A Constant Rank Theorem for Quasiconcave Solutions of Fully Nonlinear Partial Differential Equations

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ABSTRACT. We prove a constant rank theorem for the second fundamental form of the convex level surfaces of solutions to equations $F(D^2u, Du, u, x) = 0$ under a structural condition introduced by Bianchini-Longinetti-Salani in [2].

1. INTRODUCTION

A function u is called *quasiconcave* if its level set $\{x \mid u(x) \geq c\}$ is convex for each constant c. The convexity of level-sets of solutions for partial differential equations was first studied by Gabriel [9] for harmonic functions u in convex ring domains of the form

$$\Omega = \Omega_0 \setminus \Omega_1, \quad \text{with boundary condition } u|_{\partial\Omega_0} = 0 \text{ and } u|_{\partial\Omega_1} = 1. \tag{1.1}$$

Lewis [15] extended the results in [9] to p-harmonic functions. Caffarelli-Spruck treated this problem for general inhomogeneous Laplace equation in [6] with the same boundary condition (1.1) in connection to a free boundary problem. Kawhol [12] proposed an approach of using quasi-concave envelop to study the level-set convexity of solutions to PDEs. Colesanti-Salani [7] carried out this approach for a class of elliptic equations. The technique was extended by Greco [10], Cuoghi-Salani [8] and Longinetti-Salani [16] for equations of type

$$F(D^2u, Du, u, x) = 0 \tag{1.2}$$

in convex ring under various structure conditions. General structure conditions on F in equation (1.2) with Dirichlet condition (1.1) have been obtained in a recent paper [2] by Bianchini-Longinetti-Salani. All these type of results are of macroscopic nature. A different direction in the study of the convexity is the microscopic convexity principles. The *constant rank theorem* for the second fundamental forms of level sets of solutions to certain type of quasilinear equations was established by Korevaar [13], see also Xu [17] for recent generalization of results in [13].

Our interest is the microscopic counterpart of Theorem 1.1 in [2] by Bianchini-Longinetti-Salani. Let Ω be a domain in $\mathbb{R}^n$; S^n denotes the space of real symmetric $n \times n$ matrices and $\Lambda \subset S^n$ is an open set, and $F = F(r, p, u, x)$ is a $C^{2,1}$ function in $\Lambda \times \mathbb{R}^n \times \mathbb{R} \times \Omega$. For each $(\theta, u) \in \mathbb{S}^{n-1} \times \mathbb{R}$ fixed, set

$$\Gamma_F = \{(A, t, x) \in \Lambda \times (0, +\infty) \times \Omega \mid F(t^{-3}A, t^{-1}\theta, u, x) \ge 0\}. \tag{1.3}$$

We will assume that F satisfies the following conditions: there is $\gamma_0 > 0$ and $c_0 \in \mathbb{R}$,

$$F^{\alpha\beta} := \left(\frac{\partial F}{\partial r_{\alpha\beta}}(r, p, u, x)\right) > 0, \quad \forall\, (r, p, u, x) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0) \times \Omega, \tag{1.4}$$

and

$$\Gamma_F \quad \text{is locally convex for each } (\theta, u) \in \mathbb{S}^{n-1} \times (-\gamma_0 + c_0, \gamma_0 + c_0). \tag{1.5}$$

Theorem 1.1. *Suppose $u \in C^{3,1}(\Omega)$ is a solution of fully nonlinear equation (1.2) such that*

$$(D^2u(x), Du(x), u(x)) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0)$$

for each $x \in \Omega$. Suppose that F satisfies conditions (1.4) and (1.5), $Du \neq 0$, and the level sets $\{x \in \Omega \mid u(x) \ge c\}$ of u are connected and locally convex for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$ for some $\gamma_0 > 0$. Then it follows that the second fundamental form of level surfaces $\{x \in \Omega \mid u(x) = c\}$ has the same constant rank for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$.

Remark 1.2. The structural condition (1.5) is a localized version of a condition introduced by Bianchini-Longinetti-Salani (condition (1.2) in [2]). Under that condition and a weaker ellipticity condition, Bianchini-Longinetti-Salani proved (Theorem 1.1 in [2]) that any solution u of equation (1.2) on convex ring $\Omega = \Omega_0 \setminus \Omega_1$ with the Dirichlet boundary condition (1.1) is quasiconcave, provided $|Du| \neq 0$. Theorem 1.1 implies the strict convexity of the level-sets in Theorem 1.1 in [2]. Also, Theorem 1.1 may yield macroscopic level-set convexity results if there is a homotopic path. As discussed in [2], condition (1.5) is satisfied by a class of elliptic operators, including Laplace operator, p-Laplace operators and Pucci's operator.

The proof of Theorem 1.1 uses the techniques developed in Bian-Guan [1] for the convexity of solutions of nonlinear partial differential equations. The convexity of level-sets is much more involved due to the distinguished gradient direction of the set $\{u = c\}$. This is also the main fact that makes the structural condition (1.5) different from the structural condition considered in [1].

The organization of the paper is as follows. In Section 2, we list some useful formulas for the second fundamental forms of level sets in terms of u, Du, D^2u. Main technique lemmas will be proved in Section 3. The proof of Theorem 1.1 is given in Section 4.

2. Preliminaries

We recall some basic notation of differential geometry of hypersurfaces in $\mathbb{R}^n$. For a hypersurface Σ given by a graph in a domain in $\mathbb{R}^{n-1}$,

$$x_n = v(x'),\ x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1},$$

one may express the first fundamental form as

$$g_{ij} = \delta_{ij} + v_{x_i} v_{x_j}, \quad \forall\, i, j \le n-1. \tag{2.1}$$

The upward normal direction $\vec{n}$ and the second fundamental form II for a graph $x_n = v(x')$ are respectively given by

$$\vec{n} = \frac{1}{\sqrt{1 + |\nabla_{x'} v|^2}} (-v_1, -v_2, \dots, -v_{n-1}, 1), \tag{2.2}$$

$$h_{ij} = \frac{v_{x_i x_j}}{W}, \quad \forall\, i, j \le n-1 \tag{2.3}$$

where $W = (1 + |\nabla_{x'} v|^2)^{1/2}$.

Definition 2.1. The graph of function $x_n = v(x')$ is *convex* with respect to *the upward normal*

$$\vec{n} = \frac{1}{\sqrt{1 + |\nabla_{x'} v|^2}} (-v_1, -v_2, \dots, -v_{n-1}, 1)$$

if the second fundamental form $II := (h_{ij})$ defined in (2.3) is nonnegative definite.

The principal curvature $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ of the graph satisfies

$$\det(h_{ij} - \kappa g_{ij}) = 0.$$

Equivalently, that κ satisfies

$$\det(a_{ij} - \kappa \delta_{ij}) = 0,$$

where a_{ij} is the symmetric Weingarten tensor defined as

$$\{a_{ij}\} = \{g^{ij}\}^{1/2}\{h_{ij}\}\{g^{ij}\}^{1/2}, \quad \forall\, i,j \le n-1;$$

here $\{g^{ij}\}$ is the inverse matrix to $\{g_{ij}\}$, and $\{g^{ij}\}^{1/2}$ is its positive square root. They are given explicitly by

$$\{g^{ij}\} = \left\{\delta_{ij} - \frac{v_{x_i}v_{x_j}}{W^2}\right\}, \quad \{g^{ij}\}^{1/2} = \left\{\delta_{ij} - \frac{v_{x_i}v_{x_j}}{W(1+W)}\right\}. \tag{2.4}$$

The Weingarten tensor of the hypersurface can be expressed as (e.g., see [5]),

$$a_{i\ell} = \sum_{j,k=1}^{n-1} \frac{1}{W}\left(v_{i\ell} - \frac{v_i v_j v_{j\ell}}{W(1+W)} - \frac{v_\ell v_k v_{ki}}{W(1+W)} + \frac{v_i v_\ell v_j v_k v_{jk}}{W^2(1+W)^2}\right), \quad \forall\, i,\ell \le n-1. \tag{2.5}$$

Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, such that $|Du| \ne 0$ in Ω. Denote the level surface of u passing through the point $x_0 \in \Omega$ as

$$\Sigma^{u(x_0)} := \{x \in \Omega \mid u(x) = u(x_0)\}.$$

We wish to express the Weingarten tensor of the level surface in terms of u, Du, D^2u.

At x_0, after proper rotation, we may assume $Du = (u_1, \dots, u_n)$ with $u_n \ne 0$. By Implicity Function Theorem, the level set $\Sigma^{u(x_0)}$ can be locally represented as a graph

$$x_n = v(x'),\ x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

For $u(x_1, \dots, x_{n-1}, x_n) \in C^2(\Omega)$, and the function $v(x')$ satisfies the following equation

$$u(x_1, x_2, \dots, x_{n-1}, v(x_1, x_2, \dots, x_{n-1})) = c. \tag{2.6}$$

Differentiate equation (2.6),

$$u_i + u_n v_i = 0, \quad v_i = -\frac{u_i}{u_n}, \quad W = (1 + |\nabla_{x'} v|^2)^{1/2} = \frac{|Du|}{|u_n|}.$$

It follows that the upward outer normal direction of the level sets is

$$\vec{n} = \frac{|u_n|}{|Du|u_n}(u_1, u_2, \dots, u_{n-1}, u_n). \tag{2.7}$$

Differentiating (2.6) one more time,

$$u_{ij} + u_{in}v_j + u_{nj}v_i + u_{nn}v_iv_j + u_nv_{ij} = 0.$$

In turn,

$$v_{ij} = -\frac{1}{u_n^3}[u_n^2u_{ij} + u_{nn}u_iu_j - u_nu_ju_{in} - u_nu_iu_{jn}]. \tag{2.8}$$

The second fundamental form II of the level surface of function u with respect to the upward normal direction (2.7) is

$$h_{ij} = -\frac{|u_n|(u_n^2u_{ij} + u_{nn}u_iu_j - u_nu_ju_{in} - u_nu_iu_{jn})}{|Du|u_n^3}. \tag{2.9}$$

Note that expression (2.9) is valid locally near $x_0 \in \Omega$, independent of constant c in (2.6).

Definition 2.2. For a function $u \in C^2(\Omega)$ with $|Du| \neq 0$ in Ω, for each $y \in \Omega$, the level surface

$$\Sigma^{u(y)} = \{x \in \Omega \mid u(x) = u(y)\}$$

is called locally convex with respect to Du near $x_0 \in \Sigma^{u(y)}$ if there is a local coordinate chart near x_0 (probably after some rotation) such that $u_n(x) > 0$ and the second fundamental form h_{ij} defined in (2.9) is nonnegative definite near x_0 with respect to the upward normal direction $\vec{n}$ defined in (2.7) for $x \in \Sigma^{u(y)}$ close to x_0.

Remark 2.3. If $\{x \in \Omega \mid u(x) \geq c\}$ is locally convex, then by Definition 2.2 the second fundamental form of Σ^c is nonnegative definite with respect to Du. For any $x_0 \in \Omega$, if $u_n(x_0) = |Du(x_0)|$ and the level set $\{x \in \Omega \mid u(x) = u(x_0)\}$ is locally convex near x_0, then (2.9) implies that the matrix $(u_{ij}(x_0))$ is nonpositive definite.

From (2.5) and (2.9),

$$a_{ij} = \sum_{k,\ell=1}^{n-1}\left(h_{ij} - \frac{u_iu_\ell h_{j\ell}}{W(1+W)u_n^2} - \frac{u_ju_\ell h_{i\ell}}{W(1+W)u_n^2} + \frac{u_iu_ju_ku_\ell h_{k\ell}}{W^2(1+W)^2u_n^4}\right). \tag{2.10}$$

With the above notation, at the point x where $u_n(x) = |\nabla u(x)| > 0$, $u_i(x) = 0$, $a_{ij,k}$ is commutative. That is, they satisfy the Codazzi property $a_{ij,k} = a_{ik,j}$, $\forall\, i, j, k \leq n-1$.

3. Estimates

Since Theorem 1.1 is of local feature, we may assume that the level surface $\Sigma^c = \{x \in \Omega \mid u(x) = c\}$ is connected for each $c \in (c_0 - \gamma_0, c_0 + \gamma_0)$. Let $\ell(x)$ be the rank of the second fundamental form of $\Sigma^{u(x)}$ at x. Denote

$$\ell = \inf_{x\in\Omega} \ell(x). \tag{3.1}$$

Since the values of $\ell(x)$ are in $\mathbb{Z}$, there is $x_0 \in \Omega$ such that $\ell(x_0) = \ell$. We will concentrate in a neighborhood of some point $x_0 \in \Omega$ such that $\ell(x_0) = \ell$. We may assume $\ell \le n-2$. We will assume $u \in C^{3,1}(\Omega)$, $u_n > 0$ *and the level surface* Σ^c *is convex with respect to normal* Du *for each* c *in a small neighborhood of* $u(x_0)$ in the rest of the paper.

Let $\mathcal{O}$ be a small open neighborhood of x_0 such that for each $x \in \mathcal{O}$, there are ℓ "good" eigenvalues of (a_{ij}) which are bounded below by a positive constant, and the other $n-1-\ell$ "bad" eigenvalues of (a_{ij}) are very small. Denote G the index set of these "good" eigenvalues and B the index set of "bad" eigenvalues. For each $x \in \mathcal{O}$ fixed, we may express (a_{ij}) in a form of (2.10), by choosing $e_1, \dots, e_{n-1}, e_n$ such that

$$|Du|(x) = u_n(x) > 0, \text{ matrix } (u_{ij}),\ i,j = 1,\dots,n-1 \text{ is diagonal at } x. \tag{3.2}$$

From (2.10), the matrix (a_{ij}), $i,j = 1,\dots,n-1$ is also diagonal at x, and without loss of generality we may assume $a_{11} \le a_{22} \le \cdots \le a_{n-1,n-1}$. There is a positive constant $C_o > 0$ such that

$$a_{n-1,n-1} \ge a_{n-2,n-2} \ge \cdots \ge a_{n-\ell,n-\ell} > C_o, \quad \forall\, x \in \mathcal{O},$$
$$G = \{n-\ell, n-\ell+1, \dots, n-1\}, \quad B = \{1, 2, \dots, n-\ell-1\}.$$

So that there is no confusion, we also denote

$$B = \{a_{11}, \dots, a_{n-\ell-1,n-\ell-1}\} \quad \text{and} \quad G = \{a_{n-\ell,n-\ell}, \dots, a_{n-1,n-1}\}. \tag{3.3}$$

Note that for any $\delta > 0$, we may choose $\mathcal{O}$ small enough such that $a_{jj}(x) < \delta$ for all $j \in B$ and $x \in \mathcal{O}$. For two functions f, h in $\mathcal{O}$, we write $h = O(f)$ if $|h(x)| \le Cf(x)$ for $x \in \mathcal{O}$ with positive constant C under control.

For each c close to $u(x_0)$, let $a = (a_{ij})$ be the symmetric Weingarten tensor of Σ^c. Set

$$p(a) = \sigma_{\ell+1}(a_{ij}), \quad q(a) = \begin{cases} \dfrac{\sigma_{\ell+2}(a_{ij})}{\sigma_{\ell+1}(a_{ij})}, & \text{if } \sigma_{\ell+1}(a_{ij}) > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Theorem 1.1 is equivalent to saying $p(a) \equiv 0$ (defined in (3.4)) in $\mathcal{O}$. For general fully nonlinear equation (1.2), as in the case for the convexity of solutions in [1], there are some technical difficulties to deal with $p(a)$ alone. A key idea introduced in [1] is to use some crucial concavity properties of function q defined in (3.4). Set

$$\varphi(a) = p(a) + q(a) \tag{3.5}$$

where p and q are as in (3.4). Theorem 1.1 is equivalent to saying $\varphi(a) \equiv 0$.

To get around $p = 0$, for $\varepsilon > 0$ sufficiently small, consider

$$\varphi_\varepsilon(a) = \varphi(a_\varepsilon), \tag{3.6}$$

where $a_\varepsilon = a + \varepsilon I$. Denote $G_\varepsilon = \{a_{ii} + \varepsilon \mid i \in G\}$, $B_\varepsilon = \{a_{ii} + \varepsilon \mid i \in B\}$.

To simplify the notation, we will drop subindex ε with the understanding that *all the estimates will be independent of* ε. In this setting, if $\mathcal{O}$ is small enough, there is $C > 0$ independent of ε such that

$$\varphi(a(z)) \ge C\varepsilon, \ \sigma_1(B(z)) \ge C\varepsilon, \quad \text{for all } z \in \mathcal{O}. \tag{3.7}$$

In what follows, we will use $i, j, \dots$ as indices running from 1 to $n - 1$ and $\alpha, \beta, \dots$ as indices running from 1 to n. Denote

$$p_\alpha = \frac{\partial p}{\partial x_\alpha}, \quad p_{\alpha\beta} = \frac{\partial^2 p}{\partial x_\alpha \partial x_\beta}, \quad F^{\alpha\beta} = \frac{\partial F}{\partial u_{\alpha\beta}}, \quad 1 \le \alpha, \beta \le n,$$

and set

$$\mathcal{H}_\varphi = \sum_{i,j \in B} |\nabla a_{ij}| + \varphi. \tag{3.8}$$

Lemma 3.1. *For any fixed $x \in \mathcal{O}$, with the coordinate chart chosen as in* (3.2) *and* (3.3),

$$p_\alpha = \sigma_\ell(G) \sum_{j \in B} a_{jj,\alpha} + O(\mathcal{H}_\varphi) \tag{3.9}$$

and

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} p_{\alpha\beta} \le -u_n^{-3} \sigma_\ell(G) \sum_{j \in B} \Big[\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta jj} u_n^2 - 6 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} u_{nj} u_n + 6 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} u_{nj}^2 \Big] + O(\mathcal{H}_\varphi). \tag{3.10}$$

Proof. For each fixed point $x \in \mathcal{O}$, in a coordinate system as in (3.2),

$$-\frac{u_{jj}}{u_n} = a_{jj} = O(\mathcal{H}_\varphi), \ \forall\, j \in B; \quad p_\alpha = \sigma_\ell(G) \sum_{j \in B} a_{jj,\alpha} + O(\mathcal{H}_\varphi). \tag{3.11}$$

By (3.11),

$$p_{\alpha\beta} = \sigma_\ell(G)\Big[\sum_{j\in B} a_{jj,\alpha\beta} - 2\sum_{i\in G,\, j\in B} \frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}}\Big] + O(\mathcal{H}_\varphi). \tag{3.12}$$

We now need to figure in the distinguished gradient direction Du in the symmetric tensor (a_{ij}). Since $u_k = 0$ at x for $k = 1, \dots, n-1$, from (2.10),

$$u_n u_{ij\alpha} = -u_n^2 a_{ij,\alpha} + u_{nj}u_{i\alpha} + u_{ni}u_{j\alpha} + u_{n\alpha}u_{ij}, \quad \forall\, i, j \le n-1, \tag{3.13}$$

and for each $j \in B$,

$$\begin{aligned} u_n^3 a_{jj,\alpha\beta} &= 2u_n u_{j\alpha}u_{nj\beta} + 2u_n u_{j\beta}u_{nj\alpha} + 2u_n u_{nj}u_{\alpha\beta j} - u_n^2 u_{\alpha\beta jj} \\ &\quad - 2u_{n\alpha}u_{nj}u_{j\beta} - 2u_{n\beta}u_{nj}u_{j\alpha} - 2u_{nn}u_{j\alpha}u_{j\beta} + O(\mathcal{H}_\varphi). \end{aligned}$$

Hence, for $j \in B$,

$$\begin{aligned} \sum_{\alpha,\beta=1}^n F^{\alpha\beta}a_{jj,\alpha\beta} = \sum_{\alpha,\beta=1}^n \frac{F^{\alpha\beta}}{u_n^3}\Big[&-u_n^2 u_{\alpha\beta jj} - 4u_{n\alpha}u_{nj}u_{j\beta} \\ &+ 4u_n u_{j\alpha}u_{nj\beta} + 2u_n u_{nj}u_{\alpha\beta j} \\ &- 2u_{nn}u_{j\alpha}u_{j\beta}\Big] + O(\mathcal{H}_\varphi). \end{aligned} \tag{3.14}$$

Using the fact that $\sum_{\alpha=1}^n F^{\alpha n}u_{n\alpha} = (\sum_{\alpha,\beta=1}^n - \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^n)F^{\alpha\beta}u_{\alpha\beta}$, $\forall\, j \in B$, we have

$$\begin{aligned} \sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{n\alpha}u_{j\beta} &= u_{nj}\Big(\sum_{\alpha,\beta=1}^n - \sum_{\beta=1}^{n-1}\sum_{\alpha=1}^n\Big)F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\varphi), \\ \sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha}u_{nj\beta} &= u_{nj}\Big(\sum_{\alpha,\beta=1}^n - \sum_{\alpha=1}^{n-1}\sum_{\beta=1}^n\Big)F^{\alpha\beta}u_{\alpha\beta j} + O(\mathcal{H}_\varphi), \end{aligned}$$

and

$$\begin{aligned} -2u_{nn}\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha}u_{j\beta} &= -2u_{nn}F^{nn}u_{nj}^2 + O(\mathcal{H}_\varphi) \\ &= -2u_{nj}^2\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta} + 4u_{nj}^2\sum_{\alpha=1}^{n-1}F^{\alpha n}u_{n\alpha} \\ &\quad + 2u_{nj}^2\sum_{\alpha,\beta=1}^{n-1}F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\varphi). \end{aligned}$$

Put above to (3.14),

$$
\begin{aligned}
(3.15)\qquad & \sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_n^3 a_{jj,\alpha\beta} \\
&= -u_n^2\sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} + 6u_n\sum_{j\in B}u_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} \\
&\quad - 6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta} - 4u_n\sum_{j\in B}u_{nj}\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} \\
&\quad + 8\sum_{j\in B}u_{nj}^2\sum_{\alpha=1}^{n-1}F^{\alpha n}u_{n\alpha} + 6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n-1}F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\varphi).
\end{aligned}
$$

By (3.13), for $j \in B$,

$$
\begin{aligned}
(3.16)\qquad u_n\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} &= u_n\sum_{\alpha=1}^{n}\Big(\sum_{i\in B}F^{\alpha i}u_{ij\alpha} + \sum_{i\in G}F^{\alpha i}u_{ij\alpha}\Big) \\
&= \sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}\left(-u_n^2 a_{ij,\alpha} + u_{i\alpha}u_{jn} + u_{j\alpha}u_{in}\right) \\
&\quad + \sum_{\alpha=1}^{n}\sum_{i\in B}F^{\alpha i}\left(u_{i\alpha}u_{jn} + u_{j\alpha}u_{in}\right) + O(\mathcal{H}_\varphi) \\
&= -u_n^2\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}a_{ij,\alpha} + u_{nj}\sum_{i\in G}F^{ii}u_{ii} \\
&\quad + 2u_{nj}\Big(\sum_{i=1}^{n-1}F^{ni}u_{ni}\Big) + O(\mathcal{H}_\varphi).
\end{aligned}
$$

Equations (3.15) and (3.16) yield that, $\forall\ j \in B$,

$$
\begin{aligned}
(3.17)\qquad \sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_n^3 a_{jj,\alpha\beta} &= -u_n^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta jj} \\
&\quad + 6u_nu_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} - 6u_{nj}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta} \\
&\quad + 4u_n^2u_{nj}\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}a_{ij,\alpha} + 2u_{nj}^2\sum_{i\in G}F^{ii}u_{ii} + O(\mathcal{H}_\varphi).
\end{aligned}
$$

From (3.17), $\forall\, j \in B$,

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\Big[a_{jj,\alpha\beta} - 2\sum_{i\in G}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}}\Big] \tag{3.18}$$
$$= -u_n^{-3}\Big[\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_n^2 u_{\alpha\beta jj} - 6u_n\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{jn}u_{\alpha\beta j}$$
$$+ 6u_{jn}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta}\Big] - 2\sum_{i\in G}\frac{1}{a_{ii}}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta}$$
$$+ 4u_n^{-1}u_{nj}\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}a_{ij,\alpha} + 2u_n^{-3}u_{nj}^2\sum_{i\in G}F^{ii}u_{ii} + O(\mathcal{H}_\varphi).$$

Claim 3.2.

$$\forall\, i, j, \quad \frac{-1}{a_{ii}}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + \frac{2u_{nj}}{u_n}\sum_{\alpha=1}^{n}F^{\alpha i}a_{ij,\alpha} + \frac{u_{nj}^2F^{ii}u_{ii}}{u_n^3} \le 0.$$

Assuming Claim 3.2, by (3.12)

$$\frac{\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}p_{\alpha\beta}}{\sigma_\ell(G)} \le -u_n^{-3}\sum_{j\in B}\sum_{\alpha,\beta=1}^{n}\Big[F^{\alpha\beta}u_n^2u_{jj\alpha\beta} - 6u_nF^{\alpha\beta}u_{jn}u_{j\alpha\beta} \tag{3.19}$$
$$+ 6u_{jn}^2F^{\alpha\beta}u_{\alpha\beta}\Big] + O(\mathcal{H}_\varphi).$$

We need to check Claim 3.2. It is equivalent to the following inequality,

$$\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} - 2u_n^{-1}u_{nj}a_{ii}\sum_{\alpha=1}^{n}F^{\alpha i}a_{ij,\alpha} + u_n^{-2}u_{nj}^2F^{ii}a_{ii}^2 \ge 0. \tag{3.20}$$

We may assume $i = 1$ and j is fixed. Set $X_0 = u_n^{-1}a_{11}u_{jn}$ and $X_\alpha = a_{1j,\alpha}$ for $1 \le \alpha \le n$, (3.20) follows from the fact that $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} F^{11} & -F^{11} & -F^{12} & \cdots & -F^{1n} \\ -F^{11} & F^{11} & F^{12} & \cdots & F^{1n} \\ -F^{21} & F^{21} & F^{22} & \cdots & F^{2n} \\ \vdots & & & & \\ -F^{n1} & F^{n1} & F^{n2} & \cdots & F^{nn} \end{bmatrix}$$

is semi-positive definite. $\square$

Lemma 3.3. *$q \in C^{1,1}(\mathcal{O})$ and for any fixed $x \in \mathcal{O}$, with the coordinate chosen as in (3.2) and (3.3),*

$$q_\alpha = \frac{\partial q}{\partial x_\alpha} = \sum_{j\in B} \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} a_{jj,\alpha} + O(\mathcal{H}_\varphi), \tag{3.21}$$

and

$$\begin{aligned} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} q_{\alpha\beta} = &-u_n^{-3} \sum_{j\in B} \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \\ &\times \sum_{\alpha,\beta=1}^n \left[F^{\alpha\beta} u_{\alpha\beta jj} u_n^2 - 6F^{\alpha\beta} u_{\alpha\beta j} u_{jn} u_n + 6F^{\alpha\beta} u_{\alpha\beta} u_{jn}^2\right] \\ &- \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^n \sum_{i\in B} F^{\alpha\beta}\left[\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j\in B} a_{jj,\alpha}\right] \\ &\times \left[\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j\in B} a_{jj,\beta}\right] \\ &- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^n \sum_{i\neq j\in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\varphi). \end{aligned} \tag{3.22}$$

Proof. The fact $q \in C^{1,1}(\mathcal{O})$ follows Corollary 2.2 in [1]. Though it was stated for nonnegative matrix function $W = (u_{ij})$ with $u \in C^{3,1}$, the proof works for any nonnegative matrix function $W \in C^{1,1}$.

Identity (3.21) follows directly from Lemma 2.4 in [1]. Again, by Lemma 2.4 in [1],

$$\begin{aligned} q_{\alpha\beta} = &\sum_{j\in B} \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \left[a_{jj,\alpha\beta} - 2\sum_{i\in G} \frac{a_{ij,\alpha} a_{ij,\beta}}{a_{ii}}\right] \\ &- \frac{1}{\sigma_1^3(B)} \sum_{i\in B} \left[\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j\in B} a_{jj,\alpha}\right] \\ &\times \left[\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j\in B} a_{jj,\beta}\right] \\ &- \frac{1}{\sigma_1(B)} \sum_{i\neq j\in B} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\varphi). \end{aligned} \tag{3.23}$$

The lemma follows from (3.18) and Claim 3.2 in the proof of Lemma 3.1. ∎

4. A Strong Maximum Principle

We start this section by discussing the structure condition imposed in Theorem 1.1. For any function $F(r, Du, u, x)$, write $F^{\alpha\beta} = \partial F/\partial r_{\alpha\beta}$, $F^{u_\ell} = \partial F/\partial u_\ell$, ...

as derivatives of F with respect to corresponding arguments. For Γ_F defined in (1.3), denote

$$\mathcal{T}\Gamma_F = \Big\{V = ((X_{\alpha\beta}), Y, (Z_i)) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \\ : \langle V, \nabla_{(A,t,x)} F(t^{-3}A, t^{-1}\theta, u, x)\rangle = 0\Big\}.$$

Lemma 4.1. *If F satisfies condition (1.5), then*

$$\begin{aligned} Q(V,V) = {} & F^{\alpha\beta,rs} X_{\alpha\beta} X_{rs} + 2F^{\alpha\beta,u_\ell}\theta_\ell X_{\alpha\beta} Y + 2F^{\alpha\beta,x_k} X_{\alpha\beta} Z_k \\ & + F^{u_\ell,u_s}\theta_\ell\theta_s Y^2 + 2F^{u_\ell,x_k}\theta_\ell Y Z_k + F^{x_i,x_j} Z_i Z_j \\ & + 2tF^{u_\ell}\theta_\ell Y^2 + 6tF^{\alpha\beta} X_{\alpha\beta} Y - 6t^{-1}F^{\alpha\beta} A_{\alpha\beta} Y^2 \\ \leqslant {} & 0, \end{aligned} \tag{4.1}$$

for every

$$(X_{\alpha\beta}, Y, (Z_i)) = ((t^{-3}\tilde{X}_{\alpha\beta} - 3t^{-4}A_{\alpha\beta}\tilde{Y}), -t^{-2}\tilde{Y}, (Z_i)), \\ \text{with } \tilde{V} = ((\tilde{X}_{\alpha\beta}), \tilde{Y}, (Z_i)) \in \mathcal{T}\Gamma_F,$$

where $F^{\alpha\beta,rs}$, $F^{\alpha\beta,u_\ell}$, etc. are evaluated at $(t^{-3}A, t^{-1}\theta, u, x)$, and the Einstein summation convention is used.

Proof. Denote $\tilde{F}(A,t,x) = F(t^{-3}A, t^{-1}\theta, u, x)$; Condition (1.5) implies that $\tilde{F}(A,t,x)$ is locally convex with respect to the normal $\nabla\tilde{F}$. That is, for each tangential vector $\tilde{V} = ((\tilde{X}_{ij}), \tilde{Y}, (\tilde{Z}_i))$:

$$\begin{aligned} & \tilde{F}^{\alpha\beta,rs}\tilde{X}_{\alpha\beta}\tilde{X}_{rs} + 2\tilde{F}^{\alpha\beta,t}\tilde{X}_{\alpha\beta}\tilde{Y} + 2\tilde{F}^{\alpha\beta,x_k}\tilde{X}_{\alpha\beta}\tilde{Z}_k \\ & \quad + \tilde{F}^{t,t}\tilde{Y}^2 + 2\tilde{F}^{t,x_k}\tilde{Y}\tilde{Z}_k + \tilde{F}^{x_i,x_j}\tilde{Z}_i\tilde{Z}_j \\ & \leq 0. \end{aligned} \tag{4.2}$$

At (A,t,x),

$$\begin{aligned} \tilde{F}^{\alpha\beta} &= t^{-3}F^{\alpha\beta}, \quad \tilde{F}^{\alpha\beta,rs} = t^{-6}F^{\alpha\beta,rs}, \quad \tilde{F}^{\alpha\beta,x_k} = t^{-3}F^{\alpha\beta,x_k}, \quad \tilde{F}^{x_i,x_j} = F^{x_i,x_j}, \\ \tilde{F}^{\alpha\beta,t} &= -3t^{-4}F^{\alpha\beta} - 3t^{-7}F^{\alpha\beta,rs}A_{rs} - t^{-5}F^{\alpha\beta,u_\ell}\theta_\ell, \\ \tilde{F}^{t} &= -3t^{-4}F^{\alpha\beta}A_{\alpha\beta} - t^{-2}F^{u_\ell}\theta_\ell, \\ \tilde{F}^{t,x_k} &= -3t^{-4}F^{\alpha\beta,x_k}A_{\alpha\beta} - t^{-2}F^{u_\ell,x_k}\theta_\ell, \\ \tilde{F}^{t,t} &= \frac{12F^{\alpha\beta}A_{\alpha\beta}}{t^5} + \frac{9F^{\alpha\beta,rs}A_{\alpha\beta}A_{rs}}{t^8} + \frac{6F^{\alpha\beta,u_\ell}A_{\alpha\beta}\theta_\ell}{t^6} + \frac{2F^{u_\ell}\theta_\ell}{t^3} + \frac{F^{u_\ell,u_s}\theta_s\theta_\ell}{t^4}. \end{aligned}$$

Equation (4.2) is equivalent to

$$
\begin{aligned}
(4.3)\quad & t^{-6}F^{\alpha\beta,rs}\tilde{X}_{\alpha\beta}\tilde{X}_{rs} - 2\Big[3t^{-4}F^{\alpha\beta} + 3t^{-7}F^{\alpha\beta,rs}A_{rs} + t^{-5}F^{\alpha\beta,u_\ell}\theta_\ell\Big]\tilde{X}_{\alpha\beta}\tilde{Y} \\
&\quad + 2t^{-3}F^{\alpha\beta,x_k}\tilde{X}_{\alpha\beta}\tilde{Z}_k - 2\Big[3t^{-4}F^{\alpha\beta,x_k}A_{\alpha\beta} + t^{-2}F^{u_\ell,x_k}\theta_\ell\Big]\tilde{Y}\tilde{Z}_k \\
&\quad + F^{x_i,x_j}\tilde{Z}_i\tilde{Z}_j + \Big[12t^{-5}F^{\alpha\beta}A_{\alpha\beta} + 9t^{-8}F^{\alpha\beta,rs}A_{\alpha\beta}A_{rs} \\
&\qquad + 6t^{-6}F^{\alpha\beta,u_\ell}A_{\alpha\beta}\theta_\ell + 2t^{-3}F^{u_\ell}\theta_\ell \\
&\qquad + t^{-4}F^{u_\ell,u_s}\theta_\ell\theta_s\Big]\tilde{Y}^2 \\
&\le 0.
\end{aligned}
$$

The left side of (4.3) can be written as

$$
\begin{aligned}
(4.4)\quad & \\
& t^{-8}F^{\alpha\beta,rs}\left(\tilde{X}_{\alpha\beta}\tilde{X}_{rs}t^2 - 6t\tilde{X}_{\alpha\beta}A_{rs}\tilde{Y} + 9A_{\alpha\beta}A_{rs}\tilde{Y}^2\right) \\
&\quad - 2t^{-6}F^{\alpha\beta,u_\ell}\theta_\ell\left[t\tilde{X}_{\alpha\beta} - 3A_{\alpha\beta}\tilde{Y}\right]\tilde{Y} + 2t^{-4}F^{\alpha\beta,x_k}\left[t\tilde{X}_{\alpha\beta} - 3A_{\alpha\beta}\tilde{Y}\right]\tilde{Z}_k \\
&\quad + t^{-4}F^{u_\ell,u_s}\theta_\ell\theta_s\tilde{Y}^2 - 2t^{-2}F^{u_\ell,x_k}\theta_\ell\tilde{Y}\tilde{Z}_k + F^{x_i,x_j}\tilde{Z}_i\tilde{Z}_j \\
&\quad + 2t^{-3}F^{u_\ell}\theta_\ell\tilde{Y}^2 - 6t^{-5}F^{\alpha\beta}\left[t\tilde{X}_{\alpha\beta} - 3A_{\alpha\beta}\tilde{Y}\right]\tilde{Y} - 6t^{-5}F^{\alpha\beta}A_{\alpha\beta}\tilde{Y}^2 \\
&= F^{\alpha\beta,rs}X_{\alpha\beta}X_{rs} + 2F^{\alpha\beta,u_\ell}\theta_\ell X_{\alpha\beta}Y + 2F^{\alpha\beta,x_k}X_{\alpha\beta}Z_k \\
&\quad + F^{u_\ell,u_s}\theta_\ell\theta_s Y^2 + 2F^{u_\ell,x_k}\theta_\ell YZ_k + F^{x_i,x_j}Z_iZ_j \\
&\quad + 2tF^{u_\ell}\theta_\ell Y^2 + 6tF^{\alpha\beta}X_{\alpha\beta}Y - 6t^{-1}F^{\alpha\beta}A_{\alpha\beta}Y^2
\end{aligned}
$$

where $X_{\alpha\beta} = t^{-4}[t\tilde{X}_{\alpha\beta} - 3A_{\alpha\beta}\tilde{Y}]$, $Y = -t^{-2}\tilde{Y}$, and $Z_i = \tilde{Z}_i$. Equation (4.1) follows from (4.3) and (4.4). ◻

Theorem 1.1 is a direct consequence of the following proposition and the strong maximum principle.

Proposition 4.2. *Suppose F, u are satisfying assumptions in Theorem 1.1. If $\ell = \ell(x_0)$ (ℓ defined in (3.1))for some point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}$ of x_0 and a positive constant C independent of φ (defined in (3.5)), such that*

$$
(4.5)\qquad \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\varphi_{\alpha\beta}(x) \le C(\varphi(x) + |\nabla\varphi(x)|), \quad \forall\, x \in \mathcal{O}.
$$

Proof. Let $u \in C^{3,1}(\Omega)$ be a solution of equation (1.2) and $(u_{ij}) \in \mathcal{S}^n$. Suppose $\ell(x_0) = \ell$ for some $x_0 \in \Omega$. We work on a small open neighborhood $\mathcal{O}$ of x_0. We may assume $\ell \le n-2$. Lemma 3.5 implies $\varphi \in C^{1,1}(\mathcal{O})$, $\varphi(x) \ge 0$, $\varphi(x_0) = 0$. For $\varepsilon > 0$ sufficient small, let φ_ε defined as in (3.5) and (3.6). For each fixed x, choose a local coordinate chart $e_1, \ldots, e_{n-1}, e_n$ so that (3.2) and

(3.3) are satisfied. We want to establish differential inequality (4.5) for φ_ε defined in (3.6) with constant C independent of ε. In what follows, we will omit the subindex ε with the understanding that all the estimates are independent of ε.

By Lemma 3.1 and Lemma 3.3

(4.6)

$$\begin{aligned}
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\varphi_{\alpha\beta} &= \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}(p_{\alpha\beta}+q_{\alpha\beta})\\
&\le -u_n^{-3}\sum_{j\in B}\left[\sigma_\ell(G)+\frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right]\\
&\quad\times\left[\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_n^2u_{jj\alpha\beta}-6u_n\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{jn}u_{j\alpha\beta}+6u_{jn}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}\right]\\
&\quad-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}\Big[\sigma_1(B)a_{ii,\alpha}-a_{ii}\sum_{j\in B}a_{jj,\alpha}\Big]\\
&\quad\times\Big[\sigma_1(B)a_{ii,\beta}-a_{ii}\sum_{j\in B}a_{jj,\beta}\Big]\\
&\quad-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{\substack{i\ne j,\\ i,j\in B}} F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta}+O(\mathcal{H}_\varphi).
\end{aligned}$$

For each $j\in B$, differentiating equation (1.2) in e_j direction at x,

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j}+F^{u_n}u_{jn}+F^{u_j}u_{jj}+F^{x_j}=0, \tag{4.7}$$

$$\begin{aligned}
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} &= -\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj}\\
&\quad-2\sum_{\alpha,\beta,\ell=1}^{n} F^{\alpha\beta,u_\ell}u_{\alpha\beta j}u_{\ell j}-2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u}u_{j\alpha\beta}u_j\\
&\quad-2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{\alpha\beta j}-\sum_{\ell,s=1}^{n} F^{u_\ell,u_s}u_{\ell j}u_{sj}\\
&\quad-2\sum_{\ell=1}^{n} F^{u_\ell,u}u_{\ell j}u_j-\sum_{\ell=1}^{n} F^{u_\ell,x_j}u_{\ell j}-F^{u,u}u_j^2\\
&\quad-2F^{u,x_j}u_j-F^{x_j,x_j}-\sum_{\ell=1}^{n} F^{u_\ell}u_{\ell jj}-F^{u}u_{jj}.
\end{aligned} \tag{4.8}$$

It follows from (3.13) that, at x

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} = -\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} - 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{nj} - 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{\alpha\beta j} - F^{u_n,u_n}u_{jn}^2 - 2F^{u_n,x_j}u_{jn} - F^{x_j,x_j} - 2\frac{F^{u_n}}{u_n}u_{jn}^2 + O(\mathcal{H}_\varphi). \tag{4.9}$$

Since $u_{\alpha\beta jj} = u_{jj\alpha\beta}$, (4.6) and (4.9) yield

(4.10)

$$\begin{aligned} F^{\alpha\beta}\varphi_{\alpha\beta} = & \sum_{j\in B} u_n^{-3}\left[\sigma_\ell(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] \\ & \times \Bigg\{\Bigg[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{j\alpha\beta} \\ & \quad + F^{u_n,u_n}u_{jn}^2 + 2F^{u_n,x_j}u_{jn} + F^{x_j,x_j} + 2\frac{F^{u_n}}{u_n}u_{jn}^2\Bigg]u_n^2 \\ & \quad + 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta}u_{jn}u_n - 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_{jn}^2\Bigg\} \\ & - \frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}\Big[\sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B}a_{jj,\alpha}\Big]\Big[\sigma_1(B)a_{ii,\beta} - a_{ii}\sum_{j\in B}a_{jj,\beta}\Big] \\ & - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{\substack{i\neq j,\\ i,j\in B}} F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + O(\mathcal{H}_\varphi). \end{aligned}$$

For each $j \in B$, set

$$\begin{aligned} S_j = \Bigg[& \sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{j\alpha\beta}u_{rsj} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} \\ & + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,x_j}u_{j\alpha\beta} + F^{u_n,u_n}u_{jn}^2 \\ & + 2F^{u_n,x_j}u_{jn} + F^{x_j,x_j} + 2\frac{F^{u_n}}{u_n}u_{jn}^2\Bigg]u_n^2 \\ & + 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta}u_{jn}u_n - 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_{jn}^2 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
X_{nn} &= u_{nnj}u_n + \frac{F^{u_j}}{F^{nn}}u_{jj}u_n;\\
X_{\alpha\beta} &= u_{\alpha\beta j}u_n, \quad \forall\,(\alpha,\beta)\neq(n,n);\\
Y &= u_{jn}u_n, \quad \text{and} \quad Z_i = \delta_{ij}u_n.
\end{aligned}$$

In the coordinate system (3.2),

$$\begin{aligned}
(D^2u(x), Du(x), u(x), x) &= (D^2u, (0,\dots,0,|Du(x)|), u, x)\\
&= (t^{-3}A, t^{-1}\theta, u, x).
\end{aligned}$$

Accordingly, the components of $\tilde{V}$ defined in Lemma 4.1 are

$$\begin{aligned}
\tilde{X}_{nn} &= \frac{u_{nnj}}{u_n^2} - \frac{3u_{nn}u_{jn}}{u_n^3} + \frac{F^{u_j}u_{jj}}{F^{nn}u_n^2};\\
\tilde{X}_{\alpha\beta} &= \frac{u_{\alpha\beta j}}{u_n^2} - \frac{3u_{\alpha\beta}u_{jn}}{u_n^3}, \quad \forall\,(\alpha,\beta)\neq(n,n);\\
\tilde{Y} &= -\frac{u_{jn}}{u_n}, \quad \tilde{Z}_i = \delta_{ij}u_n.
\end{aligned}$$

At $(t^{-3}A, t^{-1}\theta, u, x)$,

$$\nabla_{(A,t,x)}F = \Big((F^{\alpha\beta}u_n^3), -3\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta}u_n - F^{u_n}u_n^2, (F^{x_i})\Big).$$

By (4.7),

$$\begin{aligned}
\frac{\langle\tilde{V}, \nabla_{(A,t,x)}F\rangle}{u_n} &= u_n^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}\left(\frac{u_{\alpha\beta j}}{u_n^2} - \frac{3u_{\alpha\beta}u_{jn}}{u_n^3}\right) + F^{u_j}u_{jj}\\
&\quad + \frac{u_{jn}}{u_n}\Big(3\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta} + F^{u_n}u_n\Big) + F^{x_j}\\
&= 0.
\end{aligned}$$

That is $\tilde{V} \in \mathcal{T}\Gamma_F$. It follows from Lemma 4.1 and the fact $u_{jj} = O(\varphi)$ for $j \in B$,

$$S_j \leq C(\varphi). \tag{4.12}$$

Condition (1.4) implies

$$(F^{\alpha\beta}) \geq \delta_0 I, \quad \text{for some } \delta_0 > 0, \text{ and } \forall\, x \in \mathcal{O}. \tag{4.13}$$

Set

$$V_{i\alpha} = \sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B} a_{jj,\alpha}.$$

Combine (4.13), (4.12) and (4.10),

(4.14)

$$F^{\alpha\beta}\varphi_{\alpha\beta} \le C\Big(\varphi + \sum_{i,j\in B}|\nabla a_{ij}|\Big) - \delta_0\left[\frac{\sum_{i\neq j\in B,\,\alpha=1}^n a_{ij\alpha}^2}{\sigma_1(B)} + \frac{\sum_{i\in B,\,\alpha=1}^n V_{i\alpha}^2}{\sigma_1^3(B)}\right].$$

By Lemma 3.3 in [1], for each $M \ge 1$, for any $M \ge |\gamma_i| \ge 1/M$, there is a constant C depending only on n and M such that, $\forall\ \alpha$,

$$\begin{aligned}\sum_{i,j\in B}|a_{ij\alpha}| \le{}& C\left(1+\frac{1}{\delta_0^2}\right)\Big(\sigma_1(B) + \Big|\sum_{i\in B}\gamma_i a_{ii\alpha}\Big|\Big)\\ &+\frac{\delta_0}{2}\left[\frac{\sum_{i\neq j\in B}|a_{ij\alpha}|^2}{\sigma_1(B)} + \frac{\sum_{i\in B}V_{i\alpha}^2}{\sigma_1^3(B)}\right].\end{aligned} \tag{4.15}$$

Set

$$\gamma_j = \sigma_\ell(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}, \quad \forall\ j \in B,$$

the Newton-MacLaurine inequality implies

$$\sigma_\ell(G) + 1 \ge \gamma_j \ge \sigma_\ell(G), \quad \forall\ j \in B.$$

We conclude from Lemma 3.1, Lemma 3.3 and (4.15) that $\sum_{i,j\in B}|\nabla a_{ij}|$ is controlled by the rest terms on the right hand side in (4.14) together with $\varphi + |\nabla\varphi|$. The proof is complete. ∎

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