogonal frame field on M^n.

For $n = 2$, Huang [23] had given the formula (4.2). Here we give a complete proof of (4.2) by using the moving frame on hypersurface. Before the proof, we recall some notations. We will adapt the following convention for indices in the rest of this section:

$$1 \leq \alpha, \beta, \ldots \leq n-1; \qquad 1 \leq i, j, \ldots \leq n; \qquad 1 \leq A, B, \ldots \leq n+1.$$

For an orthogonal frame field $\{X; e_A\}$ in $\mathbb{R}^{n+1}$, we have:

$$dX = \omega_A e_A; \quad de_A = \omega_{AB} e_B, \tag{4.3}$$

where $\{\omega_A\}$ is the dual of $\{e_A\}$ and $\{\omega_{AB}\}$ is the connection. Then the structure equations are:

$$d\omega_A = \omega_{AB} \wedge \omega_B; \quad d\omega_{AB} = \omega_{AC} \wedge \omega_{CB}. \tag{4.4}$$

If we choose e_{n+1} to be the normal vector field of M^n. Then restricted to M^n, we have $\omega_{n+1} = 0$, and hence by (4.4):

$$\omega_{n+1,i} \wedge \omega_i = 0. \tag{4.5}$$

So Cartan's lemma implies

$$\omega_{n+1,i} = h_{ij}\omega_j, \qquad h_{ij} = h_{ji}, \tag{4.6}$$

where $\mathbf{B} = (h_{ij})$ is the second fundamental form of M^n. The following formulas are well known:

$$\begin{aligned}
X_i &= e_i, \\
X_{ij} &= -h_{ij}e_{n+1} \qquad \text{(Gauss formula)}, \\
e_{n+1,i} &= h_{ij}e_j \qquad \text{(Weingarten equation)}, \\
h_{ijk} &= h_{ikj} \qquad \text{(Codazzi formula)}, \\
R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \text{(Gauss equation)}, \\
h_{ijkl} &= h_{ijlk} + h_{im}R_{mjkl} + h_{jm}R_{mikl},
\end{aligned} \tag{4.7}$$

where R_{ijkl} is the curvature tensor. For a smooth function u on M^n we also have the following Ricci formula for the third covariant derivatives:

$$u_{ijk} = u_{ikj} + u_m R_{mijk}. \tag{4.8}$$

Proof of Proposition 4.1

First, we check that the right hand side of (4.2) is independent of the choice of the tangential frame fields $\{X; e_i\}$ of M^n. And then we can just prove (4.2) under a special tangential frame field.

Let T be an orthogonal transformation of two tangential frame fields, i.e., $(\bar{e}_1, \ldots, \bar{e}_n) = (e_1, \ldots, e_n)T$. Then

$$(\bar{u}_1, \ldots, \bar{u}_n) = (u_1, \ldots, u_n)T \tag{4.9}$$

where $\bar{\nabla} u = (\bar{u}_1, \ldots, \bar{u}_n)$ is the gradient of u with respect to $\{\bar{e}_1, \ldots, \bar{e}_n\}$. Also we have, for the dual frame field and the connections:

$$(\bar{\omega}_1, \ldots, \bar{\omega}_n) = (\omega_1, \ldots, \omega_n)T, \tag{4.10}$$

and

$$(\bar{\omega}_{1,n+1}, \ldots, \bar{\omega}_{n,n+1}) = (\omega_{1,n+1}, \ldots, \omega_{n,n+1})T. \tag{4.11}$$

Furthermore we have, for the second fundamental form:

$$\mathbf{B} = T^{-1}\bar{\mathbf{B}}T. \tag{4.12}$$

Obviously $\sigma_k(\mathbf{B})$ and $|\nabla u|$ are invariants under the transformation of T. Then the following equalities show that the right hand side of (4.2) is independent of the choice of $\{e_1, \dots, e_n\}$:

$$\begin{aligned}\frac{\partial\sigma_k(\mathbf{B})}{\partial h_{ij}}u_iu_j &= \frac{\partial\sigma_k(\bar{\mathbf{B}})}{\partial\bar{h}_{ml}}\frac{\partial\bar{h}_{ml}}{\partial h_{ij}}u_iu_j \\ &= \frac{\partial\sigma_k(\bar{\mathbf{B}})}{\partial\bar{h}_{ml}}T_{im}T_{jl}\bar{u}_pT^{pi}\bar{u}_qT^{qj} \\ &= \frac{\partial\sigma_k(\bar{\mathbf{B}})}{\partial\bar{h}_{ml}}\delta_{pm}\delta_{ql}\bar{u}_p\bar{u}_q \\ &= \frac{\partial\sigma_k(\bar{\mathbf{B}})}{\partial\bar{h}_{ml}}\bar{u}_m\bar{u}_l.\end{aligned} \tag{4.13}$$

Now we adapt the above frame field so that along the level set Σ_c, the e_α's are its tangential vectors. Furthermore, we choose another frame field $\tilde{e}_A$ in $\mathbb{R}^{n+1}$ such that $\tilde{e}_\alpha = e_\alpha$, $\tilde{e_n}$ lies in the level hyperplane Π_c and normal to Σ_c. With respect to this frame field, we have the structure equations of Σ_c as:

$$d\tilde{\omega}_i = \tilde{\omega}_{ij} \wedge \tilde{\omega}_j; \quad d\tilde{\omega}_{ij} = \tilde{\omega}_{il} \wedge \tilde{\omega}_{lj}. \tag{4.14}$$

And restricted to Σ_c, $\tilde{\omega}_n = 0$, which implies

$$\tilde{\omega}_{n\alpha} = \tilde{h}_{\alpha\beta}\tilde{\omega}_\beta, \qquad \tilde{h}_{\alpha\beta} = \tilde{h}_{\beta\alpha}, \tag{4.15}$$

where $\tilde{h}_{\alpha\beta}$ is the second fundamental form of Σ_c in Π_c (with respect to the unit normal $\tilde{e}_n$).

Clearly e_n, e_{n+1} and $\tilde{e}_n, \tilde{e}_{n+1}$ are in the same hyperplane orthogonal to the e_α's. Let ϕ be the angle from e_n to $\tilde{e}_n$. Then we have

$$\tilde{e}_n = e_n \cos\phi + e_{n+1}\sin\phi; \quad \tilde{e}_{n+1} = -\tilde{e}_n \sin\phi + e_{n+1}\cos\phi. \tag{4.16}$$

Accordingly

$$\tilde{\omega}_n = \omega_n \cos\phi + \omega_{n+1}\sin\phi; \quad \tilde{\omega}_{n+1} = -\omega_n \sin\phi + \omega_{n+1}\cos\phi; \quad \tilde{\omega}_\alpha = \omega_\alpha. \tag{4.17}$$

Taking exterior differentiating to (4.17) , using (4.4) and (4.17) again, we get

$$d\tilde{\omega}_n = (d\phi + \omega_{n,n+1}) \wedge \tilde{\omega}_{n+1} + [(\cos\phi)\omega_{n\alpha} + (\sin\phi)\omega_{n+1,n}] \wedge \omega_\alpha, \tag{4.18}$$

and

$$d\tilde{\omega}_{n+1} = (-d\phi + \omega_{n+1,n}) \wedge \tilde{\omega}_n + [(\cos\phi)\omega_{n+1,\alpha} - (\sin\phi)\omega_{n\alpha}] \wedge \omega_\alpha. \tag{4.19}$$

Notice that when restricted to Σ_c, $\tilde{\omega}_n = \tilde{\omega}_{n+1} = 0$. Comparing (4.18)$\sim$(4.19) with (4.14), we have

$$\tilde{\omega}_{n\alpha} = (\cos\phi)\omega_{n\alpha} + (\sin\phi)\omega_{n+1,\alpha}, \tag{4.20}$$

and

$$\tilde{\omega}_{n+1,\alpha} = (-\sin\phi)\omega_{n\alpha} + (\cos\phi)\omega_{n+1,\alpha}. \tag{4.21}$$

On the other hand, $\langle\tilde{e}_\alpha,\xi\rangle=0$ holds on Σ_c. And since $d(\langle\tilde{e}_\alpha,\xi\rangle)=\langle\tilde{\omega}_{\alpha,A}\tilde{e}_A,\xi\rangle$, we have $\tilde{\omega}_{\alpha,n+1}=0$. This together with(4.20)∼(4.21) implies

$$\tilde{\omega}_{n\alpha}=\frac{\cos^2\phi}{\sin\phi}\omega_{n+1,\alpha}+\sin\phi(\omega_{n+1,\alpha})=\frac{1}{\sin\phi}\omega_{n+1,\alpha}=\frac{1}{\sin\phi}(h_{\alpha\beta}\omega_\beta+h_{\alpha n}\omega_n). \tag{4.22}$$

Combining this with (4.15) gives

$$\tilde{h}_{\alpha\beta}=\frac{1}{\sin\phi}h_{\alpha\beta};\qquad h_{\alpha n}=0. \tag{4.23}$$

Finally, from the definition of the height function u, we can see $u_i=e_i(\langle X,\xi\rangle)=\langle e_i,\xi\rangle$, in particular, $u_n=\langle e_n,\xi\rangle$. Note that $\tilde{e}_{n+1}=\xi$ or $\tilde{e}_{n+1}=-\xi$, then the second equation of (4.16) becomes $\pm\xi=-e_n\sin\phi+e_{n+1}\cos\phi$. Hence $u_n=\mp\sin\phi$ and $\langle\xi,e_{n+1}\rangle=\pm\cos\phi$. By the decomposition $\xi=\sum_1^n\langle\xi,e_i\rangle e_i+\langle\xi,e_{n+1}\rangle e_{n+1}$ we deduce that $1=|\nabla u|^2+\cos^2\phi$ and therefore $|\nabla u|=\pm\sin\phi$. With $\tilde{e}_n$ chosen suitably we may assume $\sin\phi>0$. So (4.23) becomes

$$\tilde{h}_{\alpha\beta}=\frac{1}{|\nabla u|}h_{\alpha\beta};\qquad h_{\alpha,n}=0. \tag{4.24}$$

From this one can easily see that

$$\begin{aligned}L_k=\sigma_k(\tilde{h}_{\alpha\beta})&=\frac{1}{|\nabla u|^k}\sigma_k(h_{\alpha\beta})=\frac{1}{|\nabla u|^{k+2}}\frac{\partial\sigma_{k+1}(\mathbf{B})}{\partial h_{nn}}u_nu_n\\&=\frac{\partial\sigma_{k+1}(\mathbf{B})}{\partial h_{ij}}u_iu_j|\nabla u|^{-(k+2)}.\end{aligned} \tag{4.25}$$

where we have used that $|u_n|=|\nabla u|$ and the last equality follows from (4.13). This completes the proof of this proposition. □

Now we prove the following constant rank theorem which is a special case of a more general theorem by Korevaar[31]. We do the calculations by using moving frame.

Theorem 4.2. (Korevaar [31]) *Let u be a smooth solution to mean curvature type equation*

$$\textstyle\sum_{i=1}^2 D_i\left(\frac{u_i}{\sqrt{1+|Du|^2}}\right)=-2(1+|Du|^2)^\alpha\qquad in\qquad\Omega\subset\mathbb{R}^2. \tag{4.26}$$

Suppose that u has convex level sets (i.e., the curves $\{u=t\}$ are convex) and $|Du|\neq0$ in Ω. If $\alpha\in(-\infty,-\frac{1}{2}]\cup[0,+\infty)$, then all the curvature of the level sets of u has the same sign throughout the connected domain Ω.

Let M^2 be the solution surface of u in $\mathbb{R}^3$. Following the notations as in Proposition 4.1, we choose $\xi=(0,0,1)$ and e_3, the unit normal of M^2, such that $\langle e_3,\xi\rangle>0$. Then

$$u=u(X)=\langle X,\xi\rangle, \tag{4.27}$$

and the level line

$$\Sigma_c \equiv \{X \in M^2 \mid u(X) = c\}$$

has curvature K in horizontal plane $\Pi_c = \{X \in \mathbb{R}^3 \mid \langle X, \xi \rangle = c\}$, which be called the level curvature.

Denote

$$\nabla u = \sum_{i=1}^{2} u_i e_i, \quad u_i = \langle X_i, \xi \rangle = \langle e_i, \xi \rangle.$$

It follows that

$$u_{ij} = \langle X_{ij}, \xi \rangle = -h_{ij} \langle e_3, \xi \rangle = -h_{ij} W, \tag{4.28}$$

and

$$W_1 = h_{1j} \langle e_j, \xi \rangle, \qquad W_{11} = h_{11j} \langle e_j, \xi \rangle - h_{1j}^2 W. \tag{4.29}$$

If $|\nabla u| \neq 0$, we then have, by (4.2):

$$K = \frac{h_{11} u_2^2 + h_{22} u_1^2 - 2h_{12} u_1 u_2}{|\nabla u|^3}, \tag{4.30}$$

or we can rewrite it as:

$$K \cdot |\nabla u|^3 = \frac{\partial \det(h_{kl})}{\partial h_{ij}} u_i u_j. \tag{4.31}$$

Using the above moving frame, we can also rewrite Equation (4.26) as

$$h_{11} + h_{22} = -f \qquad \text{on} \qquad M^2, \tag{4.32}$$

where $f = 2(1 + |Du|^2)^\alpha = 2W^{-2\alpha}$.

Proof of Theorem 4.2

Since the level lines of u are all convex, $K(X) \geq 0 \; \forall \, X \in M^2$. Set

$$Z = \{X \in M^2 | K(X) = 0\}.$$

If Z is empty, then we are done. Otherwise we will show $Z = M^2$. Clearly Z is closed. We need only to show that Z is open. As in Korevaar-Lewis [32], our idea is to show

$$\triangle K \leq C_1 K + C_2 |\nabla K|$$

in some small neighborhood of any point $p \in Z$, where $\triangle$ is the Beltrami-Laplacian on the surface M^2 and C_1, C_2 are controlled constants. Since

$$K(p) = 0 = \min_{X \in M^2} K(X),$$

then the strong maximum principle tells us that $K(p) \equiv 0$ in the small neighborhood of p. Hence Z is open and therefore $Z = M^2$.

We will divide the proof into three steps.

Step 1.

Take Laplacian on the both sides of Equation (4.31)

$$\triangle(K|\nabla u|^3) = \triangle(\frac{\partial \det(h_{kl})}{\partial h_{ij}} u_i u_j),$$
$$\triangle(K|\nabla u|^3) = \triangle K \cdot |\nabla u|^3 + 2\nabla K \cdot \nabla |\nabla u|^3 + K \cdot \triangle |\nabla u|^3.$$

Therefore

$$\triangle K \cdot |\nabla u|^3 = \triangle(\frac{\partial \det(h_{kl})}{\partial h_{ij}} u_i u_j),$$

where the equality holds in the sense of module K and ∇K. We carry out the computations at any fixed point q in some small neighborhood of the fixed point p where $K(p) = 0$. At this point q, we can choose a suitable frame work such that $u_1 = 0$ and $u_2 = |\nabla u| > 0$.

In the following, all the calculations work at q and all the equalities should be understood in the sense of module K and ∇K.

Since

$$K|\nabla u|^3 = \frac{\partial \det(h_{kl})}{\partial h_{ij}} u_i u_j.$$

It follows that

$$K|\nabla u|^3 = h_{11}|\nabla u|^2,$$

and hence

$$h_{11} = 0. \tag{4.33}$$

Take first derivative of Equation (4.31)

$$(K|\nabla u|^3)_k = (\frac{\partial det(h_{kl})}{\partial h_{ij}} u_i u_j)_k,$$

i.e.,

$$\begin{aligned} K_k \cdot |\nabla u|^3 + K \cdot (|\nabla u|^3)_k &= \frac{\partial^2 det(h_{kl})}{\partial h_{ij}\partial h_{pq}} h_{pqk} u_i u_j + 2\frac{\partial \det(h_{kl})}{\partial h_{ij}} u_{ik} u_j \\ &= |\nabla u|^2 h_{11k} + 2|\nabla u|[h_{11}u_{2k} - h_{12}u_{1k}] \\ &= |\nabla u|^2 h_{11k} - 2|\nabla u| h_{12} u_{1k} \\ &= |\nabla u|^2 h_{11k} + 2|\nabla u| h_{12} h_{1k} W. \end{aligned}$$

Therefore, when $k = 1$ and $k = 2$, we have the following two relations:

$$h_{111} = 0, \tag{4.34}$$
$$h_{112}|\nabla u| = -2h_{12}^2 W. \tag{4.35}$$

Take second derivatives of Equation (4.31), we have

$$\begin{aligned} \triangle K \cdot |\nabla u|^3 \ = \ & \frac{\partial^2 det(h_{kl})}{\partial h_{ij}\partial h_{pq}} h_{pqkk} u_i u_j + 4\frac{\partial^2 det(h_{kl})}{\partial h_{ij}\partial h_{pq}} h_{pqk} u_{ik} u_j \\ & +2\frac{\partial \det(h_{kl})}{\partial h_{ij}} u_{ikk} u_j + 2\frac{\partial \det(h_{kl})}{\partial h_{ij}} u_{ik} u_{jk} \\ =: & \ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \end{aligned} \tag{4.36}$$

We will calculate each term in the above equality.

Step 2.

Now we estimate the four terms on the right hand side of (4.36) by using the above listed formulas and identities from (4.33) to (4.35). For the first term, we obtain

$$\begin{aligned}
\mathrm{I} := & |\nabla u|^2 h_{11kk} = |\nabla u|^2 h_{k11k} \\
= & |\nabla u|^2 [h_{k1k1} + h_{km} R_{m11k} + h_{1m} R_{mk1k}] \\
= & |\nabla u|^2 [h_{kk11} + h_{km}(h_{m1}h_{1k} - h_{mk}h_{11})] \\
= & |\nabla u|^2 [h_{kk11} + h_{22}h_{12}^2] \\
= & -|\nabla u|^2 f_{11} + |\nabla u|^2 h_{22} h_{12}^2,
\end{aligned}$$

Now for the second term, we have

$$\begin{aligned}
\mathrm{II} := & 4|\nabla u|[u_{2k}h_{11k} - u_{1k}h_{12k}] \\
= & 4|\nabla u|[u_{22}h_{112} - u_{12}h_{122}] \\
= & 8h_{22}h_{12}^2 W^2 - 4|\nabla u| h_{12} W f_1.
\end{aligned}$$

For the third term, it follows that

$$\begin{aligned}
\mathrm{III} := & 2|\nabla u|[h_{11}u_{2kk} - h_{12}u_{1kk}] \\
= & -2|\nabla u| h_{12} u_{k1k} \\
= & -2|\nabla u| h_{12}[u_{kk1} + u_m R_{mk1k}] \\
= & -2|\nabla u| h_{12}(f_1 W + f W_1) \\
= & -2|\nabla u| h_{12} W f_1 - 2|\nabla u|^2 h_{12}^2 f.
\end{aligned}$$

For the fourth term, we have

$$\begin{aligned}
\mathrm{IV} := & 2[h_{11}u_{2k}^2 + h_{22}u_{1k}^2 - 2h_{12}u_{1k}u_{2k}] \\
= & -2h_{22}h_{12}^2 W^2.
\end{aligned}$$

From the above four terms, we can see

$$\begin{aligned}
\triangle K \cdot |\nabla u|^3 = & -|\nabla u|^2 f_{11} - |\nabla u|^2 f h_{12}^2 - 8 f h_{12}^2 W^2 - 4|\nabla u| h_{12} W f_1 \\
& -2|\nabla u| h_{12} W f_1 - 2|\nabla u|^2 h_{12}^2 f + 2 f h_{12}^2 W^2.
\end{aligned}$$

From

$$f = 2W^{-2\alpha},$$

we have

$$f_1 = -4\alpha W^{-2\alpha-1} W_1 = -4\alpha W^{-2\alpha-1} h_{12} |\nabla u|,$$

and

$$\begin{aligned}
f_{11} = & (-4\alpha)(-2\alpha - 1) W^{-2\alpha-2} W_1^2 - 4\alpha W^{-2\alpha-1} W_{11} \\
= & 4\alpha(2\alpha + 1) W^{-2\alpha-2} h_{12}^2 |\nabla u|^2 - 4\alpha W^{-2\alpha-1}[|\nabla u| h_{112} - h_{12}^2 W] \\
= & 4\alpha(2\alpha + 1) W^{-2\alpha-2} h_{12}^2 |\nabla u|^2 + 12\alpha W^{-2\alpha} h_{12}^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\triangle K \cdot |\nabla u|^3 W^{2\alpha+2}\\
&= [-4\alpha(2\alpha+1)|\nabla u|^4 - 12\alpha|\nabla u|^2 W^2 - 12W(-2\alpha)W^2|\nabla u|^2\\
&\quad - 6|\nabla u|^2 W^2 - 12W^4]h_{12}^2\\
&= [-4\alpha(2\alpha+1)|\nabla u|^4 + 6(2\alpha-1)|\nabla u|^2 W^2 - 12W^4]h_{12}^2\\
&= -2[(2\alpha+3)(2\alpha+1)|\nabla u|^4 - 3(2\alpha+3)|\nabla u|^2 + 6]h_{12}^2,
\end{aligned}$$

where we have used

$$W^2 = 1 - |\nabla u|^2.$$

Set $t = |\nabla u|^2 \in [0,1]$, it follows that

$$\triangle K \cdot |\nabla u|^3 W^{2\alpha+2} = -2[(2\alpha+3)(2\alpha+1)t^2 - 3(2\alpha+3)t + 6]h_{12}^2.$$

Step 3.

Set

$$P(t) = (2\alpha+3)(2\alpha+1)t^2 - 3(2\alpha+3)t + 6 \tag{4.37}$$

Then we only need to determine α so that $P(t) \geq 0$.

First we search for the necessary condition for α so that $P(t) \geq 0$:

$$P(0) = 6 > 0,$$

$$P(1) = 4\alpha^2 + 8\alpha + 3 - 3(2\alpha+3) + 6 = 2\alpha(2\alpha+1) \geq 0.$$

Solving this inequality we get necessary condition :

$$\alpha \in (-\infty, -\frac{1}{2}] \cup [0, +\infty).$$

By some careful observation, it is also sufficient. □

5 Proof of Theorem 2.14

We will prove Theorem 2.14 in this section. Our proof use the moving frame which we introduce in section 4.

Proof of Theorem 2.14:

In the rest of this section, the repeated indices are summed from 1 to 3, unless otherwise stated.

Denote by K the Gauss curvature of level sets of the minimum hypersurface M^3 with height function u. At any point $X \in M^3$, we will deduce the following inequality

$$\Delta K(X) \leq 0 \quad \mathrm{mod}\ \{\nabla K(X)\} \tag{5.1}$$

where we modify the terms containing the gradient of K with locally bounded coefficients. Then by the strong maximum principle we get the result as desired.

From (4.2), we have the following formula of K:

$$K := F^{ij} u_i u_j |\nabla u|^{-4}, \tag{5.2}$$

with $F := \det(h_{ij})$ and $F^{ij} := \frac{\partial F}{\partial h_{ij}}$. We also use the following notations in the rest of this section:

$$F^{ij,rs} := \frac{\partial^2 F}{\partial h_{ij} \partial h_{rs}}; \qquad F^{ij,rs,pq} := \frac{\partial^3 F}{\partial h_{ij} \partial h_{rs} \partial h_{pq}}.$$

Now by differentiating both sides of $|\nabla u|^4 K = F^{ij} u_i u_j$ we get:

$$(|\nabla u|^4 K)_\alpha = |\nabla u|^4 K_\alpha + (|\nabla u|^4)_\alpha K, \tag{5.3}$$

and

$$(F^{ij} u_i u_j)_\alpha = F^{ij,rs} h_{rs\alpha} u_i u_j + 2F^{ij} u_{i\alpha} u_j. \tag{5.4}$$

Differentiate once again we can get respectively:

$$(|\nabla u|^4 K)_{\alpha\alpha} = |\nabla u|^4 K_{\alpha\alpha} + 2(|\nabla u|^4)_\alpha K_\alpha + (|\nabla u|^4)_{\alpha\alpha} K, \tag{5.5}$$

and

$$\begin{aligned} (F^{ij} u_i u_j)_{\alpha\alpha} = {} & F^{ij,rs,pq} h_{pq\alpha} h_{rs\alpha} u_i u_j + F^{ij,rs} h_{rs\alpha\alpha} u_i u_j \\ & + 4F^{ij,rs} h_{rs\alpha} u_{i\alpha} u_j + 2F^{ij} u_{i\alpha\alpha} u_j + 2F^{ij} u_{i\alpha} u_{j\alpha}. \end{aligned} \tag{5.6}$$

Hence

$$\begin{aligned} & |\nabla u|^4 K_{\alpha\alpha} + 2(|\nabla u|^4)_\alpha K_\alpha \\ & = F^{ij,rs,pq} h_{pq\alpha} h_{rs\alpha} u_i u_j + F^{ij,rs} h_{rs\alpha\alpha} u_i u_j + 4F^{ij,rs} h_{rs\alpha} u_{i\alpha} u_j \\ & \quad + 2F^{ij} u_{i\alpha\alpha} u_j + 2F^{ij} u_{i\alpha} u_{j\alpha} - (|\nabla u|^4)_{\alpha\alpha} K, \end{aligned} \tag{5.7}$$

while

$$\begin{aligned} -K(|\nabla u|^4)_{\alpha\alpha} &= -F^{ij} u_i u_j |\nabla u|^{-4} (4|\nabla u|^2 u_k u_{k\alpha})_\alpha \\ &= -8F^{ij} u_i u_j |\nabla u|^{-4} u_k u_{k\alpha} u_l u_{l\alpha} - 4F^{ij} u_i u_j |\nabla u|^{-2} u_{k\alpha} u_{k\alpha} \\ & \quad -4F^{ij} u_i u_j |\nabla u|^{-2} u_k u_{k\alpha\alpha}. \end{aligned} \tag{5.8}$$

At any fixed point $X \in M^3$, we may assume $u_1 = u_2 = 0, h_{12} = h_{21} = 0$ by a suitable choice of the frame work $\{e_1, e_2, e_3\}$. In the following, all the calculations will be done at this fixed point. Then inserting (5.8) into (5.7) yields:

$$\begin{aligned} |\nabla u|^4 K_{\alpha\alpha} + 2(|\nabla u|^4)_\alpha K_\alpha \; = {} & u_3^2 F^{33,rs,pq} h_{pq\alpha} h_{rs\alpha} + 4u_3 F^{i3,rs} h_{rs\alpha} u_{i\alpha} \\ & + u_3^2 F^{33,rs} h_{rs\alpha\alpha} + 2u_3 F^{i3} u_{i\alpha\alpha} - 4u_3 F^{33} u_{3\alpha\alpha} \\ & + 2F^{ij} u_{i\alpha} u_{j\alpha} - 4F^{33} u_{k\alpha} u_{k\alpha} - 8F^{33} u_{3\alpha} u_{3\alpha}, \\ := {} & \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \end{aligned} \tag{5.9}$$

Where we denote

$$\mathrm{I} := u_3^2 F^{33,rs,pq} h_{pq\alpha} h_{rs\alpha}, \tag{5.10}$$

$$\mathrm{II} := 4u_3F^{i3,rs}h_{rs\alpha}u_{i\alpha}, \tag{5.11}$$

$$\mathrm{III} := u_3^2F^{33,rs}h_{rs\alpha\alpha}, \tag{5.12}$$

$$\mathrm{IV} := 2u_3F^{i3}u_{i\alpha\alpha} - 4u_3F^{33}u_{3\alpha\alpha}, \tag{5.13}$$

and

$$\mathrm{V} := 2F^{ij}u_{i\alpha}u_{j\alpha} - 4F^{ij}u_{k\alpha}u_{k\alpha} - 8F^{ij}u_{3\alpha}u_{3\alpha}. \tag{5.14}$$

To compute the above terms in more detail, the third derivatives of u need to be handled. For this, we do some preparations to find the relationships of them.

By (5.3) and (5.4) we have:

$$\begin{aligned} |\nabla u|^4K_\alpha &= -(|\nabla u|^4)_\alpha K + F^{ij,rs}h_{rs\alpha}u_iu_j + 2F^{ij}u_{i\alpha}u_j \\ &= -4u_3F^{33}u_{3\alpha} + u_3^2F^{33,rs}h_{rs\alpha} + 2u_3F^{i3}u_{i\alpha} \\ &= -4u_3h_{11}h_{22}u_{3\alpha} + u_3^2h_{22}h_{11\alpha} + u_3^2h_{11}h_{22\alpha} \\ &\quad +2u_3h_{11}h_{22}u_{3\alpha} - 2u_3h_{11}h_{32}u_{2\alpha} - u_3h_{22}h_{31}u_{1\alpha} \\ &= u_3^2h_{22}h_{11\alpha} + u_3^2h_{11}h_{22\alpha} \\ &\quad -2u_3h_{11}h_{22}u_{3\alpha} - 2u_3h_{11}h_{32}u_{2\alpha} - u_3h_{22}h_{31}u_{1\alpha}, \end{aligned} \tag{5.15}$$

i.e.,

$$\begin{aligned} u_3^2h_{11}h_{22\alpha} = &-u_3^2h_{22}h_{11\alpha} + 2u_3h_{11}h_{22}u_{3\alpha} \\ &+2u_3h_{11}h_{32}u_{2\alpha} + u_3h_{22}h_{31}u_{1\alpha} + |\nabla u|^4K_\alpha. \end{aligned} \tag{5.16}$$

Since the level sets of u are all strictly convex, we have $K > 0$. But now $K = F^{33}u_3u_3|\nabla u|^{-4} = h_{11}h_{22}|\nabla u|^{-2}$, hence $h_{11}h_{22} > 0$. Let $\alpha = 1, 2, 3$ respectively in (5.16), by $|u_3| = |\nabla u| > 0$ we deduce:

$$\begin{cases} u_3h_{122} = -\frac{h_{22}}{h_{11}}u_3h_{111} + 4wh_{13}h_{22} + \frac{u_3^3}{h_{11}}K_1 \\ u_3h_{222} = -\frac{h_{22}}{h_{11}}u_3h_{112} + 4wh_{22}h_{23} + \frac{u_3^3}{h_{11}}K_2 \\ u_3h_{223} = -\frac{h_{22}}{h_{11}}u_3h_{113} + 2\frac{h_{22}}{h_{11}}wh_{13}^2 + 2wh_{22}h_{33} + 2h_{23}^2 + \frac{u_3^3}{h_{11}}K_3. \end{cases} \tag{5.17}$$

Obviously, all the coefficients of the first derivative of K in (5.17) are locally bounded. So we omit these terms when we submit with (5.17) in the following, i.e., all equalities or inequalities will be understood in the sense of mod$\{\nabla K\}$.

Now we compute (5.10):

$$\begin{aligned} \mathrm{I} &:= u_3^2F^{33,rs,pq}h_{pq\alpha}h_{rs\alpha} \\ &= u_3^2F^{33,rr,ss}h_{rr\alpha}h_{ss\alpha} + u_3^2F^{33,rs,sr}h_{rs\alpha}h_{sr\alpha} \\ &= 2u_3^2h_{11\alpha}h_{22\alpha} - 2u_3^2h_{12\alpha}h_{21\alpha}. \end{aligned} \tag{5.18}$$

Submitting (5.17) into the above terms respectively yields:

$$\begin{aligned} \mathrm{I}_1 &:= 2u_3^2h_{111}h_{122} - 2u_3^2h_{122}^2 \\ &= 2u_3h_{111}(-\tfrac{h_{22}}{h_{11}}u_3h_{111} + 4wh_{13}h_{22}) - 2(-\tfrac{h_{22}}{h_{11}}u_3h_{111} + 4wh_{13}h_{22})^2 \\ &= -2\tfrac{h_{22}}{h_{11}}(1 + \tfrac{h_{22}}{h_{11}})u_3^2h_{111}^2 + 8(1 + 2\tfrac{h_{22}}{h_{11}})h_{13}h_{22}wu_3h_{111} - 32w^2h_{13}^2h_{22}^2, \end{aligned} \tag{5.19}$$

$$\begin{aligned} \mathrm{I}_2 &:= 2u_3^2h_{112}h_{222} - 2u_3^2h_{112}^2 \\ &= 2u_3h_{112}(-\tfrac{h_{22}}{h_{11}}u_3h_{112} + 4wh_{22}h_{23}) - 2u_3^2h_{112}^2 \\ &= -2(1 + \tfrac{h_{22}}{h_{11}})u_3^2h_{112}^2 + 8h_{22}h_{23}wu_3h_{112}, \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \mathrm{I}_3 := & \; 2u_3^2 h_{113}h_{223} \\ = & \; 2u_3h_{113}(-\tfrac{h_{22}}{h_{11}}u_3h_{113} + 2\tfrac{h_{22}}{h_{11}}wh_{13}^2 + 2wh_{22}h_{33} + 2h_{23}^2) \\ = & \; -2\tfrac{h_{22}}{h_{11}}u_3^2h_{113}^2 + 4(\tfrac{h_{22}}{h_{11}}h_{13}^2 + h_{23}^2 + h_{22}h_{33})wu_3h_{113}. \end{aligned} \tag{5.21}$$

For (5.11) we compute:

$$\begin{aligned} \mathrm{II} := & \; 4u_3F^{i3,rs}h_{rs\alpha}u_{i\alpha} \\ = & \; 4wu_3F^{33,rr}h_{rr\alpha}h_{3\alpha} + 4wu_3F^{13,rs}h_{rs\alpha}h_{1\alpha} + 4wu_3F^{23,rs}h_{rs\alpha}h_{2\alpha}, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \mathrm{II}_1 := & \; 4wu_3F^{33,rr}h_{rr\alpha}h_{3\alpha} \\ = & \; 4wu_3h_{22}h_{3\alpha}h_{11\alpha} + 4wu_3h_{11}h_{3\alpha}h_{22\alpha}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} \mathrm{II}_2 := & \; 4wu_3F^{13,rs}h_{rs\alpha}h_{1\alpha} \\ = & \; 4wu_3F^{13,31}h_{31\alpha}h_{1\alpha} \\ & +4wu_3F^{13,22}h_{22\alpha}h_{1\alpha} + 4wu_3F^{13,21}h_{21\alpha}h_{1\alpha} \\ = & \; 4wu_3h_{13}h_{22}h_{111} - 4wu_3h_{13}(h_{11}-h_{22})h_{122} \\ & +4wu_3h_{11}h_{23}h_{112} - 4wu_3h_{11}h_{22}h_{113} \\ & -4wu_3h_{13}^2h_{223} + 4wu_3h_{13}h_{23}h_{123}, \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} \mathrm{II}_3 := & \; 4wu_3F^{23,rs}h_{rs\alpha}h_{2\alpha} \\ = & \; 4wu_3F^{23,32}h_{32\alpha}h_{2\alpha} + 4wu_3F^{23,11}h_{11\alpha}h_{2\alpha} + 4wu_3F^{23,12}h_{12\alpha}h_{2\alpha} \\ = & \; 4wu_3(h_{11}-h_{22})h_{23}h_{112} - 4wu_3h_{23}^2h_{113} + 4wu_3h_{13}h_{22}h_{122} \\ & +4wu_3h_{11}h_{23}h_{222} - 4wu_3h_{11}h_{22}h_{223} + 4wu_3h_{13}h_{23}h_{123}. \end{aligned} \tag{5.25}$$

Inserting (5.23)-(5.25) into (5.22) shows:

$$\begin{aligned} \mathrm{II} := & \; 4u_3F^{i3,rs}h_{rs\alpha}u_{i\alpha} \\ = & \; 8wu_3h_{13}h_{22}h_{111} + 8wu_3h_{11}h_{23}h_{112} - 4wu_3(2h_{11}h_{22} + h_{22}^2 + h_{23}^2)h_{113} \\ & +8wu_3h_{13}h_{22}h_{122} + 8wu_3h_{11}h_{23}h_{222} \\ & -4wu_3(h_{11}^2 + 2h_{11}h_{22} + h_{13}^2)h_{223} + 8wu_3h_{13}h_{23}h_{123}. \end{aligned} \tag{5.26}$$

By inserting (5.17) into the above we can see:

$$\begin{aligned} \mathrm{II} := & \; 4u_3F^{i3,rs}h_{rs\alpha}u_{i\alpha} \\ = & \; 8(1-\tfrac{h_{22}}{h_{11}})wu_3h_{13}h_{22}h_{111} + 8(h_{11}-h_{22})wu_3h_{23}h_{112} \\ & +4wu_3h_{113}(-h_{11}h_{22} + \tfrac{h_{22}}{h_{11}}h_{13}^2 + h_{22}^2 - h_{23}^2 + 4h_{11}h_{22}) + 8wu_3h_{13}h_{23}h_{123} \\ & +8w^2h_{11}^3h_{22} + 24w^2h_{11}^2h_{22}^2 - 8w^2h_{11}^2h_{23}^2 \\ & +16w^2h_{11}h_{22}^3 + 16w^2h_{11}h_{22}h_{23}^2 + 24w^2h_{13}^2h_{22}^2 \\ & -8w^2h_{13}^2h_{23}^2 - 8w^2\tfrac{h_{22}}{h_{11}}h_{13}^4. \end{aligned} \tag{5.27}$$

Submitting (5.19)+(5.20)+(5.21) into (5.18) and combining with (5.27) yields:

$$\begin{aligned}
\mathrm{I}+\mathrm{II} := & \; u_3^2 F^{33,rs,pq} h_{pq\alpha} h_{rs\alpha} + 4u_3 F^{i3,rs} h_{rs\alpha} u_{i\alpha} \\
= & \; I_1 + I_2 + I_3 - 2u_3^2 h_{123}^2 + II \\
= & \; -2\tfrac{h_{22}}{h_{11}}(1+\tfrac{h_{22}}{h_{11}}) u_3^2 h_{111}^2 - 2(1+\tfrac{h_{22}}{h_{11}}) u_3^2 h_{112}^2 \\
& -2\tfrac{h_{22}}{h_{11}} u_3^2 h_{113}^2 - 2u_3^2 h_{123}^2 \\
& +8(2+\tfrac{h_{22}}{h_{11}}) h_{13} h_{22} w u_3 h_{111} + 8 h_{11} h_{23} w u_3 h_{112} \\
& +4 w u_3 h_{113} (2\tfrac{h_{22}}{h_{11}} h_{13}^2 - 2h_{11} h_{22}) + 8 w u_3 h_{13} h_{23} h_{123} \\
& +8w^2 h_{11}^3 h_{22} + 24 w^2 h_{11}^2 h_{22}^2 - 8w^2 h_{11}^2 h_{23}^2 \\
& +16 w^2 h_{11} h_{22}^3 + 16 w^2 h_{11} h_{22} h_{23}^2 - 8w^2 h_{13}^2 h_{22}^2 \\
& -8w^2 h_{13}^2 h_{23}^2 - 8w^2 \tfrac{h_{22}}{h_{11}} h_{13}^4 .
\end{aligned} \tag{5.28}$$

For (5.12) direct calculations show:

$$\begin{aligned}
\mathrm{III} := & \; u_3^2 F^{33,rs} h_{rs\alpha\alpha} \\
= & \; u_3^2 h_{22} h_{11\alpha\alpha} + u_3^2 h_{11} h_{22\alpha\alpha} \\
= & \; u_3^2 h_{22} (h_{1\alpha\alpha 1} + h_{1m} R_{m\alpha 1\alpha} + h_{\alpha m} R_{m11\alpha}) \\
& + u_3^2 h_{11} (h_{2\alpha\alpha 2} + h_{2m} R_{m\alpha 2\alpha} + h_{\alpha m} R_{m22\alpha}) \\
= & \; u_3^2 h_{22} \Big(H_{11} + h_{1m}(h_{m1} h_{\alpha\alpha} - h_{m\alpha} h_{\alpha 1}) + h_{\alpha m}(h_{m1} h_{1\alpha} - h_{m\alpha} h_{11}) \Big) \\
& + u_3^2 h_{11} \Big(H_{22} + h_{2m}(h_{m2} h_{\alpha\alpha} - h_{m\alpha} h_{\alpha 2}) + h_{\alpha m}(h_{m2} h_{2\alpha} - h_{m\alpha} h_{22}) \Big) \\
= & \; -4u_3^2 h_{11}^3 h_{22} - 4u_3^2 h_{11}^2 h_{22}^2 - 4u_3^2 h_{11} h_{13}^2 h_{22} \\
& -4u_3^2 h_{11} h_{22}^3 - 4u_3^2 h_{11} h_{22} h_{23}^2 .
\end{aligned} \tag{5.29}$$

For (5.13) we compute:

$$\begin{aligned}
\mathrm{IV} := & \; 2u_3 F^{i3} u_{i\alpha\alpha} - 4u_3 F^{33} u_{3\alpha\alpha} \\
= & \; -2u_3 h_{13} h_{22} u_{1\alpha\alpha} - 2u_3 h_{11} h_{23} u_{2\alpha\alpha} - 2u_3 h_{11} h_{22} u_{3\alpha\alpha},
\end{aligned} \tag{5.30}$$

while

$$\begin{aligned}
u_{i\alpha\alpha} & = u_{\alpha\alpha i} + u_m R_{m\alpha i\alpha} \\
& = (h_{\alpha\alpha} w)_i + u_m (h_{mi} h_{\alpha\alpha} - h_{m\alpha} h_{\alpha i}) \\
& = -u_3 h_{3\alpha} h_{\alpha i}.
\end{aligned} \tag{5.31}$$

Inserting (5.31) into (5.30) deduces:

$$\begin{aligned}
\mathrm{IV} := & \; 2u_3 F^{i3} u_{i\alpha\alpha} - 4u_3 F^{33} u_{3\alpha\alpha} \\
= & \; 2u_3^2 h_{11}^3 h_{22} + 4u_3^2 h_{11}^2 h_{22}^2 - 2u_3^2 h_{11}^2 h_{23}^2 + 2u_3^2 h_{11} h_{13}^2 h_{22} \\
& +2u_3^2 h_{11} h_{22}^3 + 2u_3^2 h_{11} h_{22} h_{23}^2 - 2u_3^2 h_{13}^2 h_{22}^2.
\end{aligned} \tag{5.32}$$

By (5.29)+(5.32) we get:

$$\begin{aligned}
\mathrm{III}+\mathrm{IV} := & \; u_3^2 F^{33,rs} h_{rs\alpha\alpha} + 2u_3 F^{i3} u_{i\alpha\alpha} - 4u_3 F^{33} u_{3\alpha\alpha} \\
= & \; -2u_3^2 h_{11}^3 h_{22} - 2u_3^2 h_{11}^2 h_{23}^2 - 2u_3^2 h_{11} h_{13}^2 h_{22} \\
& -2u_3^2 h_{11} h_{22}^3 - 2u_3^2 h_{11} h_{22} h_{23}^2 - 2u_3^2 h_{13}^2 h_{22}^2.
\end{aligned} \tag{5.33}$$

For (5.14) we compute:

$$\begin{aligned}
\mathrm{V} := & \; 2F^{ij} u_{i\alpha} u_{j\alpha} - 4F^{33} u_{k\alpha} u_{k\alpha} - 8F^{33} u_{3\alpha} u_{3\alpha} \\
:= & \; V_1 + V_2,
\end{aligned} \tag{5.34}$$

where

$$\begin{aligned} \mathrm{V}_1 &:= 2F^{ij}u_{i\alpha}u_{j\alpha} \\ &= 2w^2(h_{11}h_{22}h_{33} - h_{13}^2\, h_{22} - h_{23}^2\, h_{11})(h_{11} + h_{22} + h_{33}) \\ &= 0, \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} \mathrm{V}_2 &:= -4F^{33}u_{k\alpha}u_{k\alpha} - 8F^{33}u_{3\alpha}u_{3\alpha} \\ &= -16w^2h_{11}^3\, h_{22} - 24w^2h_{11}^2\, h_{22}^2 - 16w^2h_{11}h_{13}^2\, h_{22} \\ &\quad -16w^2h_{11}h_{22}^3 - 16w^2h_{11}h_{22}h_{23}^2. \end{aligned} \tag{5.36}$$

Submitting (5.35)+(5.36) into (5.34) we get:

$$\begin{aligned} \mathrm{V} &:= \mathrm{V}_1 + \mathrm{V}_2 \\ &= -16w^2h_{11}^3\, h_{22} - 24w^2h_{11}^2\, h_{22}^2 - 16w^2h_{11}h_{13}^2\, h_{22} \\ &\quad -16w^2h_{11}h_{22}^3 - 16w^2h_{11}h_{22}h_{23}^2. \end{aligned} \tag{5.37}$$

Finally, by inserting (5.28)+(5.33)+(5.37) into (5.9) we arrive at:

$$\begin{aligned} |\nabla u|^4K_{\alpha\alpha} &:= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} \\ &= -2\tfrac{h_{22}}{h_{11}}(1 + \tfrac{h_{22}}{h_{11}})\Big(u_3h_{111} - \tfrac{2w(2h_{11}+h_{22})}{h_{11}+h_{22}}h_{11}h_{13}\Big)^2 \\ &\quad -2(1 + \tfrac{h_{22}}{h_{11}})\Big(u_3h_{112} - \tfrac{2wh_{11}}{h_{11}+h_{22}}h_{11}h_{23}\Big)^2 \\ &\quad -2\tfrac{h_{22}}{h_{11}}\Big(u_3h_{113} - 2w(h_{13}^2 - h_{11}^2)\Big)^2 \\ &\quad -2(u_3h_{123} - 2wh_{13}h_{23})^2 - \tfrac{8w^2h_{11}}{h_{11}+h_{22}}h_{13}^2\, h_{22}^2 - \tfrac{8w^2h_{22}}{h_{11}+h_{22}}h_{11}^2\, h_{23}^2 \\ &\quad -2u_3^2(h_{11}^3\, h_{22} + h_{11}^2\, h_{23}^2 + h_{11}h_{13}^2\, h_{22} + h_{11}h_{22}^3 \\ &\quad +h_{11}h_{22}h_{23}^2 + h_{13}^2\, h_{22}^2). \end{aligned} \tag{5.38}$$

Since $K = F^{33}u_3u_3|\nabla u|^{-4} = h_{11}h_{22}|\nabla u|^{-2} > 0$, and hence from $h_{11}h_{22} > 0$ and (5.38) it follows that

$$\Delta K(X) \leq 0 \quad \mathrm{mod}\ \{\nabla K(X)\}. \tag{5.39}$$

This is just (5.1). The proof is completed. □

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