PRINCIPAL CURVATURE ESTIMATES FOR THE LEVEL SETS OF HARMONIC FUNCTIONS AND MINIMAL GRAPHS IN $\mathbb{R}^3$

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Abstract. We give a sharp lower bound for the principal curvature of the
level sets of harmonic functions and minimal graphs defined on convex rings in
$\mathbb{R}^3$ with homogeneous Dirichlet boundary conditions.

1. **Introduction.** The convexity of the level sets of the solutions of elliptic partial
differential equations has been studied for a long time. For instance, Ahlfors [1] contains the well-known result that level curves of Green function on simply connected
convex domain in the plane are the convex Jordan curves. In 1931, Gergen [10]
proved the star-shapeness of the level sets of Green function on 3-dimensional star-
shaped domain. For the minimal annulus whose boundary consists of two closed
convex curves in parallel planes P_1 and P_2, in 1956 Shiffman [22] proved that the
intersection of the surface with any plane P, between P_1 and P_2, is a convex Jordan
curve. For elliptic partial differential equations on domains in $\mathbb{R}^n$, the convexity of
level set was first considered by Gabriel [9] in 1957. He proved that the level sets
of the Green function on a 3-dimensional convex domain are strictly convex. Later,
in 1977, Lewis [15] extended Gabriel's result to p-harmonic functions in higher dimensions and obtained the following theorem.

Theorem 1.1 (Gabriel [9], Lewis [15]). *Let u satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & in & \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on & \partial\Omega_0, \\ u = 1 & on & \partial\Omega_1, \end{cases} \tag{1.1}$$

*where $1 < p < +\infty$, Ω_0 and Ω_1 are bounded convex domains in $\mathbb{R}^n, n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$.
Then all the level sets of u are strictly convex.*

For the minimal graphs, Korevaar (see Remark 13 in [14]) proved the following
result.

Theorem 1.2 (Korevaar [14]). *Let u satisfy*

$$\begin{cases} \text{div}(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0 & in \quad \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on \quad \partial \Omega_0, \\ u = 1 & on \quad \partial \Omega_1, \end{cases} \qquad (1.2)$$

where Ω_0 and Ω_1 are bounded convex domains in $\mathbb{R}^n, n \geq 2, \bar{\Omega}_1 \subset \Omega_0$. Then all the level sets of u are strictly convex.

In 1982, Caffarelli-Spruck [6] generalized the Lewis [15] results to a class of semi-linear elliptic partial differential equations. A good survey of this subject is given by Kawohl [13]. For more recent related extensions, please see the papers by Bianchini-Longinetti-Salani [3] and Bian-Guan-Ma-Xu [2].

Now we turn to the curvature estimates of the level sets of the solutions of elliptic partial differential equations. For 2-dimensional harmonic functions and minimal surfaces with convex level curves, Ortel-Schneider [20], Longinetti [17] and [18] proved that the curvature of the level curves attains its minimum on the boundary (see also Talenti [23] for related results). Jost-Ma-Ou [12] proved that the Gaussian curvature of the convex level sets of 3-dimensional harmonic function attains its minimum on the boundary. For the other related results and their application to free boundary problem, please see the papers by Rosay-Rudin [21], Dolbeault-Monneau [8].

In this paper, using the strong maximum principle, we obtain a sharp principal curvature estimates for the level set of lower dimensional p-harmonic functions and minimal graphs defined on convex ring. Our theorems are the principal curvature counterpart of the Gaussian curvature estimates in [12].

Now we state our theorems.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $3 \leq n \leq 5$, $1 < p < +\infty$ and u be a p-harmonic function in Ω, i.e.*

$$\text{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad in \quad \Omega. \qquad (1.3)$$

Assume $|\nabla u| > 0$ in Ω. If $\frac{n+1}{2} \leq p \leq 3$ and the level sets of u are strictly convex with respect to normal ∇u, then the smallest principal curvature of the level sets of u cannot attain its minimum in Ω, unless it is constant.

Using Theorem 1.1 and Theorem 1.3, we have the following corollary.

Corollary 1.4. *Let u be the solution of the following boundary value problem*

$$\begin{cases} \text{div}(|\nabla u|^{p-2} \nabla u) = 0 & in \quad \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on \quad \partial \Omega_0, \\ u = 1 & on \quad \partial \Omega_1, \end{cases} \qquad (1.4)$$

where Ω_0 and Ω_1 are smooth bounded convex domains in $\mathbb{R}^3$, $\bar{\Omega}_1 \subset \Omega_0$. If $3 \leq n \leq 5$ and $\frac{n+1}{2} \leq p \leq 3$, then the principal curvature of the level sets of u attains its minimum on $\partial \Omega$.

Now we turn on the 3-dimensional minimal graphs.

Theorem 1.5. *Let Ω be a bounded domain in $\mathbb{R}^3$ and u be a minimal graph over Ω, i.e., u satisfy the minimal surface equation*

$$\text{div}(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0 \quad in \quad \Omega. \qquad (1.5)$$

Assume $|\nabla u| > 0$ in Ω. If the level sets of u are strictly convex with respect to normal ∇u, then the smallest principal curvature of the level sets of u cannot attain its minimum in Ω, unless it is constant.

Similarly, we have the following corollary.

Corollary 1.6. *Let u satisfy*

$$\begin{cases} \text{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & in \quad \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on \quad \partial\Omega_0, \\ u = 1 & on \quad \partial\Omega_1, \end{cases} \tag{1.6}$$

where Ω_0 and Ω_1 are smooth bounded convex domains in $\mathbb{R}^3$, $\bar{\Omega}_1 \subset \Omega_0$. Then the principal curvature of the level sets of u attains its minimum on $\partial\Omega$.

Let (a_{ij}) be the symmetry curvature matrix on the strictly convex level sets defined in (2.4), and let (a^{ij}) be its inverse matrix. We consider the auxiliary function

$$\varphi(x,\xi) = a^{ij}(x)\xi_i\xi_j, \quad where \quad \xi = (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}, \quad |\xi| = 1.$$

We shall derive the following differential inequality

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\varphi_{\alpha\beta} \geq 0 \qquad \text{mod} \qquad \nabla\varphi \quad \text{in} \quad \Omega, \tag{1.7}$$

where

$$F^{\alpha\beta}(\nabla u) = |\nabla u|^2\delta_{\alpha\beta} + (p-2)u_\alpha u_\beta,$$

or

$$F^{\alpha\beta}(\nabla u) = (1+|\nabla u|^2)\delta_{\alpha\beta} - u_\alpha u_\beta,$$

is the associated elliptic operator in (1.1) or (1.2). In (1.7), we have suppressed the terms containing the gradient of $\nabla\varphi$ with locally bounded coefficients, then we apply the strong maximum principle to obtain the results.

Now let us mention that three dimensional harmonic function always has very special properties. The famous theorem of Lewy [16] states that if u is a harmonic function on a domain in $\mathbb{R}^3$ and the map $x \longrightarrow \nabla u(x)$ is a homeomorphism, then $x \longrightarrow \nabla u(x)$ is a diffeomorphism. In 1991, Gleason-Wolff [11] extended this results to higher dimensions, but needed some extra conditions, and gave a counterexample in the higher dimensional case without these additional conditions.

In section 2, we first give brief definition on the convexity of the level sets, then obtain the curvature matrix a_{ij} of the level sets of a function, which appeared in [2]. The main technique in the proof of theorems consists in rearranging the second and third derivatives terms using the equation and the first derivatives condition for φ. In the 3-dimensional case, we get "good" sign for the second and third derivatives terms, which allows us to reach our conclusions.

2. **The curvature formulas of level sets.** In this section, we shall give the brief definition on the convexity of the level sets, then introduce the curvature matrix (a_{ij}) of the level sets of a function, which appeared in [2]. Firstly, we recall some fundamental notations in classical surface theory. Assume a surface $\Sigma \subset \mathbb{R}^n$ is given by the graph of a function v in a domain in $\mathbb{R}^{n-1}$:

$$x_n = v(x'), x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

Definition 2.1. We define the graph of function $x_n = v(x')$ is convex with respect to the upward normal $\vec{\nu} = \frac{1}{W}(-v_1, -v_2, \cdots, -v_{n-1}, 1)$ if the second fundamental form $b_{ij} = \frac{v_{ij}}{W}$ of the graph $x_n = v(x')$ is nonnegative definite, where $W = \sqrt{1 + |\nabla v|^2}$.

The principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of the graph of v, being the eigenvalues of the second fundamental form relative to the first fundamental form. We have the following well-known formula.

Lemma 2.2 ([5]). *The principal curvature of the graph $x_n = v(x')$ with respect to the upward normal $\vec{\nu}$ are the eigenvalues of the symmetric curvature matrix*

$$a_{il} = \frac{1}{W}\left\{ v_{il} - \frac{v_i v_j v_{jl}}{W(1+W)} - \frac{v_l v_k v_{ki}}{W(1+W)} + \frac{v_i v_l v_j v_k v_{jk}}{W^2(1+W)^2} \right\}, \qquad (2.1)$$

where the summation convention over repeated indices is employed.

Now we give the definition of the convex level sets of the function u. Let Ω be a domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$, its level sets can be usually defined in the following sense.

Definition 2.3. Assume $|\nabla u| > 0$ in Ω, we define the level set of u passing through the point $x_o \in \Omega$ as $\Sigma^{u(x_o)} = \{x \in \Omega | u(x) = u(x_o)\}$.

Now we shall locally work near the point x_o where $|\nabla u(x_o)| > 0$. Without loss of generality, we assume $u_n(x_o) \neq 0$. By implicit function theorem, locally the level set $\Sigma^{u(x_o)}$ could be represented as a graph

$$x_n = v(x'), x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1},$$

and $v(x')$ satisfies the following equation

$$u(x_1, x_2, \cdots, x_{n-1}, v(x_1, x_2, \cdots, x_{n-1})) = u(x_o).$$

Then the first fundamental form of the level set is $g_{ij} = \delta_{ij} + \frac{u_i u_j}{u_n^2}$, and $W = (1 + |\nabla v|^2)^{\frac{1}{2}} = \frac{|\nabla u|}{|u_n|}$. The upward normal direction of the level set is

$$\vec{\nu} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, \cdots, u_{n-1}, u_n). \qquad (2.2)$$

Let

$$h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \qquad (2.3)$$

then the second fundamental form of the level set of function u is

$$b_{ij} = \frac{v_{ij}}{W} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}.$$

Definition 2.4. For the function $u \in C^2(\Omega)$ we assume $|\nabla u| > 0$ in Ω. Without loss of generality we can let $u_n(x_o) \neq 0$ for $x_o \in \Omega$. We define locally the level set $\Sigma^{u(x_o)} = \{x \in \Omega | u(x) = u(x_o)\}$ is convex with respect to the upward normal direction $\vec{\nu}$ if the second fundamental form b_{ij} is nonnegative definite.

Remark 2.5. If we let ∇u be the upward normal of the level set $\Sigma^{u(x_o)}$ at x_o, then $u_n(x_o) > 0$ by (2.2). And from the definition 2.4, if the level set $\Sigma^{u(x_o)}$ is convex with respect to the normal direction ∇u, then the matrix $(h_{ij}(x_o))$ is nonpositive definite.

Now we obtain the representation of the curvature matrix (a_{ij}) of the level sets of the function u with the derivative of the function u,

$$a_{ij} = \frac{1}{|\nabla u|u_n^2}\left\{ -h_{ij} + \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} + \frac{u_j u_l h_{il}}{W(1+W)u_n^2} \right.$$
$$\left. - \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4} \right\}. \tag{2.4}$$

From now on we denote

$$B_{ij} = \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} + \frac{u_j u_l h_{il}}{W(1+W)u_n^2}, \quad C_{ij} = \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}, \tag{2.5}$$

and

$$A_{ij} = -h_{ij} + B_{ij} - C_{ij}, \tag{2.6}$$

then the symmetric curvature matrix of the level sets of u could be represented as

$$a_{ij} = \frac{1}{|\nabla u|u_n^2}\left[-h_{ij} + B_{ij} - C_{ij} \right] = \frac{1}{|\nabla u|u_n^2} A_{ij}. \tag{2.7}$$

3. **Principal curvature estimates of level set of p-harmonic function.** In this section, we prove the Theorem 1.3. We study the following equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad \Omega, \tag{3.1}$$

and we shall prove this theorem using strong minimum principle.

In the following proof, the Greek indices $(\alpha, \beta, \gamma, \delta, ...)$ run from 1 to n, the Roman indices $(i, j, k, l, ...)$ run from 1 to $n-1$.

Denote

$$F^{\alpha\beta}(\nabla u) = |\nabla u|^2 \delta_{\alpha\beta} + (p-2)u_\alpha u_\beta. \tag{3.2}$$

Then equation (3.1) is equivalent to

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} = 0. \tag{3.3}$$

Proof of Theorem 1.3:

Since the level sets of u are strictly convex with respect to normal ∇u, then the curvature matrix (a_{ij}) of the level sets is positive definitive in Ω. Let (a^{ij}) be the inverse matrix of (a_{ij}). We consider the auxiliary function

$$\varphi(x,\xi) = a^{ij}(x)\xi_i\xi_j, \quad \text{where} \quad \xi = (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1}, \quad |\xi| = 1.$$

We shall derive the following differential inequality

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 0 \qquad \text{mod} \qquad \nabla\varphi \quad \text{in} \quad \Omega, \tag{3.4}$$

where we modify the terms of the gradient of φ with locally bounded coefficients. Then by the standard strong maximum principle, we get the result immediately.

In order to prove (3.4) at an arbitrary point $x_o \in \Omega$, as in Caffarelli-Friedman [4], we choose the normal coordinate at x_o. We have mentioned in Remark 2.5, since the level sets of u are strictly convex with respect to normal ∇u, by rotating the coordinate system suitably by T_{x_o}, we may assume that $u_i(x_o) = 0$, $1 \leq i \leq n-1$ and $u_n(x_o) = |\nabla u| > 0$. And we can further assume $\xi = e_1$, the matrix $(u_{ij}(x_o))$ $(1 \leq i, j \leq n-1)$ is diagonal and $u_{ii}(x_o) < 0$. Consequently we can choose T_{x_o} to vary smoothly with x_o. If we can establish (3.4) at x_o under the above assumption,

then go back to the original coordinates we find that (3.4) remain valid with new locally bounded coefficients on $\nabla\varphi$ in (3.4), depending smoothly on the independent variable. Thus it remains to establish (3.4) under the above assumptions.

Now we write $\varphi(x) = a^{11}$. From now on, *all the calculations will be done at the fixed point x_o.*

Taking the first derivative of φ, we get

$$\varphi_\alpha = -\sum_{k,l=1}^{n-1} a^{1k}a^{1l}a_{kl,\alpha}, \tag{3.5}$$

it follows that

$$a_{11,\alpha} = -a_{11}^2\varphi_\alpha. \tag{3.6}$$

Taking derivative of equation (3.5) once more, we have

$$\varphi_{\alpha\beta} = \sum_{k,l,p,q=1}^{n-1} a^{1p}a^{kq}a^{1l}a_{kl,\alpha}a_{pq,\beta} + \sum_{k,l,p,q=1}^{n-1} a^{1k}a^{1p}a^{lq}a_{kl,\alpha}a_{pq,\beta}$$
$$- \sum_{k,l=1}^{n-1} a^{1k}a^{1l}a_{kl,\alpha\beta},$$

therefore

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\varphi_{\alpha\beta} = (a^{11})^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\left(2\sum_{k=1}^{n-1} a^{kk}a_{1k,\alpha}a_{1k,\beta} - a_{11,\alpha\beta}\right). \tag{3.7}$$

In order to get (3.4), we only need to prove

$$u_n^3 a_{11}^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\varphi_{\alpha\beta}$$
$$= u_n^3\left(2\sum_{k=1}^{n-1}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}a^{kk}a_{1k,\alpha}a_{1k,\beta} - \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}a_{11,\alpha\beta}\right) \tag{3.8}$$
$$\geq 0 \quad \mod \quad \nabla\varphi,$$

where we modify the terms of the gradient of φ with locally bounded coefficients. We shall prove (3.8) in two steps.

Step 1: We first calculate the term $u_n^3\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}a_{11,\alpha\beta}$, it will be completed in (3.35).

By (2.7), it follows that

$$A_{11} = |\nabla u|u_n^2 a_{11}. \tag{3.9}$$

Let $E = |\nabla u|u_n^2$. Then at x_o we have

$$E_\alpha = 3u_n^2 u_{n\alpha},$$
$$E_{\alpha\beta} = 5u_n u_{n\alpha}u_{n\beta} + 3u_n^2 u_{\alpha\beta n} + u_n\sum_{\gamma=1}^{n} u_{\alpha\gamma}u_{\beta\gamma}. \tag{3.10}$$

Taking derivative of equation (3.9), we get

$$A_{11,\alpha} = E_\alpha a_{11} + E a_{11,\alpha} = 3u_n^2 u_{n\alpha}a_{11} + u_n^3 a_{11,\alpha}. \tag{3.11}$$

By (2.3), it follows that

$$h_{11,\alpha} = 2u_n u_{n\alpha} u_{11} + u_n^2 u_{11\alpha} + u_{nn\alpha} u_1^2 + 2u_{nn} u_1 u_{1\alpha} \\ - 2u_{n\alpha} u_1 u_{1n} - 2u_n u_{1\alpha} u_{1n} - 2u_n u_1 u_{1n\alpha}. \tag{3.12}$$

Then at x_o, we have

$$A_{11,\alpha} = - h_{11,\alpha} + B_{11,\alpha} - C_{11,\alpha} = -h_{11,\alpha} \\ = - u_n^2 u_{11\alpha} - 2u_n u_{n\alpha} u_{11} + 2u_n u_{1\alpha} u_{1n}.$$

Since at x_o, $a_{11} = -\frac{u_{11}}{u_n}$, by (3.11) we have the following important relation for the third derivative of u,

$$u_n u_{11\alpha} = 2u_{1\alpha} u_{1n} + u_{11} u_{n\alpha} - u_n^2 a_{11,\alpha}. \tag{3.13}$$

Taking derivatives twice of equation (3.9), we have

$$\sum_{\alpha,\beta=1}^{n} u_n^3 F^{\alpha\beta} a_{11,\alpha\beta} \\ = \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} A_{11,\alpha\beta} - \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} E_{\alpha\beta} a_{11} - 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} E_{\alpha} a_{11,\beta} \\ = \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} A_{11,\alpha\beta} - 6u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha} a_{11,\beta} \\ + u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \left(\sum_{\gamma=1}^{n} u_{\gamma\alpha} u_{\gamma\beta} + 5u_{n\alpha} u_{n\beta} + 3u_n u_{n\alpha\beta} \right). \tag{3.14}$$

In the following calculation, we mainly treat the last two lines of (3.14). By (2.6),

$$A_{11,\alpha\beta} = -h_{11,\alpha\beta} + B_{11,\alpha\beta} - C_{11,\alpha\beta}.$$

Taking the first and second derivatives of equation (2.5), we have

$$B_{11,\alpha\beta} = \sum_{l=1}^{n-1} \frac{2u_{1\alpha} u_{l\beta} h_{1l} + 2u_{1\beta} u_{l\alpha} h_{1l}}{W(1+W)u_n^2},$$

hence

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} B_{11,\alpha\beta} = \sum_{l=1}^{n-1} \sum_{\alpha,\beta=1}^{n} \frac{4F^{\alpha\beta} u_{1\alpha} u_{l\beta} h_{1l}}{W(1+W)u_n^2} = 2u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{1\alpha} u_{1\beta}. \tag{3.15}$$

Similarly, we have

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} C_{11,\alpha\beta} = 0. \tag{3.16}$$

By (3.12), it follows that

$$h_{11,\alpha\beta} = 2u_{11} u_{n\beta} u_{n\alpha} + 2u_n u_{n\alpha\beta} u_{11} + 2u_n u_{n\alpha} u_{11\beta} + u_n^2 u_{11\alpha\beta} \\ + 2u_n u_{n\beta} u_{11\alpha} + 2u_{nn} u_{1\beta} u_{1\alpha} - 2u_{n\alpha} u_{1\beta} u_{1n} - 2u_{n\beta} u_{1\alpha} u_{1n} \\ - 2u_n u_{1\alpha\beta} u_{1n} - 2u_n u_{1\alpha} u_{1n\beta} - 2u_n u_{1\beta} u_{1n\alpha}. \tag{3.17}$$

From (3.14)–(3.17), it follows that

$$
\begin{aligned}
u_n^3 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} a_{11,\alpha\beta} = &- u_n^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta 11} + 2u_n u_{n1} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta 1} \\
&+ 4u_n \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1n\beta} - 2u_n u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha\beta} \\
&- 4u_n \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{11\beta} - 2u_{nn} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1\beta} \\
&+ 4u_{1n} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{1\beta} + 2u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1\beta} \\
&- 2u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{n\beta} - 6u_n^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} a_{11,\beta} \\
&+ u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \left(\sum_{\gamma=1}^n u_{\gamma\alpha} u_{\gamma\beta} + 5u_{n\alpha} u_{n\beta} + 3u_n u_{n\alpha\beta} \right) \\
= &\, T_1 + T_2 + T_3 + T_4 - 6u_n^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} a_{11,\beta},
\end{aligned}
\tag{3.18}
$$

where

$$
T_1 = 3u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{n\beta} + u_{11} \sum_{\alpha,\beta,\gamma=1}^n F^{\alpha\beta} u_{\gamma\alpha} u_{\gamma\beta},
$$

$$
T_2 = 4u_n \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1n\beta} - 4u_n \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{11\beta},
$$

$$
\begin{aligned}
T_3 = &\, 2u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1\beta} + 4u_{1n} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{1\beta} \\
&- 2u_{nn} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{1\alpha} u_{1\beta},
\end{aligned}
$$

$$
\begin{aligned}
T_4 = &\, u_n u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha\beta} + 2u_n u_{1n} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta 1} \\
&- u_n^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta 11}.
\end{aligned}
$$

We first treat the term T_1. From (3.2), we have

$$
\begin{aligned}
T_1 &= 3u_{11} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{n\alpha} u_{n\beta} + u_{11} \sum_{\alpha,\beta,\gamma=1}^n F^{\alpha\beta} u_{\gamma\alpha} u_{\gamma\beta} \\
&= u_n^2 u_{11} \left[\sum_{i=1}^{n-1} u_{ii}^2 + (p+3) \sum_{i=1}^{n-1} u_{ni}^2 + 4(p-1) u_{nn}^2 \right].
\end{aligned}
\tag{3.19}
$$

For the term T_2, by (3.13) we have

$$
\begin{aligned}
T_2 &= 4u_n \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{1\alpha} u_{1n\beta} - 4u_n \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha} u_{11\beta} \\
&= 4u_n^2 \left[2u_{11}u_{1n}^2 + u_{11}^2 u_{nn} + (p-1)u_n u_{1n} u_{1nn} \right] - 4u_n^4 u_{11} a_{11,n} \\
&\quad - 4u_n^2 \left[3u_{11}u_{1n}^2 + u_{11} \sum_{i=2}^{n-1} u_{ni}^2 + (p-1)(2u_{1n}^2 u_{nn} + u_{11} u_{nn}^2) \right] \\
&\quad + 4u_n^2 \sum_{\alpha=1}^{n} F^{\alpha\alpha} u_{n\alpha} a_{11,\alpha}.
\end{aligned}
\tag{3.20}
$$

For the term T_3,

$$
\begin{aligned}
T_3 &= 2u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{1\alpha} u_{1\beta} - 2u_{nn} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{1\alpha} u_{1\beta} \\
&\quad + 4u_{1n} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha} u_{1\beta} \\
&= 2u_n^2 u_{11} \left[u_{11}^2 + (p-1)u_{1n}^2 \right] - 2u_n^2 u_{nn} \left[u_{11}^2 + (p-1)u_{1n}^2 \right] \\
&\quad + 4u_n^2 u_{1n} \left[u_{1n} u_{11} + (p-1)u_{1n} u_{nn} \right].
\end{aligned}
\tag{3.21}
$$

It follows that

$$
\begin{aligned}
T_2 + T_3 &= 2u_n^2 u_{11}^3 + 2(p-1)u_n^2 u_{11} u_{1n}^2 + 2u_n^2 u_{11}^2 u_{nn} \\
&\quad - 6(p-1)u_n^2 u_{1n}^2 u_{nn} - 4u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 \\
&\quad - 4(p-1)u_n^2 u_{11} u_{nn}^2 + 4(p-1)u_n^3 u_{1n} u_{nn1} \\
&\quad - 4u_n^4 u_{11} a_{11,n} + 4u_n^2 \sum_{\alpha=1}^{n} F^{\alpha\alpha} u_{n\alpha} a_{11,\alpha}.
\end{aligned}
\tag{3.22}
$$

Now we work on the term T_4, we will first calculate $u_n u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha\beta}$ in T_4. At the fixed point x_o, equation (3.1) becomes

$$
F^{\alpha\beta} u_{\alpha\beta} = u_n^2 \triangle u + (p-2)u_n^2 u_{nn} = 0,
$$

i.e.

$$
(p-1)u_{nn} = -\sum_{i=1}^{n-1} u_{ii}.
\tag{3.23}
$$

By (3.2), we have

$$
(F^{\alpha\beta})_n = 2u_n u_{nn} \delta_{\alpha\beta} + (p-2)u_{n\alpha} u_\beta + (p-2)u_\alpha u_{n\beta}.
$$

Together with (3.23), it follows that

$$
\begin{aligned}
u_n u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha\beta} &= -u_n u_{11} \sum_{\alpha,\beta=1}^{n} (F^{\alpha\beta})_n u_{\alpha\beta} \\
&= -u_n u_{11} \sum_{\alpha,\beta=1}^{n} \left[2u_n u_{nn} \delta_{\alpha\beta} + (p-2)u_{n\alpha} u_\beta + (p-2)u_\alpha u_{n\beta} \right] u_{\alpha\beta} \\
&= -2(p-2)u_n^2 u_{11} u_{1n}^2 - 2(p-2)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{in}^2.
\end{aligned}
\tag{3.24}
$$

For the term $2u_n u_{1n} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 1}$ in T_4, by (3.13) we have

$$
\begin{aligned}
2u_n u_{1n} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 1} &= 2u_n u_{1n} \sum_{\alpha,\beta=1}^{n} \left[|\nabla u|^2 \delta_{\alpha\beta} + (p-2)u_\alpha u_\beta \right] u_{\alpha\beta 1} \\
&= 2u_n u_{1n} \left[u_n^2 u_{111} + u_n^2 \sum_{i=2}^{n-1} u_{ii1} + (p-1)u_n^2 u_{1nn} \right] \\
&= 6u_n^2 u_{11} u_{1n}^2 + 2u_n^3 u_{1n} \sum_{i=2}^{n-1} u_{ii1} \\
&\quad + 2(p-1)u_n^3 u_{1n} u_{nn1} - 2u_n^4 u_{1n} a_{11,1}.
\end{aligned}
\tag{3.25}
$$

For the term $-u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 11}$ in T_4. By (3.2), we have

$$
(F^{\alpha\beta})_1 = 2u_n u_{n1} \delta_{\alpha\beta} + (p-2)u_{1\alpha} u_\beta + (p-2)u_\alpha u_{1\beta}
\tag{3.26}
$$

and

$$
\begin{aligned}
(F^{\alpha\beta})_{11} = 2\sum_{\gamma=1}^{n} u_{1\gamma}^2 \delta_{\alpha\beta} + 2u_n u_{11n} \delta_{\alpha\beta} + (p-2)u_{11\alpha} u_\beta \\
+ (p-2)u_\alpha u_{11\beta} + 2(p-2)u_{1\alpha} u_{1\beta}.
\end{aligned}
\tag{3.27}
$$

Taking derivatives of equation (3.3) twice, we can get

$$
\sum_{\alpha,\beta=1}^{n} \left[(F^{\alpha\beta})_1 u_{\alpha\beta} + F^{\alpha\beta} u_{\alpha\beta 1} \right] = 0,
$$

$$
\sum_{\alpha,\beta=1}^{n} \left[(F^{\alpha\beta})_{11} u_{\alpha\beta} + 2(F^{\alpha\beta})_1 u_{\alpha\beta 1} + F^{\alpha\beta} u_{\alpha\beta 11} \right] = 0.
$$

Then we have

$$
-u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 11} = u_n^2 \sum_{\alpha,\beta=1}^{n} (F^{\alpha\beta})_{11} u_{\alpha\beta} + 2u_n^2 \sum_{\alpha,\beta=1}^{n} (F^{\alpha\beta})_1 u_{\alpha\beta 1}.
\tag{3.28}
$$

By (3.13) and (3.26), it follows that

$$
\begin{aligned}
2u_n^2 \sum_{\alpha,\beta=1}^{n} (F^{\alpha\beta})_1 u_{\alpha\beta 1} = {}& (8p-4)u_n^2 u_{11} u_{n1}^2 + 4(p-2)u_n^2 u_{11}^2 u_{nn} \\
& + 4u_n^3 u_{n1} \sum_{i=2}^{n-1} u_{ii1} + 4(p-1)u_n^3 u_{n1} u_{nn1} \\
& - 4u_n^4 u_{n1} a_{11,1} - 4(p-2)u_n^4 u_{11} a_{11,n}.
\end{aligned}
\tag{3.29}
$$

By (3.13), (3.23) and (3.27), we obtain

$$
\begin{aligned}
u_n^2 \sum_{\alpha,\beta=1}^{n} (F^{\alpha\beta})_{11} u_{\alpha\beta} = {}& 2(p-1)u_n^2 u_{11}^3 + 2u_n^2 u_{11}^2 u_{nn} + 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} \\
& + 10(p-2)u_n^2 u_{11} u_{n1}^2 + 2(p-2)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 \\
& - 2(p-2)u_n^4 \sum_{i=1}^{n-1} u_{ni} a_{11,i}.
\end{aligned}
\tag{3.30}
$$

Combining (3.28)–(3.30), we get

$$
\begin{aligned}
-u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 11} = {}& 2(p-1)u_n^2 u_{11}^3 + (4p-6)u_n^2 u_{11}^2 u_{nn} \\
& + 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} + (18p-24)u_n^2 u_{11} u_{n1}^2 \\
& + 2(p-2)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 + 4u_n^3 u_{n1} \sum_{i=2}^{n-1} u_{ii1} \\
& + 4(p-1)u_n^3 u_{n1} u_{nn1} - 4u_n^4 u_{n1} a_{11,1} \\
& - 4(p-2)u_n^4 u_{11} a_{11,n} - 2(p-2)u_n^4 \sum_{i=1}^{n-1} u_{ni} a_{11,i}.
\end{aligned}
\tag{3.31}
$$

By (3.24), (3.25) and (3.31), we obtain

$$
\begin{aligned}
T_4 = {}& u_n u_{11} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{n\alpha\beta} + 2u_n u_{1n} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 1} - u_n^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta 11} \\
= {}& 2(p-1)u_n^2 u_{11}^3 + (4p-6)u_n^2 u_{11}^2 u_{nn} + 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} \\
& + (16p-14)u_n^2 u_{11} u_{1n}^2 + 6u_n^3 u_{1n} \sum_{i=2}^{n-1} u_{ii1} + 6(p-1)u_n^3 u_{1n} u_{nn1} \\
& - 4(p-2)u_n^4 u_{11} a_{11,n} - 6u_n^4 u_{1n} a_{11,1} - 2(p-2)u_n^4 \sum_{i=1}^{n-1} u_{ni} a_{11,i}.
\end{aligned}
\tag{3.32}
$$

Noticed that (3.23), by (3.19), (3.22) and (3.32), we have

$$
u_n^3 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} a_{11,\alpha\beta} = (2p-3)u_n^2 u_{11}^3 - 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} + (19p-7)u_n^2 u_{11} u_{1n}^2
$$

$$
+ (p-1)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 + u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ii}^2 + 6u_n^2 u_{1n}^2 \sum_{i=2}^{n-1} u_{ii}
$$

$$
+ 10(p-1)u_n^3 u_{1n} u_{nn1} + 6u_n^3 u_{1n} \sum_{i=2}^{n-1} u_{ii1} + G(\nabla a_{11}),
$$

$$(3.33)$$

where

$$
G(\nabla a_{11}) = -2(p-2)u_n^4 \sum_{i=1}^{n-1} u_{ni} a_{11,i} - 4(p-1)u_n^4 u_{11} a_{11,n}
$$

$$
- 6u_n^4 u_{1n} a_{11,1} - 2u_n^2 \sum_{\alpha=1}^n F^{\alpha\alpha} u_{n\alpha} a_{11,\alpha}.
$$

Let us use the equation (3.1) to substitute the term u_{nn1} in (3.33). We take derivative of (3.1) with respect to x_1 to get

$$
(p-1)u_n^2 u_{nn1} = -u_n^2 \sum_{i=1}^{n-1} u_{ii1} - 2u_n u_{n1} \triangle u - 2(p-2)u_n \sum_{\alpha=1}^n u_{1\alpha} u_{n\alpha},
$$

by (3.13), it follows that

$$
(p-1)u_n^2 u_{nn1} = -u_n^2 \sum_{i=2}^{n-1} u_{ii1} + (1-2p)u_n u_{11} u_{1n} + u_n^3 a_{11,1}. \tag{3.34}
$$

Hence, by (3.33)–(3.34) we have

$$
u_n^3 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} a_{11,\alpha\beta} = (2p-3)u_n^2 u_{11}^3 - 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} + (3-p)u_n^2 u_{11} u_{1n}^2
$$

$$
+ (p-1)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 + u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ii}^2 \tag{3.35}
$$

$$
+ 6u_n^2 u_{1n}^2 \sum_{i=2}^{n-1} u_{ii} - 4u_n^3 u_{1n} \sum_{i=2}^{n-1} u_{ii1} + H(\nabla a_{11}),
$$

where

$$
H(\nabla a_{11}) = -2(p-2)u_n^4 \sum_{i=1}^{n-1} u_{ni} a_{11,i} - 4(p-1)u_n^4 u_{11} a_{11,n}
$$

$$
+ 4u_n^4 u_{1n} a_{11,1} - 2u_n^2 \sum_{\alpha=1}^n F^{\alpha\alpha} u_{n\alpha} a_{11,\alpha}.
$$

$$(3.36)$$

Step 2: The end of the theorem.
Now we calculate the following term in (3.8)

$$
2u_n^3 \sum_{k=1}^{n-1} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} a^{kk} a_{1k,\alpha} a_{1k,\beta}.
$$

By (2.3) and (2.6), we get for $2 \leq i \leq n-1$,

$$u_n^2 a_{1i,i} = -u_n u_{ii1} + u_{1n} u_{ii}. \tag{3.37}$$

Since $a_{kk} = -\frac{u_{kk}}{u_n} > 0$, by (3.37)

$$2u_n^3 \sum_{k=1}^{n-1} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a^{kk} a_{1k,\alpha} a_{1k,\beta} \geq 2u_n^3 \sum_{i=2}^{n-1} F^{ii} a^{ii} a_{1i,i}^2$$

$$= 2 \sum_{i=2}^{n-1} \frac{1}{-u_{ii}} u_n^2 (-u_n u_{ii1} + u_{1n} u_{ii})^2. \tag{3.38}$$

By (3.35) and (3.38), we obtain

$$u_n^3 a_{11}^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \varphi_{\alpha\beta}$$

$$\geq -(3-p)u_n^2 u_{11} u_{1n}^2 - (p-1)u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ni}^2 - (2p-3)u_n^2 u_{11}^3$$

$$- u_n^2 u_{11} \sum_{i=2}^{n-1} u_{ii}^2 + 2u_n^2 u_{11}^2 \sum_{i=2}^{n-1} u_{ii} - 6u_n^2 u_{1n}^2 \sum_{i=2}^{n-1} u_{ii}$$

$$+ 2 \sum_{i=2}^{n-1} \frac{1}{-u_{ii}} u_n^2 (-u_n u_{1ii} + u_{1n} u_{ii})^2 + 4u_n^3 u_{1n} \sum_{i=2}^{n-1} u_{1ii} - H(\nabla a_{11}) \tag{3.39}$$

$$= -u_n^2 u_{11} \left[(3-p)u_{1n}^2 + (p-1) \sum_{i=2}^{n-1} u_{ni}^2 \right]$$

$$- u_n^2 u_{11} \left[(2p-3)u_{11}^2 - 2 \sum_{i=2}^{n-1} u_{11} u_{ii} + \sum_{i=2}^{n-1} u_{ii}^2 \right]$$

$$+ 2 \sum_{i=2}^{n-1} \frac{u_n^2}{-u_{ii}} (2u_{1n} u_{ii} - u_n u_{1ii})^2 - H(\nabla a_{11})$$

where $H(\nabla a_{11})$ is the term involving $\nabla\varphi$ with locally bounded coefficients.

Recall that the level sets are strictly convex with respect to the normal direction ∇u, we have $u_{ii} < 0$ for $1 \leq i \leq n-1$. Hence, for $p \geq \frac{n+1}{2}$, we have

$$- u_n^2 u_{11} \left[(2p-3)u_{11}^2 - 2 \sum_{i=2}^{n-1} u_{11} u_{ii} + \sum_{i=2}^{n-1} u_{ii}^2 \right]$$

$$\geq -u_n^2 u_{11} \sum_{i=2}^{n-1} (u_{11} - u_{ii})^2 \geq 0.$$

We also need $p - 1 \geq 0$ and $3 - p \geq 0$, i.e. $1 \leq p \leq 3$.

Hence, for $\frac{n+1}{2} \leq p \leq 3$ we obtain

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 0 \qquad \mathrm{mod} \quad \nabla\varphi. \tag{3.40}$$

We complete the proof of the Theorem 1.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. **Principal curvature estimates of level set of minimal graphs.** In this section, we study the following equation

$$\text{div}(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^3, \tag{4.1}$$

and prove the Theorem 1.5.

We prove this theorem as in the last section. In the following proof, the Greek indices $(\alpha, \beta, \gamma, \delta, ...)$ run from 1 to 3, the Roman indices $(i, j, k, l, ...)$ run from 1 to 2. Denote

$$F^{\alpha\beta}(\nabla u) = (1 + |\nabla u|^2)\delta_{\alpha\beta} - u_\alpha u_\beta. \tag{4.2}$$

Then equation (4.1) is equivalent to

$$\sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta} = 0. \tag{4.3}$$

Proof of Theorem 1.5:

Since the level sets of u are strictly convex with respect to the normal direction ∇u, the curvature matrix (a_{ij}) of the level sets is positive definite in Ω. Let (a^{ij}) be the inverse matrix of (a_{ij}). As in the last section, Let $\varphi = a^{11}$. We will derive the following differential inequality

$$\sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 0 \qquad \text{mod} \quad \nabla\varphi \quad \text{in} \quad \Omega, \tag{4.4}$$

here and in the following we modify the terms involving $\nabla\varphi$ with locally bounded coefficients. Then by the standard strong maximum principle, we get the result immediately.

In order to prove (4.4) at an arbitrary point $x_o \in \Omega$, as in last section we choose the normal coordinate at x_o. We may assume $u_i(x_o) = 0$, $1 \leq i \leq 2$ and $u_3(x_o) = |\nabla u| > 0$. We can further assume the matrix $(u_{ij}(x_o))$ $(1 \leq i, j \leq 2)$ is diagonal and $u_{ii}(x_o) < 0$. It remains to establish (4.4) under the above assumptions.

From now on, *all the calculation will be done at the fixed point x_o.*

Similar to the proof of Theorem 1.3 in the last section, we have

$$u_3^3 a_{11}^2 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 2u_3^3 F^{22} a^{22} a_{12,2}^2 - u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta}.$$

Since $a^{22} = -\frac{u_3}{u_{22}}$, we have

$$u_3^4 a_{22} a_{11}^2 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 2F^{22}(u_3^2 a_{12,2})^2 + u_3^3 u_{22} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta}. \tag{4.5}$$

In the following, we shall prove

$$2F^{22}(u_3^2 a_{12,2})^2 + u_3^3 u_{22} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta} \geq 0 \qquad \text{mod} \quad \nabla\varphi \quad \text{in} \quad \Omega, \tag{4.6}$$

Let us prove (4.6) in two steps.

Step 1: We first calculate the term $u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta}$, it will be completed in (4.25).

As in the proof of (3.13), we can obtain

$$u_3 u_{11\alpha} = 2u_{1\alpha} u_{13} + u_{11} u_{3\alpha} - u_3^2 a_{11,\alpha}. \tag{4.7}$$

Similar to (3.18), we have

$$u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta} = T_1 + T_2 + T_3 + T_4 - 6u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha}, \qquad (4.8)$$

where the terms T_1, T_2, T_3, T_4 are the same as in (3.18) for $n = 3$.

For the term T_1, by (4.2), we have

$$\begin{aligned}
T_1 &= 3u_{11} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{3\alpha} u_{3\beta} + u_{11} \sum_{\alpha,\beta,\gamma=1}^{3} F^{\alpha\beta} u_{\gamma\alpha} u_{\gamma\beta} \\
&= (5 + 4u_3^2) u_{11} u_{13}^2 + (5 + 4u_3^2) u_{11} u_{23}^2 + 4u_{11} u_{33}^2 \\
&\quad + (1 + u_3^2) u_{11}^3 + (1 + u_3^2) u_{11} u_{22}^2.
\end{aligned} \qquad (4.9)$$

For the term T_2, by (4.7) we have

$$\begin{aligned}
T_2 &= 4u_3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{1\alpha} u_{13\beta} - 4u_3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{3\alpha} u_{11\beta} \\
&= 4(1 + u_3^2) u_{11}^2 u_{33} - 4(1 + u_3^2) u_{11} u_{13}^2 - 4(1 + u_3^2) u_{11} u_{23}^2 \\
&\quad - 4u_{11} u_{33}^2 - 8u_{33} u_{13}^2 + 4u_3 u_{13} u_{133} \\
&\quad + 4u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha} - 4u_3^2 (1 + u_3^2) u_{11} a_{11,3}.
\end{aligned} \qquad (4.10)$$

For the term T_3,

$$\begin{aligned}
T_3 &= 2u_{11} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{1\alpha} u_{1\beta} + 4u_{13} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{3\alpha} u_{1\beta} - 2u_{33} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{1\alpha} u_{1\beta} \\
&= 2u_{11} \big[(1 + u_3^2) u_{11}^2 + u_{13}^2\big] - 2u_{33} \big[(1 + u_3^2) u_{11}^2 + u_{13}^2\big] \\
&\quad + 4u_{13}^2 \big[(1 + u_3^2) u_{11} + u_{33}\big].
\end{aligned} \qquad (4.11)$$

By (4.10)–(4.11), we obtain

$$\begin{aligned}
T_2 + T_3 &= 2(1 + u_3^2) u_{11}^3 + 2(1 + u_3^2) u_{11}^2 u_{33} + 2u_{11} u_{13}^2 \\
&\quad - 4(1 + u_3^2) u_{11} u_{23}^2 - 4u_{11} u_{33}^2 - 6u_{33} u_{13}^2 \\
&\quad + 4u_3 u_{13} u_{133} + 4u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha} \\
&\quad - 4u_3^2 (1 + u_3^2) u_{11} a_{11,3}.
\end{aligned} \qquad (4.12)$$

Now we work on the term T_4, by (4.7) we get

$$\begin{aligned}
2u_3 u_{13} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta 1} &= 2u_3 u_{13}[(1 + u_3^2) u_{111} + (1 + u_3^2) u_{122} + u_{133}] \\
&= 6(1 + u_3^2) u_{11} u_{13}^2 + 2u_3(1 + u_3^2) u_{13} u_{122} \\
&\quad + 2u_3 u_{13} u_{133} - 2u_3^2 (1 + u_3^2) u_{13} a_{11,1}.
\end{aligned} \qquad (4.13)$$

Since

$$(F^{\alpha\beta})_3 = 2u_3 u_{33} \delta_{\alpha\beta} - u_{\alpha 3} u_\beta - u_\alpha u_{\beta 3},$$

we have

$$
\begin{aligned}
u_3 u_{11} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta3} &= - u_3 u_{11} \sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_3 u_{\alpha\beta} \\
&= 2u_3^2 u_{11} u_{13}^2 + 2u_3^2 u_{11} u_{23}^2 - 2u_3^2 u_{11}^2 u_{33} \\
&\quad - 2u_3^2 u_{11} u_{22} u_{33}.
\end{aligned}
\tag{4.14}
$$

For the term $-u_3^2 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta11}$ in T_4, by (4.2) we have

$$
(F^{\alpha\beta})_1 = 2u_3 u_{13}\delta_{\alpha\beta} - u_{\alpha1} u_{\beta} - u_{\alpha} u_{\beta1},
\tag{4.15}
$$

$$
\begin{aligned}
(F^{\alpha\beta})_{11} = 2\sum_{\gamma=1}^{3} u_{\gamma1}^2 \delta_{\alpha\beta} + 2u_3 u_{113}\delta_{\alpha\beta} \\
- u_{\alpha11} u_{\beta} - u_{\alpha} u_{\beta11} - 2u_{\alpha1} u_{\beta1}.
\end{aligned}
\tag{4.16}
$$

Taking derivative of equation (4.3) twice, we get

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_1 u_{\alpha\beta} + \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta1} = 0, \\
\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_{11} u_{\alpha\beta} + 2\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_1 u_{\alpha\beta1} + \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta11} = 0.
\end{aligned}
\tag{4.17}
$$

By (4.7) and (4.16), we have

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_{11} u_{\alpha\beta} &= (6u_{22} - 4u_{11})u_{13}^2 - 2u_{11} u_{23}^2 + 2u_{11}^2(u_{22} + 2u_{33}) \\
&\quad + 2u_{11} u_{22} u_{33} - 2u_3^2 \triangle u a_{11,3} + 2u_3^2 \sum_{\alpha=1}^{3} u_{3\alpha} a_{11,\alpha}.
\end{aligned}
\tag{4.18}
$$

Similarly, by (4.7) and (4.15), we have

$$
\begin{aligned}
2\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_1 u_{\alpha\beta1} &= 4u_{11} u_{13}^2 - 4u_{11}^2 u_{33} + 4u_3 u_{13} u_{221} \\
&\quad - 4u_3^2 u_{13} a_{11,1} + 4u_3^2 u_{11} a_{11,3}.
\end{aligned}
\tag{4.19}
$$

Therefore combining (4.17)–(4.19), we have

$$
\begin{aligned}
-\sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta11} &= \sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_{11} u_{\alpha\beta} + 2\sum_{\alpha,\beta=1}^{3} (F^{\alpha\beta})_1 u_{\alpha\beta1} \\
&= 6u_{22} u_{13}^2 + 2u_{11} u_{22} u_{33} + 2u_{11}^2 u_{22} \\
&\quad - 2u_{11} u_{23}^2 + 4u_3 u_{13} u_{122} - 2u_3^2 \triangle u a_{11,3} \\
&\quad + 2u_3^2 \sum_{\alpha=1}^{3} u_{3\alpha} a_{11,\alpha} - 4u_3^2 u_{13} a_{11,1} + 4u_3^2 u_{11} a_{11,3}.
\end{aligned}
\tag{4.20}
$$

By (4.13), (4.14) and (4.20), we have

$$
\begin{aligned}
T_4 = {} & u_3 u_{11} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta3} - u_3^2 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta11} \\
& + 2u_3 u_{13} \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} u_{\alpha\beta1} \\
= {} & (6 + 8u_3^2) u_{11} u_{13}^2 + 6u_3^2 u_{22} u_{13}^2 - 2u_3^2 u_{11}^2 u_{33} + 2u_3^2 u_{11}^2 u_{22} \\
& + (2u_3 + 6u_3^3) u_{13} u_{122} + 2u_3 u_{13} u_{133} - 2u_3^4 \triangle u a_{11,3} \\
& + 2u_3^4 \sum_{\alpha=1}^{3} u_{3\alpha} a_{11,\alpha} - (2u_3^2 + 6u_3^4) u_{13} a_{11,1} + 4u_3^4 u_{11} a_{11,3}.
\end{aligned}
\tag{4.21}
$$

Collecting (4.9), (4.12) and (4.21), we can obtain

$$
\begin{aligned}
u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta} = {} & T_1 + T_2 + T_3 + T_4 - 6u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha} \\
= {} & 3(1 + u_3^2) u_{11}^3 + 2u_{11}^2 u_{33} + 2u_3^2 u_{11}^2 u_{22} \\
& + (13 + 12u_3^2) u_{11} u_{13}^2 + u_{11} u_{23}^2 + (1 + u_3^2) u_{11} u_{22}^2 \\
& + 6u_3^2 u_{22} u_{13}^2 - 6u_{33} u_{13}^2 + (2u_3 + 6u_3^3) u_{13} u_{122} \\
& + 6u_3 u_{13} u_{133} - 2u_3^4 \triangle u a_{11,3} - 4u_3^2 u_{11} a_{11,3} \\
& + 2u_3^4 \sum_{\alpha=1}^{3} u_{3\alpha} a_{11,\alpha} - (2u_3^2 + 6u_3^4) u_{13} a_{11,1} \\
& - 2u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha}.
\end{aligned}
\tag{4.22}
$$

To simplify the above formula, we use the equation (4.1) to substitute the terms u_{33} and u_{133} in (4.22). From the equation (4.1), we have

$$
u_{33} = -(1 + u_3^2)(u_{11} + u_{22}).
\tag{4.23}
$$

Taking derivative of equation (4.3) with respect to x_1, we get

$$
u_{133} = -2u_3 u_{13} u_{22} - (1 + u_3^2) u_{111} - (1 + u_3^2) u_{122}.
$$

By (4.7) and (4.23), we have

$$
\begin{aligned}
u_3 u_{13} u_{133} = {} & -2u_3^2 u_{13}^2 u_{22} - u_3(1 + u_3^2) u_{13} u_{122} \\
& - 3(1 + u_3^2) u_{11} u_{13}^2 + (1 + u_3^2) u_3^2 u_{13} a_{11,1}.
\end{aligned}
\tag{4.24}
$$

Inserting (4.23) and (4.24) into (4.22), it follows that

$$
\begin{aligned}
u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta} = {} & (1 + u_3^2) u_{11}^3 - 2u_{11}^2 u_{22} + u_{11} u_{13}^2 + u_{11} u_{23}^2 \\
& + (1 + u_3^2) u_{11} u_{22}^2 + 6u_{13}^2 u_{22} - 4u_3 u_{13} u_{122} + L(\nabla a_{11}),
\end{aligned}
\tag{4.25}
$$

where

$$L(\nabla a_{11}) = -2u_3^4 \triangle u a_{11,3} - 4u_3^2 u_{11} a_{11,3} + 4u_3^2 u_{13} a_{11,1}$$
$$+ 2u_3^4 \sum_{\alpha=1}^{3} u_{3\alpha} a_{11,\alpha} - 2u_3^2 \sum_{\alpha=1}^{3} F^{\alpha\alpha} u_{3\alpha} a_{11,\alpha}$$

is the term involving $\nabla\varphi$ with locally bounded coefficients.

Step 2: The end of the theorem.

Similar to (3.37) in the last section, we have

$$u_3^2 a_{12,2} = -u_3 u_{122} + u_{13} u_{22}. \tag{4.26}$$

By (4.25) and (4.26), we have

$$2F^{22}(u_3^2 a_{12,2})^2 + u_{22} u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta}$$
$$= 2(1+u_3^2)(-u_3 u_{122} + u_{13} u_{22})^2 - 4u_3 u_{13} u_{22} u_{122}$$
$$+ u_{11} u_{22} u_{13}^2 + (1+u_3^2) u_{11}^3 u_{22} + 6u_{13}^2 u_{22}^2$$
$$+ u_{11} u_{22} u_{23}^2 + (1+u_3^2) u_{11} u_{22}^3 - 2u_{11}^2 u_{22}^2$$
$$= 2u_3^2(-u_3 u_{122} + u_{13} u_{22})^2 + 2(2u_{13} u_{22} - u_3 u_{122})^2$$
$$+ u_3^2 u_{11} u_{22}(u_{11}^2 + u_{22}^2)$$
$$+ u_{11} u_{22}[(u_{11} - u_{22})^2 + u_{13}^2 + u_{23}^2] \qquad \text{mod} \quad \nabla\varphi. \tag{4.27}$$

Since $u_{11} < 0$ and $u_{22} < 0$, we have

$$2F^{22}(u_3^2 a_{12,2})^2 + u_{22} u_3^3 \sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} a_{11,\alpha\beta} \geq 0 \qquad \text{mod} \quad \nabla\varphi.$$

From the observation in (4.5), it follows that

$$\sum_{\alpha,\beta=1}^{3} F^{\alpha\beta} \varphi_{\alpha\beta} \geq 0 \qquad \text{mod} \quad \nabla\varphi.$$

We complete the proof of Theorem 1.5. $\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.1. Using another auxiliary function, recently Chang-Ma-Yang [7] and Ma-Ou-Zhang [19] got the lower bound estimates of the principal curvature and the Gaussian curvature for the convex level sets of higher-dimensional harmonic functions in convex rings.

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